K_0 of Rings & Projective Modules

Ring possibly non commutative with 1 Ring $*$ defined by

$\downarrow$ $[P][Q] = [P \otimes Q]$

$K_0 R = \mathbb{Z}\{[P] \mid [P] \text{ iso class of projective module}\} / \langle [P]+[Q] - [P\oplus Q] \rangle$

① Serre-Swan Thm X smooth manifold $\rightsquigarrow C^\infty(M)$

$\{\text{Vector bundles on } X\} \simeq \{\text{f.g modules} / C^\infty(M)\}$

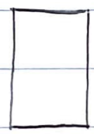
② Thm R local ring & M f.g module $/R$

M free $\Leftrightarrow$ M flat $\Leftrightarrow$ M Projective

$X \quad C$

$\uparrow$

$R = \Gamma(X, \mathcal{O}_X)$



$\mathbb{A}^n_K$.

free module $/R \longleftrightarrow$ trivial VB$/R$.

$\{\text{VB on } X\} \longleftrightarrow \{\text{f.g. Projective module} / R\}$

EX (1). K field. $K_0(K) \simeq \mathbb{Z}$

(2) R local Ring $\Rightarrow K_0(R) \simeq \mathbb{Z}$

(3) R PID $\Rightarrow K_0(R) \simeq \mathbb{Z}$.

(by str. Thm) + (submodule of free module is free / PID)

(4) $M_n(K) = R \longrightarrow K_0 R = \mathbb{Z} \leftarrow [K^n]$ Not R itself $[R] = n$

Thm : $M_n(K)$ is a semisimple K-algebra. $\exists!$ simple module K^n

Assume we have $f: R \longrightarrow R'$ Then Projective $P \longrightarrow P \otimes_R R'$.

$\rightsquigarrow f_*: K_0 R \longrightarrow K_0 R'$

Note: $f: \mathbb{Z} \to R \rightsquigarrow f_*: K_0\mathbb{Z} \to K_0 R$

$$K_0\mathbb{Z} \cong \mathbb{Z}.$$

(5). V: ∞-dim VS/K. $R = \mathrm{End}(V) \rightsquigarrow f_*: K_0\mathbb{Z} \to K_0R$. is 0 (zero) homomorphism

if R is commutative $f_*: \mathbb{Z} \to K_0R$ splits $p: R \to R/m$.

$$K_0\mathbb{Z} \longrightarrow K_0R \longrightarrow K_0(R/m) \overset{\text{rank map}}{\cong} \mathbb{Z}$$

Thm (Morita Equiv)

$K_0 R \simeq K_0(M_n(R))$.

Can K_0R has Torsion?

(6) $R = \mathbb{Z}[\sqrt{-5}]$

$I = (3, 2+\sqrt{-5})$ $\quad I \oplus I$ is free

$\rightsquigarrow [R] - [I] \in K_0R$ is 2-torsion

<u>Dedekind Domain</u>

Def: A dedekind domain is 1-dimensional Noetherian normal domain

$\Leftrightarrow$ R is a dedekind domain if it is domain and R_p is a DVR. ($p \neq 0$)

Thm. [Unique factorization of ideals]

$0 \subsetneq I \subsetneq R \rightsquigarrow \exists!\ I = P_1^{c_1} \cdots P_r^{c_r}$ where P_i are prime ideals

EX: ① $\mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2, 3, \cancel{4} \bmod 4 \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{if } d \equiv 1 \bmod 4 \end{cases}$

(d square free)

② Every PID e.g. $K[x]$

③
$$\begin{tikzcd} L \arrow[r, dash] \arrow[d, dash, "\text{fin.}"'] & B \arrow[d, dash, "\text{Integral closure}"] \\ K \arrow[r, dash] & A \end{tikzcd}$$

$\mathrm{Frac}(A)$ DD

Def: A class group of R. {monoid of Proper ideals in R} / {$I \sim J$ iff $\exists$ $a, b \in R \setminus \{0\}$ st $aI = bJ$}

$\Downarrow$ denote $Cl(R)$

$\downarrow$ Dedekind domain

Prop. I proper ideal of R. $\Rightarrow$ I is projective R-module

Thm M. f.g R-module.

M is projective $\longleftrightarrow$ M_p is free over R_p $\forall p$ prime.

Thm [FT of module over DD].

M: f.g module / DD R

$\rightsquigarrow M = (\bigoplus_{i=1}^{n} R/I_i) \oplus R^{r-1} \oplus I$

Cor: $K_0(R) \simeq \mathbb{Z} \oplus Cl(R)$

$$M \text{ Projective} \xRightarrow[\text{Torsion free}]{\text{FT}} M \simeq R^{r-1} \oplus I$$

$$[M] \longrightarrow (r-1, I)$$

Ex: $R = \mathbb{C}[x,y]/(y^2-x^3-ax-b)$ $\quad -4a^3-27b^2 \neq 0$

$\updownarrow$

$E \setminus \{0\}$. Fact: $Cl(R) \simeq E(\mathbb{C}) \simeq S^1 \times S^1$

(7) $K_0(\varinjlim R_\alpha) = \varinjlim K_0(R_\alpha)$

$\quad \quad \quad \quad \; \| $

$\quad \quad \quad \quad R$

$\varinjlim_K M_{2^k}(F)$ $\quad M_{2^k}(F) \longrightarrow M_{2^{k+1}}(F) \Rightarrow K_0(M_{2^k}(F)) \longrightarrow K_0(M_{2^{k+1}}(F))$

$$A \longrightarrow \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \qquad \simeq \mathbb{Z} \qquad \simeq \mathbb{Z}$$

$$K_0(R) = \varinjlim_K \mathbb{Z}^{\times 2} = \mathbb{Z}[\tfrac{1}{2}]$$

(8) $K_0(R_1 \times R_2) = K_0(R_1) \times K_0(R_2)$