Topics: • Symplectic mnfds, Lagrangian submnfds, Morse-Smale, Arnold conjecture, Floer homology.

Prerequisites: Definition of smth mnfd, vector fields, integral curves differential forms, Lie derivatives, de Rham coh, vect bundles (C3.3 notes)

References: Canas da Silva, "Lectures on Symplectic geometry" [CdS]
McDuff, Salamon, "Introduction to symplectic topology" [MS]
[CdS] more accessible, [MS] comprehensive.

These cover background on symplectic geometry, some more:

- McDuff, Salamon, "J-holomorphic curves & quantum cohomology" [MSJ].
- Nicolaescu, "An Invitation to Morse theory" (background on Morse theory).
- Pascaleff's course on Lagrangian Floer homology (2014, M 392C, see website for more references)

Introductions: Donaldson, "What is... a pseudoholomorphic curve?"
Joyce, Symplectic geometry handout, lectures 9-16.
[MS] §4.5: a more detailed version

Defn.: symplectic mnfd (M, ω), ω is a non-degenerate^{①}, closed^{②} 2-form:

① if $V \in C^\infty(TM)$: $\omega(V, W) = 0 \ \forall W \in C^\infty(TM) \Rightarrow V = 0$.

② $d\omega = 0$.

Properties: • each T_pM is a vect space w/ symplectic form, so (lin alg $\Rightarrow$) dim M even. T_pM has basis $\{e_1, \dots e_n, f_1, \dots f_n\}$ st $\omega(e_i, f_i) = 1$, others $= 0$.

Locally $\omega = \sum e_i^* \wedge f_i^*$, $\omega^n = n!\, e_1^* \wedge f_1^* \wedge \dots \wedge e_n^* \wedge f_n^*$ non-vanishing vol form.

Examples: • T^*M, $\omega = -d\tau$, locally $\tau = \sum y_i (dx_i)$, y_i "$= dx_i$".

• $M = \mathbb{R}^n \Rightarrow \mathbb{R}^{2n}$, w/ $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. • Riemann surfaces

• $\mathbb{CP}^n$, smooth proj. var of $\mathbb{C}$ (Fubini-Study form, [MS] 4.3.3)

Nonexample: S^{2k} for $k > 1$. ($H^2(S^{2k}) = 0$, but $\omega^k \neq 0$ in $H^{2k}(S^{2k})$.)

Slogan: "Symplectic structures are rigid."

Lma: ([MS] Lma 3.2.1; [CdS] Thm. 7.4)

M^{2n}, $Q \subset M$ cmpct submnfd, $\omega_0, \omega_1 \in \Omega^2(M)$ s.t. on Q $\omega_0|_Q = \omega_1|_Q$, non-degenerate. Then "$\exists$ loc isotopy $\omega_0 \Rightarrow \omega_1$":

$\exists$ open $N_0, N_1 \supset Q$, and diffeomorphism $\psi: N_0 \to N_1$ st $\psi|_Q = id$, $\psi^* \omega_1 = \omega_0$.

Thm. ([MS] Thm 3.2.2, [CdS] Thm. 8.1) (M, ω), $p \in M \Rightarrow \exists$ chart $p \in U$, $\omega|_U = \sum_i dx_i \wedge dy_i$. Pf.: Take chart $p \in U'$, $\omega_0 = \omega|_{U'}$, and $\omega_1 = \sum_i dx_i \wedge dy_i$ on U'. Apply the lemma to $Q = \{p\}$.

Thm. ([MS] Thm 3.2.4, [CdS] Thm. 7.3) Let ω_t, $0 \leq t \leq 1$ be symplectic forms st $[\omega_i] \in H^2(M)$ constant, then "ω_t are isotopic": $\exists\, \rho: M\times\mathbb{R}\to M: \rho_t^*\omega_t = \omega_0$ for $t\in[0,1]$.

Lagrangian submnfds

Submnfd $N\subset M$ is Lagrangian if $\forall p\in N$, $T_pN\subset T_pM$ is Lagrangian: $W\subset V$ Lagrangian if $\{v\in V: \omega(v,W)=0\}=W$.

Cor: $\dim M = 2n \Rightarrow \dim N = n$.

Example: $(M,\omega) \Rightarrow (M\times M, (-\omega,\omega))$ symplectic, $\Delta\subset M\times M$ Lagrangian.

$(x_i, y_i)\in\mathbb{R}^{2n}$, $\omega = \sum_i dx_i\wedge dy_i \Rightarrow \{y_i=0\}\subset\mathbb{R}^{2n}$ Lagrangian.

$\mathbb{RP}^n\subset\mathbb{CP}^n$, $\{[z_0,\dots z_n]: |z_0| = \dots = |z_n|\}\subset\mathbb{CP}^n$ (Clifford torus)

Maslov index for Lagrangian subspaces

$\mathcal{L}(n) := \{$Lagrangian subspaces of $\mathbb{R}^{2n}$, w/ $\omega_0 := \sum_i dx_i\wedge dy_i\}$

Let $X, Y\in M_n(\mathbb{R})$, image of $\begin{pmatrix}X\\Y\end{pmatrix}: \mathbb{R}^{2n}\to\mathbb{R}^n$ Lagrangian iff $X^TY = Y^TX$. Columns of $\begin{pmatrix}X\\Y\end{pmatrix}$ orthonormal $\Leftrightarrow X+iY$ unitary.

For $L\in\mathcal{L}(n)$, pick unitary $\begin{pmatrix}X\\Y\end{pmatrix}$, then $\begin{pmatrix}X & -Y\\ Y & X\end{pmatrix}$ takes $\{(x,y)\in\mathbb{R}^{2n}: y=0\}$ to L. $\Rightarrow$ Fact: $\mathcal{L}(n)\cong U(n)/O(n)$, mnfd of dim $\frac{n(n+1)}{2}$. ([MS], §2.3) $\Rightarrow \pi_1(\mathcal{L}(n))\cong\mathbb{Z}$ (MSE 767351)

Given $\Lambda\in\mathcal{L}(n)$, write $\Lambda = \operatorname{im} Z$, $Z:\mathbb{R}^{2n}\to\mathbb{R}^n$, $Z=\begin{pmatrix}X\\Y\end{pmatrix}$, st $U := X+iY\in U(n)$. Define $\rho(\Lambda) := \det(U)^2\in S^1$.

Maslov index: $\mu: \pi_1(\mathcal{L}(n))\to\mathbb{Z}$, $\gamma\mapsto\deg(S^1\xrightarrow{\gamma}\mathcal{L}(n)\xrightarrow{\rho}S^1)$.

([MS], §2.2) Fact: well-defined (homotopy invariant), "twisted by $Sp(2n;\mathbb{R})$":
$\exists$ Maslov index $\pi_1(Sp(2n;\mathbb{R})) \to \mathbb{Z}$, for $\gamma \in \pi_1(\mathcal{L}(n))$, $\Phi \in \pi_1(Sp(2n))$:
$\mu(\Phi(\gamma)) = \mu(\gamma) + 2\mu(\Phi)$, additive for $\mathcal{L}(n_1) \times \mathcal{L}(n_2) \subset \mathcal{L}(n_1+n_2)$,
normalized: $S^1 \to \mathcal{L}(1)$ given by straight lines through origin has Maslov index 1.

Maslov class. Let $L \subset (\mathbb{R}^{2n}, \omega_0)$ be Lagrangian, loop $\gamma: S \to L$ defines $\mu_L: \pi_1(L) \to \mathbb{Z}$, $\gamma \mapsto \mu(S^1 \to \mathcal{L}(n),\ t \mapsto T_{\gamma(t)}L)$. μ_L is the Maslov class of L. If μ_L is non-zero, min Maslov number of L is +ve generator of $\mu_L(\pi_1(L)) \subset \mathbb{Z}$. o/w set it $= \infty$.

An alternative defn uses "intersection #" of loop γ w/ Maslov cycle ($\subset \mathcal{L}(n)$), see [MS], p. 53-4.

Rigidity for Lagrangian submfd:

Thm ([CdS] Thm 8.4) Suppose ω_0, ω_1 are symplectic forms on M, $L \subset M$ is Lagrangian wrt both, then $\exists\ L \subset \mathcal{N}_0, \mathcal{N}_1$ w/ diffeo $\varphi: \mathcal{N}_0 \to \mathcal{N}_1$ st $\varphi^*\omega_1|_{\mathcal{N}_1} = \omega_0|_{\mathcal{N}_0}$.

Pf: Last lemma w/ $Q = L$.

Thm ([MS] Thm 3.4.13, [CdS] Thm 9.3) (M,ω), $L \subset M$ cmpct Lagrangian, $\exists$ nbhds $L \subset \mathcal{N}_0 \subset T^*L$ ($L \overset{0}{\hookrightarrow} T^*L$), $L \subset \mathcal{N}_1 \subset M$ and diffeo $\mathcal{N}_0 \xrightarrow{\psi} \mathcal{N}_1$ st $\psi^*\omega_{T^*L} = \omega$. (Key: $NL \cong T^*L$.)

Hamiltonian

$X \mapsto \iota_X\omega$ sets up $C^\infty(TM) \cong \Omega^1(M)$, Equiv to Hamilton equation for T^*M (say $\mathbb{R}^{2n}$).

$H: M \to \mathbb{R} \Rightarrow X_H$ dual to dH (ie, $\iota_{X_H}\omega = dH$).

Cartan's formula $\mathcal{L}_X\omega = d\iota_X\omega + \iota_X d\omega \Rightarrow$ H constant, ω preserved under flow/integral curves of X_H.

ω, J, and g. ([MS] §4, [CdS] §12-13)

- An almost complex structure on M is bundle automorphism $J: TM \to TM$ st $J^2 = -id$.
  - Fact: $\exists J \Rightarrow M$ oriented (T_pM $\mathbb{C}$-vector space), even dim.
- (M, ω). J is ω-tame if $\omega(v, Jv) > 0$ for $0 \neq v \in TM$,
  ω-cmptble if $\omega(v, w) = \omega(Jv, Jw)$ and ω-tame
  $\Rightarrow g(v, w) := \omega(v, Jw)$ is a Riemannian metric on M.
- A metric g on M cmptble w/ J if $g(v, w) = g(Jv, Jw)$, then $\omega(v, w) := g(Jv, w)$ non-degenerate 2-form (not ness. closed).
- ω and any metric $g \Rightarrow \exists J$ cmptble w/ ω and g.
  Locally: both ω, g sets up $V \xrightarrow{\simeq} V^*$, so $\exists A \in GL(V)$: ([CdS], Prop 12.3)
  $\omega(u, v) = g(Au, v)$, ω skew $\Rightarrow A$ skey-symm, and
  AA^* positive symmetric $\Rightarrow \exists \sqrt{AA^*}$, and $J := (\sqrt{AA^*})^{-1}A$.
  J unique upto choice of g. Fact: $\{\omega\text{-cmptbl } J\}$, $\{J\text{-cmptbl } \omega\}$ contractible. Fixing $g, \omega \mapsto J$ is hmtpy equiv. ([MS], 2.6.4)
- J is integrable if $\exists$ complex mnfd structure on M, w/ J arising as mult. by $\sqrt{-1}$.
- J extends to $TM \otimes_{\mathbb{R}} \mathbb{C}$ by $J(v \otimes z) := Jv \otimes z$, $J^2 = -id \Rightarrow$ $\pm\sqrt{-1}$ eigenspaces $T_{1,0} \oplus T_{0,1} = TM_{\mathbb{C}}$, and $TM \simeq T_{1,0} \simeq \overline{T_{0,1}}$ (as $\mathbb{C}$-vect bundles w/ $J := \sqrt{-1}$)

Function $M \to \mathbb{C}$ is J-holomorphic if df is $\mathbb{C}$-linear, ie., $df(Jv) = i\, df(v)$. In complex analysis also say "$\bar\partial f = 0$", to make this precise: define $T^{1,0} = (T_{1,0})^* \subset T^*M$, $\underbrace{T^{1,0} \oplus T^{0,1}}_{\text{eigenspaces of dual map}} = T^*M$,

$$T^*M \xrightarrow{\pi^{1,0}} T^{1,0}, \quad T^*M \xrightarrow{\pi^{0,1}} T^{0,1}; \qquad \underbrace{C^\infty(M) \xrightarrow{d} T^*M \xrightarrow{\pi^{0,1}} T^{0,1}}_{\bar\partial}.$$

Taking Λ-powers, $\Lambda^{l,m} := (\Lambda^l T^{1,0}) \wedge (\Lambda^m T^{0,1})$, $\Omega^{l,m} := \Gamma(\Lambda^{l,m})$, can define $\partial : \Lambda^{l,m} \to \Lambda^{l+1,m}$, $\bar\partial : \Lambda^{l,m} \to \Lambda^{l,m+1}$, J integrable $\Leftrightarrow$ $\underbrace{d = \partial + \bar\partial}_{\text{holds for } \Omega^0, \text{ nontrivial for } \Omega^{>0}}$.

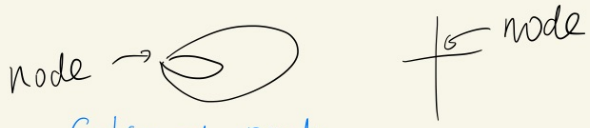
Example: On Riemann surface, $T^{1,0} = \{dz\}$, $T^{0,1} = \{d\bar z\}$, $\Omega^2(\Sigma) = \{dz \wedge d\bar z\} = \Omega^{1,1}$ (no $\Omega^{0,2}$, $\Omega^{2,0}$ components!)

J-holomorphic curves

Let (Σ, j) be a closed Riemann surface, $j \in \mathrm{Aut}(T\Sigma)$ (almost) complex structure. A J-holomorphic curve on (M, ω) is a J-holomorphic map $\Sigma \xrightarrow{u} M$, ie., $\bar\partial_J u := \underbrace{\tfrac{1}{2}(du + J \circ du \circ j) = 0}_{\text{an (elliptic) PDE for } u}$.

Goal: Construct a moduli of J-holom. curves, discuss smthness, dim, ori/fundamental class, write down invariants which "counts" J-holom curves (satisfying some constraints).

Notation: $\mathcal{M}_{g,m}(M, J, \beta) = \{ u : \Sigma \to M$, Σ of genus g w/ m marked pts $\vec z$, u is J-holom, $u^*[\Sigma] = \beta \in H_2(M, \mathbb{Z}) \}$.



← Cultural rmk

Thm: (i) Let M be cmpct. Upon allowing nodal Riemann surfaces, (loc $\{xy=0\}$) + $\mathrm{Aut}(\Sigma, \vec z, u)$ fin, get $\overline{\mathcal{M}}_{g,m}(M, J, \beta)$ Hausdorff cmpct.

(ii) For generic J, $\{(\Sigma, \vec z, u) \in \overline{\mathcal{M}}_{g,m}(M, J, \beta) : \Sigma$ non-sing & $u: \Sigma \to M$ emb$\}$ is a smth, ori mnfd,

$\dim = 2(c_1(M)\cdot\beta + (n-3)(1-g) + m)$. $c_1(M) = c_1(TM)$, $\dim M = 2n$.

For our purposes, focus on smth + $m=0$ case:

Ambient space: $\mathcal{X} = \mathrm{Map}(\Sigma, M, \beta)$ ← "Somewhere injective"

Fact: $\mathcal{X}$ ∞-dim mnfd w/ $T_u\mathcal{X} = C^\infty(u^*TM)$.

Let $\mathcal{E} \to \mathcal{X}$ have fibre $\Omega^{0,1}(u^*TM) = C^\infty(\Lambda^{0,1}(\Sigma) \otimes u^*TM)$.

Or: $\mathcal{E}_u = \{\varphi \in \Gamma(\mathrm{Hom}(T\Sigma, u^*TM)) : \varphi \circ j = -J \circ \varphi\}$ ($T^*\Sigma$ dual to $T\Sigma$.)

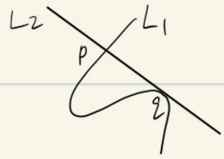
J-holom translates to $\bar\partial_J u = 0$, where

$\bar\partial_J : \mathcal{X} \to \mathcal{E}$, $u \mapsto \bar\partial_J u \in \mathcal{E}_u$, the $(0,1)$-part of du.

Define $\mathcal{M}(\beta, J) = \bar\partial_J^{-1}(0) \subset \mathcal{X}$, see this as $\{\bar\partial_J$ section$\} \cap \{0$ section$\}$.

For $\mathcal{M}(\beta, J)$ to be mnfd hope the intersection is "transverse".

$$D_u = \left(T_u\mathcal{X} \xrightarrow{d\bar\partial_J(u)} \underbrace{T_{(u, \bar\partial_J(u))}\mathcal{E}}_{T_u\mathcal{X} \oplus \mathcal{E}_u} \to \mathcal{E}_u\right).$$



$T_pL_1 + T_pL_2 = T_p\mathbb{R}^2$,

$T_qL_1 + T_qL_2 \subsetneq T_q\mathbb{R}^2$.

If D_u surjective, then $d\bar\partial_J(T_u\mathcal{X}) + \underbrace{T_u\mathcal{X}}_{\text{tan space of 0-section}} = T_{(u,0)}\mathcal{E} \Rightarrow$ "transverse".

For this reason, call J regular if $\forall \beta \in H_2$, $u \in \mathcal{M}(J, \beta)$: D_u onto.

To do analysis, need to introduce more functions ("Sobolev completion").

- J can be of class C^l,
- $\mathcal{X}^{k,p} := \{u : \Sigma \to M : k\text{-th derivative of class } L^p, u_*[\Sigma] = \beta\}$.
- $D_u : W^{k,p}(u^*TM) \to W^{k-1,p}(\Lambda^{0,1}\Sigma \otimes u^*TM)$

Sobolev functions instead of smth functions

On Σ, use $z = x+iy$, C-R equation is $\frac{\partial u}{\partial x} = -J \cdot \frac{\partial u}{\partial y}$,

so elliptic (top degree part non-vanishing),

$D_u = \bar{\partial}_J$ + 0-th order terms + elliptic regularity ([MSJ], §3.3 + Appendix B.2-3) $\Rightarrow D_u$ Fredholm $\begin{cases} \text{ran}(D_u) \text{ closed} \\ \ker(D_u), \text{coker}(D_u) \text{ fin dim.} \end{cases}$

Lma ("Implicit function thm") If D_u onto, Q_u right inverse + bounds on $\|Q_u\|$, $\|du\|_{L^p}$, and $\text{vol}(\Sigma)$, then for $\xi \in \ker D_u$ with small norm, $\exists\, \hat{\xi} = Q_u y$ st $\bar{\partial}_J(\exp_u(\xi + Q_u y)) = 0$, w/ bound on $\|Q_u y\|$, ie, also small norm. ([MSJ], Thm 3.3.4)

Cor: $\ker D_u \to \mathcal{M}(A,J)$ given by $\xi \mapsto \exp_u(\xi + Q_u y)$ charts, $\mathcal{M}(A,J)$ mnfd w/ $T_u\mathcal{M}(A,J) \cong \ker D_u$, and D_u onto $\Rightarrow \dim\ker D_u = \dim\ker D_u - \dim\text{coker } D_u := \text{ind}(D_u)$.

Recall: $D_u = \bar{\partial}_J + (\text{l.o. terms})$ homotope to $0 \Rightarrow \text{ind } D_u = \text{ind } \bar{\partial}_J$, where $\bar{\partial}_J$ determines a holom structure on u^*TM, w/ index $= 2(n(1-g) + \langle c_1(TM), \beta\rangle)$. (Riemann-Roch over Riemann surface)

rank TM

V: holom. vect bundle o/ Riemann surface $\leftarrow$ [$ch(V) = rk(V) + c_1(V)$, $Td = 1 + c_1(TM)/2$, $\text{index} = \int_M ch(V)\, Td$, and $\text{ind}_{\mathbb{C}} D_u = 2\, \text{ind}_{\mathbb{R}} D_u$.

Now we have seen $\mathcal{M}(\beta, J)$ a mnfd and know $\dim_{\mathbb{R}} \mathcal{M}(\beta, J)$, other aspects of the moduli space:

- Which J are regular (ie, D_u surjective for all u, β)?

([MSJ], Thm 3.1.2, pf at p.36)

Consider $\mathcal{M}(\beta, J) \to \mathcal{J} = \{\omega\text{-cmptbl almost cx structures}\}$

Upon modification (considering C^{ℓ} instead of C^{∞}), regular J are the "regular values" of the map, hence $\{\text{reg } J\}$ contains a countable intersection of open dense (Sard-Smale).

- How does $\mathcal{M}(\beta, J)$ depend ("deform") on J?

Given $J_0, J_1 \in \mathcal{J}$, write $\mathcal{J}(J_0, J_1)$ for smth paths of almost cx structures from J_0 to J_1. If J_0, J_1 regular, then $\exists$ dense subset $\mathcal{J}^{reg}(\beta, J_0, J_1) \subset \mathcal{J}(J_0, J_1)$, st $\forall J_\lambda \in \mathcal{J}^{reg}(\beta, J_0, J_1)$, $\mathcal{M}(\beta, \{J_\lambda\}) = \bigcup_\lambda \mathcal{M}(\beta, J_\lambda)$ is smth, $\dim = \dim \mathcal{M}(\beta, J_\lambda) + 1$, an (ori - see later!) cobordism between $\mathcal{M}(\beta, J_i)$, $i = 0, 1$.