ADVANCED ALGEBRAIC TOPOLOGY

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Contents

1. Intro to (Co)bordism Theory

1.1. **Overview.**

1.2. **Colimits of topological spaces and abelian groups.**

1.3. **Classifying spaces.** A $\mathbb{R}^n$-vector bundle ξ over a space X is a family of n-dimensional vector spaces parameterized by X. More precisely,

Definition 1.1. we have map $E \xrightarrow{p} X$ such that

(1) for any $x \in X$, $p^{-1}(x) \cong \mathbb{R}^n$;
(2) any $x \in X$ has a neighborhood U with $p^{-1}(U) \cong U \times \mathbb{R}^n$, and the map to X is projection to the first factor;
(3) given two such neighborhoods U_1 and U_2 with $U_1 \cap U_2 \neq \varnothing$,

$$(U_1 \cap U_2) \times \mathbb{R}^n \underset{u}{\xrightarrow{\cong}} p^{-1}(U_1 \cap U_2) \underset{v}{\xleftarrow{\cong}} (U_1 \cap U_2) \times \mathbb{R}^n.$$

Let $h_{12} = v^{-1} \circ u$. For any $u \in U_1 \cap U_2$ and $x \in \mathbb{R}^n$, $h_{12}(u, x) = (u, \eta_{12})(u)(x)$, where $\eta_{12}(u) \in \mathrm{GL}_n(\mathbb{R})$. This data gives a map

$$U_1 \cap U_2 \xrightarrow{\eta_{12}} \mathrm{GL}_n(\mathbb{R})$$

called the **transition map**. Transition maps are compatible in a suitable way you would expect.

Example 1.2. Let $E = X \times \mathbb{R}^n$. This is a trivial n-bundle, which we shall denote by $o(n)$. When $n = 1$, it is called a line bundle. In this case, η_{12} is the constant map.

Example 1.3. Let $X = S^1$, $n = 1$, E be the interior of the Möbius band. This is a twist of the trivial line bundle, and so $E \not\cong S^1 \times \mathbb{R}$.

1.4. **Characteristic classes.**

1.5. **Eilenberg-MacLane spaces.**

1.6. **Tangential structures.** Let H, G be Lie groups and $\rho : H \to G$ be a homomorphism.

Definition 1.4. (1) Let $Q \to M$ be a principal H-bundle. The **associated principal G-bundle** $Q_\rho \to M$ is the quotient

$$Q_\rho = (Q \times G)/H,$$

where H acts freely on the right of $Q \times G$ by $(q, g) \cdot h = (q \cdot h, (\phi(h))^{-1}(g))$, where $q \in Q$, $g \in G$, and $h \in H$.

(2) Let $P \to M$ be a principal G-bundle. A **reduction** to H is a pair (Q, θ) consisting of a principal H-bundle $Q \to M$ and an isomorphism

$$\begin{tikzcd} Q_\rho \arrow[rr, "\theta"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

of principal G-bundles.

Example 1.5. Recall the determinant homomorphism

$$\mathrm{GL}_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^{\neq 0}.$$

Let $\mathrm{GL}_n^+(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ denote the subgroup $\det^{-1}(\mathbb{R}^{>0})$. An orientation of a real rank n vector bundle $V \to M$ is a reduction of structure group of $B(V) \to M$ to the group $\mathrm{GL}_n^+(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$, where $B(V) \to M$ is the associated principal $\mathrm{GL}_n(\mathbb{R})$-bundle, or the **frame bundle**.

Definition 1.6. The **spin group** $\mathrm{Spin}(n)$ is the double cover group of $SO(n)$.

Example 1.7. $\mathrm{Spin}(1) = \mathbb{Z}/2$, $\mathrm{Spin}(2) = S^1$, $\mathrm{Spin}(3) = S^3$. Note that $\mathrm{Spin}(3)$ is a nontrivial double cover of $SO(2)$.

Definition 1.8. Let $V \to M$ be a real vector bundle of rank n with a metric. A **spin structure** on V is a reduction of structure group of the orthonormal frame bundle $B(V) \to M$ along $\rho : \mathrm{Spin}(n) \to O(n)$. Here ρ is the projection $\mathrm{Spin}(n) \to SO(n)$ followed by the inclusion $SO(n) \to O(n)$. Therefore, it can be thought of in two steps: an orientation followed by a lift to the double cover.

1.7. **Spectra.** Recall that we have an adjoint pair of sets $\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}_*(X, \Omega Y)$.

Definition 1.9. A **prespectrum** T is a sequence $\{T_n\}_{n\geq 0}$ of pointed spaces and maps $\sigma_n : \Sigma T_n \to T_{n+1}$. T is an **Ω-prespectrum** if the adjoints $\tilde{\sigma}_n : T_n \to \Omega T_{n+1}$ of structure maps are weak homotopy equivalence. T is further a **spectrum** if $\tilde{\sigma}_n$, $n \geq 0$ are homeomorphism.

Example 1.10. Let X be a pointed space. The **suspension prespectrum** of X, denoted $\Sigma^\infty X$, is defined by setting $(\Sigma^\infty X)_n = \Sigma^n X$ with structure maps being the identity maps. If $X = S^0$, $\Sigma^\infty S^0 = \mathbb{S}$ is the sphere prespectrum.

Example 1.11. Let A be an abelian group. The set of Eilenberg-MacLane spaces $\{K(A,n)\}_{n\geq 0}$ with the obvious structure maps forms an Ω-prespectrum.

One might wonder if every prespectrum becomes a spectrum after some process. This leads to the concept of *spectrification*. Let T be a prespectrum. If the adjoint structure maps $\tilde{\sigma}_n : T_n \to \Omega T_{n+1}$ are inclusions, then we define $(LT)_n$ to be the colimit of $T_n \xrightarrow{\tilde{\sigma}_n} \Omega T_{n+1} \xrightarrow{\Omega\tilde{\sigma}_{n+1}} \Omega^2 T_{n+2} \to \cdots$. Namely,

$$(LT)_n = \mathrm{colim}_{q\to\infty} \Omega^q T_{n+q}.$$

$\{LT\}_{n\geq 0}$ is then a spectrum.

Remark 1.12. Every map is a cofibration up to homotopy equivalence. In general, we can define colimits not only for inclusions, but also for cofibrations.

The spectrification function $L : \mathsf{PreSp} \to \mathsf{Sp}$ is the left adjoint to the forgetful functor $U : \mathsf{Sp} \to \mathsf{PreSp}$. In particular, for a spectrum S and a prespectrum T, we have

$$[LT, S] = [T, US].$$

Example 1.13. Let X be a pointed space. Consider its suspension prespectrum $\Sigma^\infty X$. Let $L\Sigma^\infty X$ be its spectrification. Then $(L\Sigma^\infty X)_0 = \mathrm{colim}_{q\to\infty}\Omega^q \Sigma^q X$, denoted by QX. In particular, for $X = S^0$, $(L\Sigma^\infty S^0)$ is the sphere spectrum $\mathbb{S}$ (note here we abuse the notation).

Example 1.14. Let A be an abelian group. The spectrification of $\{K(A,n)\}_{n\geq 0}$ is the **Eilenberg-MacLane spectrum** HA.

Definition 1.15. Let T be a prespectrum. Its **homotopy groups** is defined to be the colimit of $\pi_{n+q} T_q \xrightarrow{\pi_{n+q}\tilde{\sigma}_q} \pi_{n+q}\Omega_{q+1} \xrightarrow{\text{adjunction}} \pi_{n+q+1} T_{q+1} \to \cdots$. Namely,

$$\pi_n T = \mathrm{colim}_{q\to\infty} \pi_{n+q} T_q.$$

The **homology groups** of T is defined similarly:

$$H_n(T; R) = \mathrm{colim}_{q\to\infty} \tilde{H}_{n+q}(T_q; R),$$

where the colimit is taken over maps

$$\tilde{H}_{n+q}(T_q; R) \xrightarrow{\Sigma} \tilde{H}_{n+q+1}(\Sigma T_q; R) \xrightarrow{(\sigma_q)_*} \tilde{H}_{n+q+1}(\Sigma T_{q+1}; R).$$

Remark 1.16. In general, we cannot define the cohomology groups in this way. Instead, we will see how it goes using representability.

Let M, N be two prespectrum. We use the notion $[M, N]_{-k}$ ($k \in \mathbb{Z}$) to denote the homotopy classes of maps from M to N of degree $-k$. To be specific, any map in $[M, N]_{-k}$ consists of a sequence of maps $M_n \to N_{n+k}$ such that the diagram commutes:

$$\begin{tikzcd}
\Sigma M_n \arrow[r] \arrow[d] & \Sigma N_{n+k} \arrow[d] \\
M_{n+1} \arrow[r] & N_{n+k+1}
\end{tikzcd}$$

where the columns are the structure maps. Two maps of prespectra of degree $-k$ are homotopic, denoted $f \sim g$, if there is a map $H : M_* \wedge I_+ \to N_{*+k}$ which restricts to $f \vee g$ along the inclusion $M_* \vee M_* \xrightarrow{i_0 \vee i_1} M_* \wedge I_+ \xrightarrow{H} N_{*+k}$, where $I_+ = [0,1] \sqcup \{*\}$, and i_0, i_1 are the inclusion maps at $0, 1 \in [0,1]$, respectively. If N is an Ω-prespectrum, then $[M, N]_{-k}$ is an abelian group. If in addition, N is a ring spectrum (i.e. a spectrum admitting a ring structure, see Definition 1.19), then $[M, N]_{-*}$ forms a graded ring. We usually write $[M, N] = [M, N]_0$.

Corollary 1.17. $\pi_d M = [\mathbb{S}, M]_d$.

Definition 1.18. Let R be a ring. Then the Eilenberg-MacLane spectrum (Example 1.14) is a ring spectrum. The **cohomology ring** of a prespectrum M with coefficients in R is defined to be $H^*(M; R) \coloneqq [M, HR]_{-*}$.

Definition 1.19. Let T be a prespectrum. T is a **ring prespectrum** if there are maps $\eta : S^0 \to T_0$ and $\phi_{m,n} : T_m \wedge T_n \to T_{m+n}$ such that the following diagrams commute up to homotopy equivalence:

$$\begin{tikzcd}
T_m \wedge \Sigma T_n \arrow[r, "\mathrm{id} \wedge \sigma_n"] \arrow[d, equal] & T_m \wedge T_{n+1} \arrow[dr, "\phi_{m,n+1}"] & \\
\Sigma(T_m \wedge T_n) \arrow[r, "\Sigma\phi_{m,n}"] \arrow[d, "(-1)^n"'] & \Sigma T_{m+n} \arrow[r, "\sigma_{m+n}"] & T_{m+n+1} \\
\Sigma T_m \wedge T_n \arrow[r, "\sigma_m \wedge \mathrm{id}"] & T_{m+1} \wedge T_n \arrow[ur, "\phi_{m+1,n}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
S^0 \wedge T_n \arrow[r, "\eta \wedge \mathrm{id}"] \arrow[dr, "\cong"'] & T_0 \wedge T_n \arrow[d, "\phi_{0,n}"] \\
& T_n
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
T_n \wedge T_0 \arrow[d, "\phi_{n,0}"'] & T_n \wedge S^0 \arrow[l, "\mathrm{id} \wedge \eta"'] \arrow[dl, "\cong"] \\
T_n &
\end{tikzcd}$$

T is said to be **associative** if the following diagrams are homotopy commutative:

$$\begin{tikzcd}
T_m \wedge T_n \wedge T_p \arrow[r, "\phi_{m,n} \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge \phi_{n,p}"'] & T_{m+n} \wedge T_p \arrow[d, "\phi_{m+n,p}"] \\
T_m \wedge T_{n+p} \arrow[r, "\phi_{m,n+p}"] & T_{m+n+p}
\end{tikzcd}$$

T is **commutative** if there are equivalences $(-1)^{mn} : T_{m+n} \to T_{m+n}$ that suspend to $(-1)^{mn}$ on ΣT_{m+n}, and the following diagrams are homotopy commutative:

$$\begin{tikzcd}
T_m \wedge T_n \arrow[r, "t"] \arrow[d, "\phi_{m,n}"'] & T_n \wedge T_m \arrow[d, "\phi_{n,m}"] \\
T_{m+n} \arrow[r, "(-1)^{mn}"] & T_{m+n}
\end{tikzcd}$$

T is a **ring spectrum** if it is already a spectrum.

Example 1.20. The sphere spectrum $\mathbb{S}$ is a ring spectrum.

Example 1.21. The Eilenberg-MacLane spectrum HR is a ring spectrum, if R is a ring.

Example 1.22. The Thom spectra $MO, MSO, M\mathrm{Spin}$ are ring spectra. We will discuss them right away.

Recall that we have a chain of inclusions

$$i_n : BO(n) \to BO(n+1).$$

For each i_n, it is the colimit of the inclusions of real Grassmannians in the columns of the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & G_{n,k} \arrow[r] \arrow[d] & G_{n,k+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & G_{n,k} \arrow[r] \arrow[d] & G_{n,k+1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

From the definition of tautological vector bundle $S(n) \to BO(n)$, there is a natural isomorphism over $BO(n)$:

$$i_n^* S(n+1) \xrightarrow{\cong} \underline{\mathbb{R}} \oplus S(n).$$

If $\mathcal{Y}$ is a stable tangential structure, then we also have maps $i_n : \mathcal{Y}(n) \to \mathcal{Y}(n+1)$. The same isomorphism can be derived in a similar manner.

Let $\xi : V \to X$ be an n-plane bundle. Apply one-point compactification to each fiber of ξ to obtain a new bundle $S(V)$ over X, whose fibers are spheres S^n with given basepoints (usually denoted ∞). These basepoints specify a cross-section $X \to S(V)$.

Definition 1.23. The **Thom space** of ξ is the quotient space $T\xi = S(V)/X$. In other words, $T(\xi)$ is obtained from V by applying fiberwise one-point compactification and then identifying all of the points at ∞ to a single basepoint (also denoted as ∞).

If we equip ξ with an Euclidean metric and denote its unit disk bundle and unit sphere bundle by $D(V)$, $S(V)$, respectively, then there is a homeomorphism between $T\xi$ and the quotient space $D(V)/S(V)$, written as $\mathrm{Thom}(X; V)$.

Proposition 1.24. *Thom spaces satisfy*

(1) $\mathit{Thom}(X \times Y; V \times W) = \mathit{Thom}(X; V) \wedge \mathit{Thom}(Y; W)$.
(2) $\mathit{Thom}(X; V \oplus \underline{\mathbb{R}}^n) = \Sigma^n \mathit{Thom}(X; V)$.
(3) $\mathit{Thom}(X; \underline{\mathbb{R}}^n) = \Sigma^n X_+$.

Here $V \to X$, $W \to Y$ are real vector bundles, and $\underline{\mathbb{R}}^n$ is the trivial real bundle of dimension n.

Proof. For (1), note that $D(V \times W) = D(V) \times D(W)$, $S(V \times W) = \partial D(V \times W) = \partial D(V) \times D(W) \cup D(V) \times \partial D(W) = S(V) \times D(W) \cup D(V) \times S(W)$. It follows that

$$\begin{aligned}\mathrm{Thom}(X \times Y; V \times W) &= D(V \times W)\big/S(V \times W)\\ &= D(V) \times D(W)\big/S(V) \times D(W) \cup D(V) \times S(W)\\ &= D(V)/S(V) \wedge D(W)/S(W) = \mathrm{Thom}(X;V) \wedge \mathrm{Thom}(Y;W).\end{aligned}$$

Let $V = \varnothing$, then $D(V)/S(V) = X_+$. Hence, (3) immediately follows from (2). For (2), we induct on n. The only case to be checked is when $n = 1$, and the rest are straightforward. Up to homeomorphism, we can replace the disk bundle of $V \oplus \underline{\mathbb{R}} \to X$ by the Cartesian product of the unit disk in $\mathbb{R}$ and the disk bundle of $V \to X$. Crushing the boundary of the disk bundle of $V \oplus \underline{\mathbb{R}} \to X$ to a point is the same crushing which one dose to form the suspension of $\mathrm{Thom}(X;V)$. The base case is then verified. □

Let $\mathcal{Y}$ be a stable tangential structure. Consider the diagram

$$\begin{tikzcd} \mathbb{R} \oplus S(n) \arrow[r] \arrow[d] & S(n+1) \arrow[d] \\ \mathcal{Y}(n) \arrow[r, "i"] & \mathcal{Y}(n+1) \end{tikzcd}$$

There are induced maps in Thom spaces by Proposition 1.24

$$s_n : \Sigma\mathrm{Thom}(\mathcal{Y}(n); S(n)) \to \mathrm{Thom}(\mathcal{Y}(n+1); S(n+1)). \tag{1.25}$$

Definition 1.26. The **Thom prespectrum** $T\mathcal{Y}$ of a stable tangential structure $\mathcal{Y}$ is defined by $T\mathcal{Y}_n \coloneqq \mathrm{Thom}(\mathcal{Y}(n); S(n))$ with structure maps given by (1.25). The **Thom spectrum** $M\mathcal{Y}$ is the spectrification $L(T\mathcal{Y})$.

Let $\{G(n)\}_{n\geq 0}$ be a sequence of Lie groups with maps $G(n) \to G(n+1)$ for each n. Let $\rho(n) : G(n) \to O(n)$ be a series of homomorphisms such that the diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & G(n) \arrow[r] \arrow[d, "\rho(n)"] & G(n+1) \arrow[r] \arrow[d, "\rho(n+1)"] & \cdots \\ \cdots \arrow[r] & O(n) \arrow[r] & O(n+1) \arrow[r] & \cdots \end{tikzcd}$$

There is a stable tangential structure $BG \to BO$ that is the colimit of the induced sequence of maps of classifying spaces

$$\begin{tikzcd} \cdots \arrow[r] & BG(n) \arrow[r] \arrow[d, "B\rho(n)"] & BG(n+1) \arrow[r] \arrow[d, "B\rho(n+1)"] & \cdots \\ \cdots \arrow[r] & BO(n) \arrow[r] & BO(n+1) \arrow[r] & \cdots \end{tikzcd}$$

The corresponding Thom spectrum is defined by MG, where $G = \mathrm{colim}_n G(n)$.

Example 1.27. The Thom spectrum of the stable framing tangential structure is $\mathbb{S}$.

1.8. **Pontryagin-Thom isomorphism.** Endow $\mathbb{R}^{n+k}$ with the standard inner product. The map $W \to W^{\perp}$ to the orthogonal subspace induces inverse diffeomorphisms $G_{n,k} \to G_{k,n}$ between Grassmannians, exchanging the tautological subbundles S with the tautological quotient bundle Q. Taking the double colimit as $n, k \to \infty$ yields an involution

$$\ell : BO \to BO.$$

If $\mathcal{X} \to BO$ is a stable tangential structure, we define its pullback by ℓ to be a new stable tangential structure

$$\begin{tikzcd} \mathcal{X}^{\perp} \arrow[r] \arrow[d, dashed] & \mathcal{X} \arrow[d] \\ BO \arrow[r, "\ell"] & BO \end{tikzcd}$$

If $f : M \to BO$ is the stable classifying map of a vector bundle $V \to M$, and there is a complementary bundle $V^{\perp} \to M$ such that $V \oplus V^{\perp} \cong \underline{\mathbb{R}^m}$, then $\ell \circ f : M \to BO$ is a stable classifying map for $V^{\perp} \to M$.

Fix a stable tangential structure $\pi : \mathcal{X} \to BO$. Let M be a smooth manifold. A stable $\mathcal{X}$*-structure* on M is an $\mathcal{X}(n+q)$-structure on $TM \to M$ for sufficiently large q, i.e. compatible classifying maps

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & S(n) \arrow[r] \arrow[d] & S(n) \arrow[r] \arrow[d] & \cdots \\ M \arrow[r] & \mathcal{X}_{n,q} \arrow[r] & \mathcal{X}_{n,q+1} \arrow[r] & \cdots \end{tikzcd}$$

where $\mathcal{X}_{n,q}$ is the pullback of $\mathcal{X}(n) \to BO(n)$ to the Grassmannian $G_{n,q} \hookrightarrow BO(n)$. Suppose $M \hookrightarrow \mathbb{A}^{n+q}$ is an embedding with normal bundle $v \to M$ of rank q. We use the Euclidean metric to identify $v \cong TM^{\perp}$, and so $TM \oplus v \cong \underline{\mathbb{R}}^{n+q}$. This leads to a classifying map

$$\begin{tikzcd} v \arrow[r] \arrow[d] & Q(q) \arrow[d] \\ M \arrow[r] & \mathcal{X}^{\perp}_{q,n} \end{tikzcd}$$

where $\mathcal{X}^{\perp}_{q,n}$ is the pullback of $\mathcal{X}^{\perp} \to BO$ to the Grassmannian $G_{q,n}$. After stabilizing, we get a classifying map of the stable normal bundle. It is simply $\ell \circ f$, where f is the stable classifying map of the tangent bundle. Note that $\ell \circ f$ is defined without choosing an embedding. In this way, we pass back and forth between stable tangential $\mathcal{X}$-structures and stable normal $\mathcal{X}^{\perp}$-structures.

Definition 1.28. Define the **$\mathcal{X}$-bordism groups**

$$\begin{aligned} \Omega_n^{\mathcal{X}}(X) = \{(N, f) \mid\ & N \text{ is an } n\text{-dimensional closed smooth manifold endowed an} \\ & \mathcal{X}\text{-structure on the stable tangent bundle}, f : N \to X \text{ continuous}\}/\sim . \end{aligned}$$

Here $(N, f) \sim (N', f')$ if there exists (M, g), where M is an $(n+1)$-dimensional compact smooth manifold with an $\mathcal{X}$-structure on the stable tangent bundle with $\partial M = N \sqcup N'$, and the $\mathcal{X}$-structures on N, N' are induced from that on M, and $g : M \to X$ is a continuous map such that $g\mid_{\partial M} = f \sqcup f'$. If $\mathcal{X} = BG$, then we write this notion $\Omega_n^G(X)$ instead.

Theorem 1.29 (Pontryagin-Thom)**.** *There is an isomorphism*

$$\phi : \pi_n(M\mathcal{X}^\perp) \to \Omega_n^{\mathcal{X}}.$$

Proof. We will construct ϕ. Note that a class in $\pi_n(M\mathcal{X}^\perp)$ is represented by

$$f : S^{n+\varepsilon} \to T\mathcal{X}_q^\perp = \mathrm{Thom}(\mathcal{X}_q^\perp; S(q))$$

for some positive $q \in \mathbb{Z}$. We choose f so that it is smooth and transverse to the zero section $Z(q) \subset \mathrm{Thom}(\mathcal{X}_q^\perp; S(q))$. Define $M \coloneqq f^{-1}(Z(q)) \subset S^{n+q}$. The normal bundle $v \to M$ to M is a rank q bundle isomorphic to the pullback of the normal bundle to $Z(q)$, which is $S(q) \to Z(q)$. It inherits the $\mathcal{X}^\perp$-structure

$$M \xrightarrow{f} Z(q) \cong \mathcal{X}^\perp(q) \to \mathcal{X}^\perp$$

on its normal bundle, so on its stable normal bundle. This is equivalent to an $\mathcal{X}$-structure on the stable tangent bundle to M. $\square$

1.9. **Serre spectral sequence.**

1.10. **Adams spectral sequence.** Before we move on to the Adams spectral sequence, we would like to revisit the Serre's program of calculating the homotopy groups. Two main tools in the process are Hurewicz theorem and Eilenberg-MacLane spaces. Suppose we are given a $(k-1)$-connected space X, then $\pi_k X \cong H_k X$. At the same time, X and $K(\pi_k(X), k)$ share the same homotopy groups. Write $p : X \to K(\pi_k(X), k)$. If we replace X by some homotopy equivalent space $\widehat{X}$ such that the diagram commutes:

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{p}"] & K(\pi_k(X), k) \\ X \arrow[u, "\simeq"] \arrow[ur, "p"'] & \end{tikzcd}$$

then we get a Serre fibration. Let X_1 denote its fiber, so we have a fiber sequence

$$X_1 \to X_0 \xrightarrow{p_0} K(\pi_k(X), k),$$

where $X_0 = \widehat{X}$. Denote $K(\pi_k(X), k)$ by K for simplicity. There is an associated long exact sequence

$$\cdots \to \pi_i X_1 \to \pi_i X_0 \xrightarrow{\pi_i(p_0)} \pi_i K \to \pi_{i-1} X_1 \to \cdots.$$

$\pi_i(p_0)$ is an isomorphism for $i = k$ and 0 for $i \neq k$. It follows that

$$\pi_i X_1 = \begin{cases} 0 & , i \leq k, \\ \pi_i X & , i > k. \end{cases}$$

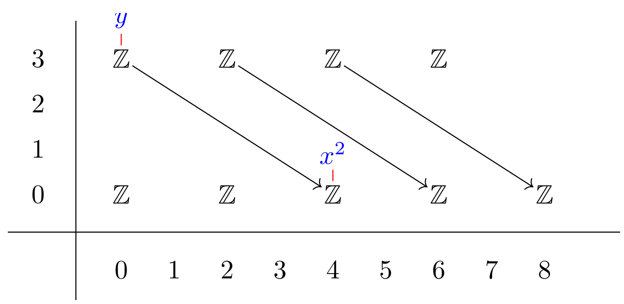
We can use the Serre SS to compute $H^* X_1$. Using the Hurewicz theorem, we can find the first nontrivial homotopy group of X_1, i.e. the second such group for $X_0 = X$.

Example 1.30. Let $X = S^2$, $\pi_2 S^2 = \mathbb{Z}$ and $\pi_1 S^2 = 0$. There is a map $S^2 \to K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$. Consider the fiber sequence

$$S^1 \simeq \Omega K(\mathbb{Z}, 2) \to X_1 \to S^2 \to K(\mathbb{Z}, 2).$$

Note that $H^* K(\mathbb{Z}, 2) = \mathbb{Z}[x]$, $x \in H^2 K(\mathbb{Z}, 2)$. Serre SS indicates $E_2^{p,q} = H^p(K(\mathbb{Z}, 2); H^q(X_1)) \Rightarrow H^* S^2$.

Figure 1.10.1: E_4-page of S^2.



From the figure, $d_4(y) = x^2$. So it is easy to derive that $d_4(x^n y) = x^{n+2}$. We conclude that

$$H^i(X_1) = \begin{cases} \mathbb{Z} & , i = 0, 3, \\ 0 & , \text{else}. \end{cases}$$

In fact, $X_1 \simeq S^3$. This is actually the Hopf fibration $S^1 \to S^3 \to S^2$. Observing the long exact sequence, one finds

$$\pi_i(S^3) = \begin{cases} \mathbb{Z} & , i < 3, \\ \pi_i(S^2) & , i \geq 3. \end{cases}$$

It follows that $\pi_3(S^2) = \mathbb{Z}$.

Proposition 1.31. *Let $F \to E \to B$ be a fiber sequence. Then $\Omega B \to F \to E$ is also a fiber sequence.*

If we have a fiber sequence $F \xrightarrow{i} E \xrightarrow{p} B$, where all spaces are n-connected, then there is a long exact sequence in dimension around $2n$:

$$\cdots \to H^i B \xrightarrow{p^*} H^i E \xrightarrow{i^*} H^i F \xrightarrow{\delta} H^{i+1} B \to \cdots$$

Suppose p^* is onto in our range of dimensions. In this case, $i^* = 0$ and J is one-to-one. The long exact sequence restricts to a short exact sequence

$$0 \to H^{i-1} F \xrightarrow{J} H^i B \xrightarrow{p^*} H^i E \to 0. \tag{1.32}$$

In Serre's approach, the first step is to study the fiber sequence $F \xrightarrow{i} S^n \xrightarrow{p} K(\mathbb{Z}, n)$. This leads to a long exact sequence

$$\cdots \to H^{n-1} F \to H^n K(\mathbb{Z}, n) \xrightarrow[p^*]{\cong} H^n S^n \to 0.$$

Like what we did in (1.32), we obtain a short exact sequence

$$0 \to H^{n+i-1} F \xrightarrow{J} H^{n+i} K(\mathbb{Z}, n) \xrightarrow{p^*} H^{n+i} S^n \to 0$$

for $0 \leq i < n - 1$. For $n > 2$, $pi_{n+1} S^n = \pi_{n+1} F = \mathbb{Z}/2$.

However, things would get really unsatisfying if we choose the coefficient to be $\mathbb{Z}/2$. In this case, p^* induced by $F \to B \to K(\mathbb{Z}/2, n+1)$ is not onto. So there is no elegant short exact sequence as in (1.32). To resolve this, Adams replaced $K(\mathbb{Z}/2, n+1)$ by another space L such that L is good enough to restart Serre's algorithm. By we mean "good", we are looking for some n-connected L with $\pi_{n+1} L = \mathbb{Z}/2$ and known cohomology, such that p^* is onto in the interested range.

Proposition 1.33. *For any $(n-1)$-connected space, there is a map $X \xrightarrow{p} L$, where L is desired good space. Moreover, p^* is onto below dimension $2m$.*

The next goal to construct this L. Consider the map $S^n \to K(\mathbb{Z}/2, n)$. We know $H^*K(\mathbb{Z}/2, n) = \mathbb{Z}/2\,[\mathrm{Sq}^I x_n]$. Here $I = (i_1, i_2, \cdots, i_\ell) \neq \varnothing$ needs to be admissible (i.e. $i_t \geq 2i_{t+1} \geq 0$ for all $1 \leq t < \ell$) with excess $e(I) = \sum_{i=1}^{\ell-1}(i_t - 2i_{t+1}) < n$. This is a certain kind of cyclic $\mathscr{A}$-module.

The Steenrod algebra $\mathscr{A}$ will be examined in great detail in Chapter 2. In short, it is a non-commutative $\mathbb{Z}/2$-algebra, generated by Sq^i for $i > 0$ subject to the Adem relations

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_{0\leq i\leq a/2} \binom{b-1-i}{a-2i} \mathrm{Sq}^{a+b-i}\mathrm{Sq}^i$$

for $a < 2b$. This relation enables us to construct higher Steenrod operators with the lower ones. For example, in the case $n \geq 0$, we have $\mathrm{Sq}^1\mathrm{Sq}^{2n} = \mathrm{Sq}^{2n+1}$ and

$$\begin{aligned}\mathrm{Sq}^2\mathrm{Sq}^{4n} &= \sum \binom{4n-1-i}{2-2i} \mathrm{Sq}^{4n+2-i}\mathrm{Sq}^i \\ &= \binom{4n-1}{2}\mathrm{Sq}^{4n+2} + \binom{4n-2}{0}\mathrm{Sq}^{4n+1}\mathrm{Sq}^1.\end{aligned}$$

Recall that in elementary number theory, we have the following result:

Theorem 1.34 (Lucas)**.** *For non-negative integers m, n, and a prime p,*

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \mod p,$$

where $m = \sum_{i=0}^k m_i p^i$ and $n = \sum_{i=0}^k n_i p^i$ are the base p expansions of m and n, respectively.

Applying to $m = \sum m_i 2^i$ and $n = \sum n_i 2^i$, where $m_i, n_i \in \{0, 1\}$ for all i, we deduce that

$$\binom{4n-1}{2} \equiv 1 \mod 2,$$

and hence,

$$\mathrm{Sq}^2\mathrm{Sq}^{4n} = \mathrm{Sq}^{4n+2} + \mathrm{Sq}^{4n+1}\mathrm{Sq}^1 = \mathrm{Sq}^{4n+2} + \mathrm{Sq}^1\mathrm{Sq}^{4n}\mathrm{Sq}^1.$$

Equivalently,

$$\mathrm{Sq}^{4n+2} = \mathrm{Sq}^2\mathrm{Sq}^{4n} + \mathrm{Sq}^1\mathrm{Sq}^{4n}\mathrm{Sq}^1.$$

In the preceding case, we actually showed that $\mathscr{A}$, as a $\mathbb{Z}/2$-algebra, is generated by $\{\mathrm{Sq}^{2^i} : i \geq 0\}$.

We can now answer the question why $F \to K(\mathbb{Z}/2, n+1)$ is not onto in H^*. Consider the short exact sequence

$$0 \to H^{n+i-1}F \to H^{n+i}K(\mathbb{Z}/2, n) \to H^{n+i}S^n \to 0.$$

A generator $x_n \in H^{n+i}K(\mathbb{Z}/2, n)$ is sent to $x_n \in H^{n+i}S^n$. Let y be the bottom class of F, corresponding to $y_{n+1} \in H^{n+1-i}F$, which is sent to $\mathrm{Sq}^2 x_n$. Then

$\mathrm{Sq}^3 y_{n+1}$ is sent to $\mathrm{Sq}^3 x_n$. There is another class $y_{n+3} \in H^{n+1-i}F$ that is mapped to $\mathrm{Sq}^4 x_n$. We immediately deduce that

$$\begin{aligned}\mathrm{Sq}^1 y_{n+3} &\mapsto \mathrm{Sq}^5 x_n\\ \mathrm{Sq}^2 y_{n+3} &\mapsto \mathrm{Sq}^6 x_n\\ \mathrm{Sq}^4 y_{n+1} &\mapsto \mathrm{Sq}^4\mathrm{Sq}^2 x_n\end{aligned}$$

$\{y_{n+2^j-1} : j > 0\}$ forms a subset of generators of $H^{n+i-1}F$. However, Sq^4 cannot be generated by lower Steenrod squares by Adem relations. This indicates that y_{n+3} cannot lie in the image of p^*, for if it did, then it corresponds to a generator $\mathrm{Sq}^4 x_n$ of $H^*K(\mathbb{Z}/2, n)$, contradicting to $e(I) < n$. To fix this, Adams chose a set $\{z_1, z_2, \cdots\}$ in H^*F that generate it as an $\mathscr{A}$-module, each of which determines a map to some $K(\mathbb{Z}/2, N_i)$. Collectively, they give a map p to $\prod_{i,N_i} K(\mathbb{Z}/2, N_i)$. Since the set $\{z_1, z_2, \cdots\}$ generates H^*F as an $\mathscr{A}$-module, p^* is onto. This $\prod_{i,N_i} K(\mathbb{Z}/2, N_i)$ is ideal candidate for the desired L.

Accepting Adam's remediation, and recursively repeating on each level, we can continue Serre's program to get the *Adams resolution*:

$$\begin{tikzcd}
X_0 = S^n \arrow[r, "p_0"] & K(\mathbb{Z}, n) = L_0 \\
X_1 \arrow[u] \arrow[r, "p_1"] & L_1 \\
X_2 \arrow[u] \arrow[r, "p_2"] & L_2 \\
\vdots \arrow[u] & \vdots
\end{tikzcd}$$

where X_i is the fiber of p_{i-1}, each p_i induces surjection in H^*, and each L_i is a product of $K(\mathbb{Z}/2, N_i)$ for some N_i, $i \geq 1$. So $H^*(L_s)$ is a free mod 2 $\mathscr{A}$-module. For each fiber sequence

$$X_{s+1} \to X_s \to L_s,$$

there is an associated short exact sequence from Serre sequence sequence ($i < n$)

$$0 \leftarrow H^{n+i}X_s \leftarrow H^{n+i}L_s \leftarrow H^{n+i-1}X_s \leftarrow 0.$$

From the discussion, it is not hard to see the following result:

Theorem 1.35. *$\pi_{n+i}(S^n)$ is a finite abelian group for $0 < i < n-1$, independent of n.*

Packaging up everything, we make the following definition.

Definition 1.36. The following diagram is called an **Adams resolution**

$$\begin{tikzcd}
S^n = X_0 \arrow[d, "f_0"] & X_1 \arrow[l] \arrow[d, "f_1"] & X_2 \arrow[l] \arrow[d, "f_2"] & X_3 \arrow[l] \arrow[d, "f_3"] & \cdots \arrow[l] \\
K(\mathbb{Z}, n) = L_0 & L_1 & L_2 & L_3 &
\end{tikzcd}$$

satisfying

(1) X_{s+1} is the fiber of f_s, and is $(n+s)$-connected;

(2) $H^*(f_s)$ is onto;
(3) $H^*(L_s)$ is a free $\mathscr{A}$-module in the stable range (below $2n+2s-2$) for $s>0$.

Recall that in homological algebra, $\mathrm{Ext}_R^*(M,N)$ is the cohomology of chain complex $\mathrm{Tot}(\hom_R(M_\bullet, N^\bullet))$, where $M_\bullet$, $N^\bullet$ are projective and injective resolutions of M,N, respectively. Let $R=\mathscr{A}$, $M=H^*X_1$, and $N=\mathbb{Z}/2$. We get

$$\mathrm{Ext}_{\mathscr{A}}^*(H^*X_1,\mathbb{Z}/2).$$

This is the E_2-term of the *Adams spectral sequence* (see his paper [2] in 1958) for finding the 2-component of π_*X_1 for large $*,n$.

One can generalize this to any spectrum X.

Theorem 1.37 (Adams spectral sequence)**.** *There is a spectral sequence with E_2-term*

$$E_2^{s,t}=\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*X,\mathbb{Z}/2)\Rightarrow \pi_{t-s}X,$$

*with the **Adams indexing** $d_n: E_n^{s,t}\to E_n^{s+n,t+n-1}$.*

Example 1.38. Let $X=H\mathbb{Z}$. Then $H^*(H\mathbb{Z};\mathbb{Z}/2)=\mathscr{A}\otimes_E\mathbb{Z}/2$, and so $E_2^{s,t}=\mathrm{Ext}_E^{s,t}(\mathbb{Z}/2,\mathbb{Z}/2)$. Replace $\mathscr{A}$ by the subalgebra E generated by Sq^1. The Adem relations tells us $\mathrm{Sq}^1\mathrm{Sq}^1=0$, so E has basis $\{1,\mathrm{Sq}^1\}$, with $|\mathrm{Sq}^1|=1$. There is a free E-resolution of $\mathbb{Z}/2$

$$\mathbb{Z}/2\leftarrow E\xleftarrow{\mathrm{Sq}^1}\Sigma E\xleftarrow{\mathrm{Sq}^1}\Sigma^2E\xleftarrow{\mathrm{Sq}^1}\cdots$$

where every unit 1 of Σ^kE is sent to Sq^1 in $\Sigma^{k-1}E$. Applying $\hom_E(-,\mathbb{Z}/2)$ to the resolution and computing the cohomology, we obtain

$$\mathrm{Ext}_E^{s,t}(\mathbb{Z}/2,\mathbb{Z}/2)=\begin{cases}\mathbb{Z}/2 & ,t=s,\\ 0 & ,t\neq s.\end{cases}$$

If E is graded $\mathbb{Z}/2$-algebra, then there is a Yoneda product structure in $\mathrm{Ext}_R^s(\mathbb{Z}/2,\mathbb{Z}/2)$, which making it a bigraded ring. See Chapter 9 of Oxford C2.2 Homological Algebra course note for hints.

Now we will discuss things in general pattern. We have seen two examples of spectral sequences due to Serre and Adams. There are others besides these in algebraic topology, and all of them can be constructed in similar ways. There are two ways to do it: by exact couples or by filtered chain complexes.

Firstly, we want to see how spectral sequences can be made from filtered chain complexes. Let C be a chain complex. An *increasing filtration* on C is a sequence of sub-chain complexes

$$F_0C\subset F_1C\subset F_2C\subset\cdots$$

with $C=\cup_iF_iC$. Naturally, we chop it in parts and get a short exact sequence for each $n>0$:

$$0\to F_{n-1}C\to F_nC\to F_nC/F_{n-1}C\to 0.$$

A reenactment can be predicted if we use the decreasing filtration with $C = \cap_i F_i C$. For simplicity, we focus on the case of increasing filtration. Consider the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_{n-2}C \arrow[r] \arrow[d, equal] & F_{n-1}C \arrow[r] \arrow[d] & F_{n-1}C/F_{n-2}C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{n-2}C \arrow[r] \arrow[d] & F_nC \arrow[r] \arrow[d] & F_nC/F_{n-2}C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F_nC/F_{n-1}C \arrow[r, equal] \arrow[d] & F_nC/F_{n-1}C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

The rows are columns are exact. The long exact sequence associated with the right column is

$$\cdots \to H_{i+1}F_n/F_{n-1} \xrightarrow{d_{n,i+1}} H_iF_{n-1}/F_{n-2} \to H_iF_n/F_{n-2} \to H_iF_n/F_{n-1} \xrightarrow{d_{n,i}} H_{i-1}F_{n-1}/F_{n-2} \to \cdots,$$

yielding a short exact sequence

$$0 \to \operatorname{coker} d_{n,i+1} \to H_iF_n/F_{n-2} \to \ker d_{n,i} \to 0.$$

If we manage to find the middle group, then we can continue to get

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_{n-3}C \arrow[r] \arrow[d, equal] & F_{n-2}C \arrow[r] \arrow[d] & F_{n-2}C/F_{n-3}C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{n-3}C \arrow[r] \arrow[d] & F_nC \arrow[r] \arrow[d] & F_nC/F_{n-3}C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F_nC/F_{n-2}C \arrow[r, equal] \arrow[d] & F_nC/F_{n-2}C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Again, from the associated long exact sequence and short exact sequence, we expect to know $\to H_iF_n/F_{n-3}$. Eventually, we learn $H_*(F_nC)$ and H_*C itself. All of the process can be encoded as a spectral sequence painlessly.

The other way is through the exact couples. Let $\mathcal{C}$ be an abelian category.

Definition 1.39. A **differential object** (E, d) in $\mathcal{C}$ is an object E with a morphism $d : E \to E$ such that $d \circ d = 0$. Given a differential object (E, d) in $\mathcal{C}$, we define the **cycle**, **boundary**, and **homology** of (E, d) to be the kernel of d, the image of d and the quotient of these two, respectively. Denote them by $Z(E)$, $B(E)$, and $H(E) = Z(E)/B(E)$, respectively.

Definition 1.40. An **exact couple** in $\mathcal{C}$ is a tuple $(D, E, i, j, k) =: (D, E)$ such that

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & D \arrow[ul, "k"] & \end{tikzcd}$$

is exact in every direction, i.e. $\ker i = \operatorname{im} k$, $\ker j = \operatorname{im} i$, and $\ker k = \operatorname{im} j$.

Given an exact couple (D, E), we can extend the diagram to get

$$\begin{tikzcd} & D \arrow[rr, "o"] \arrow[dr, "j"] & & D \arrow[dr, "j"] & \\ E \arrow[ur, "k"] \arrow[rr, "d"] & & E \arrow[ur, "k"] \arrow[rr, "d"] & & E \end{tikzcd}$$

Let $d = jk : E \to E$. It is straightforward that $d^2 = 0$, so (E, d) is a differential object.

Definition 1.41. Given an exact couple as above, the **derived couple** is a diagram

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

where $D' = \operatorname{im} i \subset D$, $E' = H(E) = \ker d / \operatorname{im} d$, and $i' = i\,|_{D'}$. For $x \in D$, j' sends $i(x)$ to $[j(x)]$, the homology class of $j(x)$. k' sends $[y]$ (for $y \in \ker d \subset E$) to $k(y)$.

Proposition 1.42. *The derived couple of an exact couple is again exact.*

We have a sequence of exact couples

$$(D, E) \rightsquigarrow (D', E') \rightsquigarrow (D'', E'') \rightsquigarrow \cdots$$

(E, d) is a differential object with $E' = H(E, d)$. Continually, (E', d') is a differential object with $E'' = H(E', d')$, $\cdots$, (E^n, d^n) is a differential object with $E^{n+1} = H(E^e, d^n)$. This leads to a spectral sequence.

Example 1.43. If we revisit the Adams resolution

$$\begin{tikzcd} X_0 \arrow[d, "f_0"] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"] & \cdots \arrow[l] \\ K_0 & K_1 & K_2 & \cdots \end{tikzcd}$$

where for each s, $X_{s+1} \xrightarrow{g_s} X_s \xrightarrow{f_s} K_s$ is a fiber sequence, then from the long exact sequence

$$\cdots \to \pi_n(X_{s+1}) \xrightarrow{(g_s)_*} \pi_n(X_s) \xrightarrow{(f_s)_*} \pi_n(K_s) \xrightarrow{h_s} \pi_{n-1}(X_{s+1}) \to \cdots$$

i.e. we have

$$\begin{tikzcd} D = \pi_* X_* \arrow[rr, "g_*"] & & \pi_* X_* = D \arrow[dl, "f_*"] \\ & E = \pi_* K_* \arrow[ul, "h"] & \end{tikzcd}$$

This is an exact couple, so it leads to a spectral sequence: the Adams spectral sequence.

Example 1.44. Consider the sequence

$$0 = K^{-1} \to K^0 \hookrightarrow K^1 \hookrightarrow K^2 \hookrightarrow K^3 \hookrightarrow \cdots \subset K,$$

where each K^i is a chain complex. This is a filtered chain complex. We have short exact sequence for each $p \geq 0$

$$0 \to K^{p-1} \to K^p \to K^p/K^{p-1} \to 0,$$

leading to a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_{p+q}K^{p-1} \arrow[r, "i"] \arrow[d, equal] & H_{p+q}K^p \arrow[r, "j"] \arrow[d, equal] & H_{p+q}\left(K^p/K^{p-1}\right) \arrow[r, "\partial = k"] \arrow[d, equal] & H_{p+q-1}K^{p-1} \arrow[r] \arrow[d, equal] & \cdots \\
& D_{p-1,q+1} & D_{p,q} & E_{p,q} & D_{p-1,q} &
\end{tikzcd}$$

This leads to an exact couple of bigraded abelian groups

$$\begin{tikzcd}
D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\
& E \arrow[ul, "\partial"] &
\end{tikzcd}$$

which is the starting point of Serre spectral sequence, if we choose the appropriate chain complex.

Definition 1.45. Let X, Y be spectra. We define the notation

$$[X, Y]_t = \operatorname{colim}[\Sigma^{n+t}X, \Sigma^n Y] = \pi_t F(X, Y),$$

where $F(-,-)$ is the function spectrum.

Theorem 1.46. *There is a spectral sequence of the form*

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X)) \Rightarrow \pi_{t-s}F(X,Y)_2^\wedge,$$

where the t-part refers to the internal grading of the modules.

Given an element $f \in [\Sigma^t X, Y]$, we get a homomorphism of $\mathscr{A}$-modules:

$$H^*(Y) \to H^*(\Sigma^t X) = \Sigma^t H^*(X),$$

and $H^*(\Sigma^t X) = [\Sigma^t X, H\mathbb{F}_2]_{-*} = [X, H\mathbb{F}_2]_{-*+t} = H^{*-t}(X)$. If M is a graded module, then $(\Sigma M)_n = M_{n-1}$. Therefore, given a stable map $\Sigma^k X \xrightarrow{f} Y$, we get $f^* \in \hom_{\mathscr{A}}(H^*(Y), H^*(\Sigma^k X))$. If f is null-homotopic, then $f^* = 0$. On the other hand, if $f^* = 0$, then it is not necessarily true that f is null-homotopic. If it is, then $C(f) = Y \vee \Sigma^{k+1}X$, $H^*C(f) = H^*(Y) \oplus H^*(\Sigma^{k+1}X)$ as $\mathscr{A}$-modules. The sequence

$$\Sigma^k X \to Y \to C(f) \to \Sigma^{k+1}X \to \cdots$$

induces a long exact sequence

$$\cdots \to H^*(\Sigma^{k+1}X) \to H^*(C(f)) \xrightarrow[\text{onto}]{p^*} H^*(Y) \xrightarrow{f^*} H^*(\Sigma^k X) \to \cdots,$$

which again implies $f^* = 0$. In this case, the long exact sequence can be compressed into a short exact sequence of graded $\mathscr{A}$-modules

$$0 \to H^*(\Sigma^{k+1}X) \to H^*(C(f)) \to H^*(Y) \to 0.$$

This is an extension of $H^*(Y)$ by $H^*(\Sigma^{k+1}X)$ in the category of $\mathscr{A}$-modules, and thus an element of $\operatorname{Ext}^1_{\mathscr{A}}(H^*(Y), H^*(\Sigma^{k+1}X))$.

Example 1.47. Consider the multiplication by 2 map on S^0, denoted as f. In mod 2 cohomology, $f^* = 0$. However, f is not null-homotopic. If we consider the extension

$$S^0 \to M(\mathbb{Z}/2) \to S^1,$$

where $M(\mathbb{Z}/2)$ is the Moore space, then we have an induced morphisms in cohomology

$$0 \to H^*(S^1) \to H^*(M(\mathbb{Z}/2)) \to H^*(S^0) \to 0.$$

But $H^*(M(\mathbb{Z}/2)) \ncong H^*(S^0) \oplus H^*(S^1)$ as $\mathscr{A}$-modules since there is no splitting $H^*(S^0) \to H^*(M(\mathbb{Z}/2))$ as $\mathscr{A}$-modules. The counterexample highlights the fact that we can detect the extension of $\mathscr{A}$-modules using the appropriate maps.

Now we are ready to construct the spectral sequence.

Definition 1.48. A **generalized Eilenberg-MacLane spectrum** is a wedge of Eilenberg-MacLane spectra.

Proposition 1.49. *If $M = H\mathbb{F}_2 \wedge X$, then for any finite spectrum Y, we have*

$$[Y, M]_* = \hom^*_{\mathscr{A}}(H^*(M), H^*(Y)).$$

Proof. By Spanier-Whitehead duality, it suffices to show the result for $Y = \mathbb{S}$. We deduce that

$$\begin{aligned}[\mathbb{S}, M]_* = \pi_*(M) = H_*X \cong (H^*X)^* &= \hom^*(H^*X, \mathbb{F}_2) \\ = \hom^*_{\mathscr{A}}(\mathscr{A} \otimes H^*X, \mathbb{F}_2) &= \hom^*_{\mathscr{A}}(H^*M, \mathbb{F}_2),\end{aligned}$$

which is the desired result. □

Let $\mathbb{S} \to H\mathbb{F}_2$ be the map giving the nonzero element of $\overline{H}^*\mathbb{S} = \mathbb{F}_2$. This is actually the unit map for the E_∞-ring structure on $H\mathbb{F}_2$. Let $\overline{H\mathbb{F}_2}$ be the fiber of $\mathbb{S} \to H\mathbb{F}_2$. For any spectrum X, smashing with the fiber sequence $\overline{H\mathbb{F}_2} \to \mathbb{S} \to H\mathbb{F}_2$ gives a new fiber sequence

$$\overline{H\mathbb{F}_2} \wedge X \to X \to X \wedge H\mathbb{F}_2.$$

This leads to the tower of fibrations for any spectrum Y:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \\
Y \wedge \left(\overline{H\mathbb{F}_2}\right)^{\wedge 2} \arrow[r] \arrow[d] & \left(Y \wedge \left(\overline{H\mathbb{F}_2}\right)^{\wedge 2}\right) \wedge H\mathbb{F}_2 \\
Y \wedge \overline{H\mathbb{F}_2} \arrow[r] \arrow[d] & \left(Y \wedge \overline{H\mathbb{F}_2}\right) \wedge H\mathbb{F}_2 \\
Y \arrow[r] & Y \wedge H\mathbb{F}_2
\end{tikzcd}$$

where $Y_n = Y \wedge \left(\overline{H\mathbb{F}_2}\right)^{\wedge n}$, and Y_{n+1} sits in a fiber sequence

$$Y_{n+1} = Y_n \wedge \overline{H\mathbb{F}_2} \to Y_n \to Y_n \wedge H\mathbb{F}_2.$$

This is the *canonical Adams resolution*. Convert this to a sequence:

$$\begin{tikzcd} Y \arrow[r] & Y \wedge H\mathbb{F}_2 \arrow[r] \arrow[d] & Y \wedge \left(\Sigma\overline{H\mathbb{F}_2}\right) \wedge H\mathbb{F}_2 \arrow[r] \arrow[d] & Y \wedge \left(\Sigma\overline{H\mathbb{F}_2}\right)^{\wedge 2} \wedge H\mathbb{F}_2 \arrow[r] \arrow[d] & \cdots \\ & Y \wedge \left(\Sigma\overline{H\mathbb{F}_2}\right) & Y \wedge \left(\Sigma\overline{H\mathbb{F}_2}\right)^{\wedge 2} & Y \wedge \left(\Sigma\overline{H\mathbb{F}_2}\right)^{\wedge 3} & \cdots \end{tikzcd}$$

where each up-right arrow is given by $\mathbb{S} \to H\mathbb{F}_2$, and $\Sigma\overline{H\mathbb{F}_2}$ is the cofiber of $\mathbb{S} \to H\mathbb{F}_2$. Applying the functor $[X, -]$ yields a spectral sequence with

$$E_1 = [X, Y \wedge \left(\Sigma\overline{H\mathbb{F}_2}\right)^{\wedge n} \wedge H\mathbb{F}_2].$$

By Proposition 1.49,

$$[X, Y \wedge \left(\Sigma\overline{H\mathbb{F}_2}\right)^{\wedge n} \wedge H\mathbb{F}_2] = \hom_{\mathscr{A}}\left(\mathscr{A} \otimes H^*\left(\Sigma\overline{H\mathbb{F}_2}\right)^{\otimes n} \otimes H^*Y, H^*X\right).$$

Proposition 1.50. *$H^*\left(\Sigma\overline{H\mathbb{F}_2}\right) = I(\mathscr{A})$, the ideal of positive degree elements.*

The map $\mathbb{S} \to H\mathbb{F}_2$ gives the augmentation $\mathscr{A} \to \mathbb{F}_2$. We know $\mathscr{A} \otimes H^*\left(\Sigma\overline{H\mathbb{F}_2}\right)^{\otimes n} \otimes H^*Y = \mathscr{A} \otimes (I(\mathscr{A}))^{\otimes n} \otimes H^*Y$.

Proposition 1.51. *In fact, $\mathscr{A} \otimes (I(\mathscr{A}))^{\otimes n} \otimes H^*Y$ is the n-th stage of the canonical projective resolution of H^*Y as an $\mathscr{A}$-module, and $H^*(Y \to Y \wedge H\mathbb{F}_2 \to \cdots)$ realizes this projective resolution.*

As a corollary, the homotopy classes $[X, Y \wedge \left(\Sigma\overline{H\mathbb{F}_2}\right)^{\wedge n} \wedge H\mathbb{F}_2]$ are $\hom_{\mathscr{A}}(P_n, H^*X)$, where P_n is the n-th stage of the projective resolution. The maps in spectra realizes the d_1-differential in the exact couple from the tower of fibrations, which is $\hom_{\mathscr{A}}(-, H^*X)$ of the maps in the projective resolution.

Corollary 1.52. *The E_2-page is $E_2 = \operatorname{Ext}_{\mathscr{A}}(H^*Y, H^*X)$.*

1.11. **Thom isomorphism.** Let $\xi : V \to X$ be an n-plane bundle. Apply one-point compactification to each fiber of ξ to obtain a new bundle $S(V)$ over X, whose fibers are spaces S^n with given basepoints, i.e. the point at ∞. These basepoints specify a cross-section $X \to S(V)$. The *Thom space* is the quotient space $T\xi = S(V)/X$. If the bundle $\xi : V \to X$ is trivial, then $V = X \times \mathbb{R}^n$, and $T\xi = \Sigma^n X_+ = X_+ \wedge S^n$. In this case,

$$H^q(X; R) \cong \tilde{H}^q(X_+; R) \cong \tilde{H}^{n+q}(T\xi; R).$$

For any n-plane bundle $\xi : V \to X$, we have a projection $\zeta : S(V) \to X$ and a quotient map $\pi : S(V) \to T\xi$. Compose their product with the diagonal map of $S(V)$, we obtain a composite map

$$S(V) \to S(V) \times S(V) \to X \times T\xi.$$

This sends all points at ∞ to points of $X \times \{\infty\}$. Therefore it factors through a map $\Delta : T\xi \to X_+ \wedge T\xi$, called the *Thom diagonal*. For a commutative ring R, we can use Δ to define a *cup product*

$$H^p(X; R) \otimes \tilde{H}^q(T\xi; R) \to \tilde{H}^{p+q}(T\xi; R),$$

and a *cap product*

$$\tilde{H}_{p+q}(T\xi; R) \otimes \tilde{H}^q(T\xi; R) \to H_p(X; R).$$

When the bundle ξ is trivial, letting $\mu \in \tilde{H}^n(X_+ \wedge S^n; R)$ be the suspension of the identity element $1 \in H^0(X; R)$, we find that $x \mapsto x \cup \mu$ specifies the isomorphism $H^q(X; R) \cong \tilde{H}^{n+q}(X_+ \wedge S^n; R) = \tilde{H}^{n+q}(T\xi; R)$.

In general, let ξ be an arbitrary bundle. On some neighborhood U of X over which ξ is trivial, we have $H^q(U; R) \cong \tilde{H}^{n+q}(T\xi\,|_U; R)$. The isomorphism depends on the local trivialization $\phi_U : U \times \mathbb{R}^n \to \xi^{-1}(U)$. One would wonder if these isomorphisms patch together to give a global isomorphism $H^q(X) \to \tilde{H}^{n+q}(T\xi)$. This is similar to the problem of patching local fundamental classes to obtain a global one, i.e. orientation.

Let b be a point in X, and S^n_b be the one-point compactification of the fiber $\xi^{-1}(b)$. Since S^n_b is the Thom space of ξ_b, we have a canonical map $\imath_b : S^n_b \to T\xi$.

Definition 1.53. Let $\xi : V \to X$ be an n-plane bundle. An R-orientation, or **Thom class**, of ξ is an element $\mu \in \tilde{H}^n(T\xi; R)$ such that, for every point $b \in X$, $\imath_b^*(\mu)$ is a generator of the free R-module $\tilde{H}^n(S^n_b)$.

Recall that if $\xi : V \to X$ is a vector bundle, then the Thom space of ξ is also defined as $T(\xi) = D(\xi)/S(\xi)$, where $D(\xi) = \{v \in V : \|v\| \le 1\}$ is the *disk bundle* of ξ, and $S(\xi) = \{v \in V : \|v\| = 1\}$ is the *sphere bundle* of ξ. We use the Serre spectral sequence to compute the homology of $T(\xi)$.

Theorem 1.54 (Thom isomorphism)**.** *Let $\xi : V \to X$ be an n-dimensional vector bundle, and R be a commutative ring. Assume that X is simply connected or char $R = 2$. Then there is an element $\mu \in \tilde{H}^n(T\xi; R)$ such that we have dual isomorphisms*

$$\Phi_* : \tilde{H}_{*+n}(T\xi; R) \cong H_*(X; R),$$
$$\Phi^* : H^*(X; R) \cong \tilde{H}_{*+n}(T\xi; R),$$

defined by

$$\Phi_*(t) = t \cap \mu,$$
$$\Phi^*(x) = x \cup \mu,$$

for $t \in \tilde{H}_*(T\xi; R)$ and $x \in H^*(X; R)$.

Theorem 1.55 (Generalized Serre spectral sequence)**.** *Let k be a commutative ring, and $(F, F') \to (E, E') \xrightarrow{\pi} B$ such that either*

(1) $E' = F' = \varnothing$, and π is a fibration or
(2) π is a relative fiber bundle, i.e. there is an open cover of B of open sets U with $(\pi^{-1}(U), \pi^{-1}(U) \cap E') \cong (U \times F, U \times F')$.

Suppose that B is a CW complex which is either simply connected or connected with char $k = 2$. Then there are spectral sequences

$$E^2_{s,t} = H_s(B; H_t(F, F'; k)) \Rightarrow H_*(E, E'; k),$$
$$E_2^{s,t} = H^s(B; H^t(F, F'; k)) \Rightarrow H^*(E, E'; k),$$

and the second SS is multiplicative.

The proof can be found in [1, Theorem 2.6.3].

Proof of Theorem 1.54. Let $\xi : (D(\xi), S(\xi)) \to X$ be a relative fiber bundle with fiber (D^n, S^{n-1}). The resulting SSS read

$$E^2_{s,t} = \begin{cases} H_s(X;R) & , t = n, \\ 0 & , t \neq n, \end{cases}$$

and

$$E_2^{s,t} = \begin{cases} H^s(X;R) & , t = n, \\ 0 & , t \neq n. \end{cases}$$

Thus,

$$\Phi_* : \tilde{H}_k(T\xi;R) \cong H_k(D(\xi), S(\xi);R) = E^\infty_{k-n,n} \cong H_{k-n}(X;R),$$
$$\Phi^* : H^{k-n}(X;R) \cong E_\infty^{k-n,n} = H^k(D(\xi), S(\xi);R) \cong \tilde{H}^k(T\xi;R).$$

Let $\mu \in \tilde{H}^n(T\xi;R)$ be corresponding to $1 \in H^0(X;R) \cong E_\infty^{0,n}$. Let $t \in \tilde{H}_*(T\xi;R)$ and $x \in H^*(X;R)$. Then clearly the preceding isomorphisms Φ_*, Φ^* are given by $\Phi_*(t) = t \cap \mu$, $\Phi^*(x) = x \cup \mu$, respectively. These isomorphisms are dual because

$$\langle \Phi^*(x), t \rangle = \langle x \cup \mu, t \rangle = \langle x, t \cap \mu \rangle = \langle x, \Phi_*(t) \rangle .$$

□

Let T be a prespectrum. Recall that $\pi_n(T) = \mathrm{colim}_q \pi_{n+q}(T_q)$. We can define the homology and cohomology groups in the same manner. That is,

$$H_n(T;R) = \mathrm{colim}_q \tilde{H}_{n+q}(T_q;R),$$

where the colimit is taken over maps

$$\tilde{H}_{n+q}(T_q;R) \xrightarrow{\Sigma} \tilde{H}_{n+q+1}(\Sigma T_q;R) \xrightarrow{(\sigma_q)_*} \tilde{H}_{n+q+1}(T_{q+1};R),$$

and

$$H^n(T;R) = \lim_q \tilde{H}^{n+q}(T_q;R),$$

where the limit is taken over the maps

$$\tilde{H}^{n+q+1}(T_{q+1};R) \xrightarrow{(\sigma_q)^*} \tilde{H}^{n+q+1}(\Sigma T_q;R) \xrightarrow{\Sigma^{-1}} \tilde{H}^{n+q}(T_q;R).$$

In fact, this definition of cohomology is not correct in general. However, this definition makes sense when R is a field and each $\tilde{H}^{n+q}(T_q;R)$ is a finite dimensional vector space over R. This is the only case needed. In this case, it is clear that $H^n(T;R)$ is the vector space dual to $H_n(T;R)$, a fact that we shall use repeatedly.

Remark 1.56. There is no cup product in $H^*(T;R)$. Note that the maps in the reverse system factor through the reduced cohomologies of suspensions, in which the cup products are identically zero. However, if T is an associative and commutative ring prespectrum, then the homology groups $H_*(T;R)$ form a graded commutative R-algebra.

The Hurewicz homomorphisms $\pi_{n+q}(T_q) \to \tilde{H}_{n+q}(T_q;\mathbb{Z})$ pass to colimit to give the stable Hurewicz homomorphism

$$h : \pi_n(T) \to H_n(T;\mathbb{Z}).$$

We may compose this with the map $H_n(T;\mathbb{Z}) \to H_n(T;R)$ induced by the unit of R, and continue to denote the composite by h. If T is an associative and commutative ring prespectrum, then $h : \pi_*(T) \to H_*(T; R)$ is a map of graded commutative rings.

For simplicity, we write H_* and H^* for homology and cohomology with coefficients in $\mathbb{Z}/2$. The Thom isomorphisms say that

$$\Phi_q : H^n(BO(q)) \to \tilde{H}^{n+q}(TO(q))$$

obtained by cupping with the Thom class $\mu_q \in \tilde{H}^q(TO(q))$. By naturality of the Thom diagonal, the map of bundles $\gamma_q \oplus \varepsilon \to \gamma_{q+1}$ gives the commutative diagram

$$\begin{tikzcd} \Sigma TO(q) \arrow[r, "\Delta"] \arrow[d, "\sigma_q"'] & BO(q)_+ \wedge \Sigma TO(q) \arrow[d, "\imath_q \wedge \sigma_q"] \\ TO(q+1) \arrow[r, "\Delta"] & BO(q+1)_+ \wedge TO(q+1) \end{tikzcd}$$

It derives a commutative diagram

$$\begin{tikzcd} H^n(BO(q+1)) \arrow[rr, "\imath_q^*"] \arrow[d, "\Phi_{q+1}"'] & & H^n(BO(q)) \arrow[d, "\Phi_q"] \\ \tilde{H}^{n+q+1}(TO(q+1)) \arrow[r, "\sigma_q"] & \tilde{H}^{n+q+1}(\Sigma TO(q)) \arrow[r, "\Sigma^{-1}"] & \tilde{H}^{n+q}(TO(q)) \end{tikzcd}$$

This implies the *stable Thom isomorphism in cohomology*:

$$\Phi^n : H^n(BO) \to H^n(TO)$$

by passing to limits. Dually, we have an isomorphism in homology:

$$\Phi_n : \tilde{H}_{n+q}(TO(q)) \to \tilde{H}_n(BO(q)),$$

here we abuse the notation Φ. Passing to colimit, we obtain the *stable Thom isomorphism in homology*:

$$\Phi_n : H_n(TO) \to H_n(BO).$$

Applying the naturality of Thom diagonal to $\gamma_q \oplus \gamma_r \to \gamma_{q+r}$, we get

$$\begin{tikzcd} TO(q) \wedge TO(r) \arrow[r, "\Delta\wedge\Delta"] \arrow[dd, "\phi_{q,r}"] & BO(q)_+ \wedge TO(q) \wedge BO(r)_+ \wedge TO(r) \arrow[d, "\mathrm{id}\wedge t\wedge \mathrm{id}"] \\ & (BO(q) \times BO(r))_+ \wedge TO(q) \wedge TO(r) \arrow[d, "(p_{q,r})_+\wedge\phi_{q,r}"] \\ TO(q+r) \arrow[r, "\Delta"] & BO(q+r)_+ \wedge TO(q+r) \end{tikzcd}$$

Passing $p_{q,r} : BO(q) \times BO(r) \to BO(q+r)$ to limit gives BO an H-space structure. It follows that $H_*(BO)$ is an $\mathbb{Z}/2$-algebra. Passing to colimit, the previous diagrams imply the following result in homology.

Proposition 1.57. *The Thom isomorphism $\Phi : H_*(TO) \to H_*(BO)$ is an isomorphism of $Z/2$-algebras.*

The next step is to determine the algebraic structure of these two homologies. Let $i : \mathbb{RP}^\infty = BO(1) \to BO$ be the inclusion, and $x_i \in H_i(\mathbb{RP}^\infty)$ be the unique non-zero element. Write $b_i = i_*(x_i)$.

Theorem 1.58. *$H_*(BO)$ is the polynomial algebra $\mathbb{Z}/2\,[b_i \mid i \geq 1]$.*

Let $a_i \in H_i(TO)$ be the element characterized by $\Phi(a_i) = b_i$.

Corollary 1.59. *$H_*(BO)$ is the polynomial algebra $\mathbb{Z}/2\,[a_i \mid i \geq 1]$.*

From the compatibility of the Thom isomorphism for $BO(1)$ and BO, we see that the a_i's come from $H_*(TO(1))$. Remember that elements of $H_{i+1}(TO(1))$ map to elements of $H_i(TO)$ as colimits; in particular, the non-zero element of $H_1(TO(n))$ maps to the identity element $1 \in H_0(TO)$.

Lemma 1.60. *$TO(1)$ and $\mathbb{RP}^\infty$ are homotopy equivalent.*

Proof. Note that $T(\gamma_1) = D(\gamma_1)/S(\gamma_1)$, and $S(\gamma_1) = S^\infty$ is contractible. The zero section $\mathbb{RP}^\infty \to D(\gamma_1)$ and the quotient map $D(\gamma_1) \to T(\gamma_1)$ are homotopy equivalences, so is their composite. □

Corollary 1.61. *For $i \geq 0$, $j_*(x_{i+1})$ maps to $a_i \in H_*(TO)$, where $j : \mathbb{RP}^\infty \to TO(1)$ and $a_0 = 1$.*

The case in cohomology is fairly easy. We already know that $H^*(BO(n)) = \mathbb{Z}/2\,[w_1, w_2, \cdots, w_n]$ and $i_q^*(\gamma_{q+1}) = \gamma_q \oplus \varepsilon$. So

$$H^*(BO) = \mathbb{Z}/2\,[w_i \mid i \geq 1]$$

as an algebra. One can also derive a coalgebra structure $\Psi : H^*(BO) \to H^*(BO) \otimes H^*(BO)$ from $p_{q,r}^*(w(\gamma_{q+r})) = w(\gamma_q \oplus \gamma_r) = w(\gamma_q) \cdot w(\gamma_r)$. Explicitly,

$$\Psi(w_k) = \sum_{i+j=k} w_i \otimes w_j.$$

The Stiefel-Whitney classes w_n can be constructed in a different way involving the Thom isomorphism Φ. To be clear, one can set

$$w_i = \Phi^{-1} \mathrm{Sq}^i \Phi(1),$$

where Sq is the Steenrod operator, where ξ is a vector bundle of rank n.

Recall that Stiefel-Whitney classes must satisfy the following axioms:

- A1 $w_0(\xi) = 1$, $w_i(\xi) = 0$ for $i > n$.
- A2 For $f : B(\xi) \to B(\eta)$ with $\xi = f^*\eta$, $w_i(\xi) = f^* w_i(\eta)$.
- A3 If ξ and η are vector bundles over the same base, then $w(\xi \oplus \eta) = w(\xi) \cup w(\eta)$, where w is the total Stiefel-Whitney class.
- A4 For the canonical line bundle γ_1^1 over $\mathbb{RP}^1$, $w_1(\gamma_1^1) \neq 0$.

We are now checking the validity of our new definition. A1 is immediate from the relations $\mathrm{Sq}^0 = \mathrm{id}$ and $\mathrm{Sq}^i(x) = 0$ for $i > |x|$. A2 is obvious. Since $T(\gamma_1^1) = \mathbb{RP}^2$ and $\mathrm{Sq}^1(x) = x^2$ for $|x| = 1$, Sq^1 is non-zero on the Thom class of γ_1^1, verifying A4. For A3, notice that for every vector bundle ξ and η, we have $T(\xi \times \eta) \simeq T(\xi) \wedge T(\eta)$. The Thom class of $\xi \times \eta$ is the tensor product of the Thom classes of ξ and η. Interpreting the Cartan's formula for the Steenrod operators externally in the cohomology of products and therefore of smash product, we arrive at the desired conclusion.

1.12. **Thom splitting.**

1.13. **Oriented bordism groups.**

1.14. **Computation of bordism groups.**

2. Steenrod Algebras

References

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