Fundamental Theorems of Algebraic K-Theory

Thm [Resolution Thm] $\mathcal{M}$: exact cat. $\mathcal{P} \subseteq \mathcal{M}$: full additive subcat, closed under ext & exact w induced exact str.

Assume (a) $\mathcal{P}$ is closed under taking kernel.

(b) For any $M \in \mathcal{M}$, $\exists$ a finite $\mathcal{P}$-resol of M.

Then $BQ\mathcal{P} \to BQ\mathcal{M}$ is an homotopy equiv.

Cor Let X be a smooth var / North. regular sep. Sch.

Then $K_i \mathrm{Vect}(X) \simeq K_i \mathrm{Coh}(X)$ $(= G_i(X))$

Thm [Localization] $\mathcal{A}$: Ab. cat, $\mathcal{B} \subseteq \mathcal{A}$: Serre subcat. $\mathcal{E} := \mathcal{A}/\mathcal{B}$.

(Serre subcat: $A \to B \to C$, $A, C \in \mathcal{B} \Rightarrow B \in \mathcal{B}$. we can def $\mathcal{A}/\mathcal{B}$.

e.g. $\mathcal{A} = \mathrm{Mod}_A^{f.g.}$ $f \in A$. $\mathcal{B} :=$ f^∞-torsion modules. $= \mathrm{Mod}_A^{f.g.}[f^\infty]$

$\rightsquigarrow \mathcal{A}/\mathcal{B} = \mathrm{Mod}_A^{f.g.}$ w no f-torsion $= \mathrm{Mod}_{A_f}^{f.g.}$)

Then $BQ(\mathcal{A}) \to BQ(\mathcal{A}/\mathcal{B})$ is a fibration w hom. fib $BQ\mathcal{B}$.

Cor $\cdots K_n(\mathrm{Mod}_A^{f.g.}[f^\infty]) \to K_i(A) \to K_i(A/f) \to K_{i-1}(\mathrm{Mod}_A^{f.g.}[f^\infty]) \to \cdots$

Thm. [Devissage] $\mathcal{E} \subset \mathcal{A}$: Ab. subcat of $\mathcal{A}$, closed under subobj/quotients/fin prod. + $\mathcal{E}$ is exact. If $\forall$ object in $C \in \mathcal{A}$ has fin. filt

$$0 = C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_1 \subseteq C_0 = C$$

where $C_i/C_{i+1} \in \mathcal{E}$, then $BQ\mathcal{E} \to BQ\mathcal{A}$ is a hom. equiv.

Cor $K_n(\mathrm{Mod}_A^{f.g.}[f^\infty]) = K_n(\mathrm{Mod}_{A/f}^{f.g.})$.

$\Rightarrow \cdots \to K_i(A/f) \to K_i(A) \to K_i(A_f) \to K_{i-1}(A/f) \to \cdots$

If A is Noetherian & regular & A/f too.

Another Ex. A: Dedekind Domain. $K := \mathrm{Frac}(A)$

$\mathcal{A} := \mathrm{Mod}_A^{f.g.}$ $\mathcal{B} :=$ cat. of fin. gen. torsion A-mod. $\rightsquigarrow \mathcal{A}/\mathcal{B} = \mathrm{Mod}_K^{f.g.}$

$+ K_i(\mathcal{B}) = \bigoplus_P K_i(A/P)$

$\Rightarrow \cdots \to \bigoplus K_i(A/P) \to K_i(A) \to K_i(K) \to \cdots$

Same for Dedekind Sch.

$\mathcal{E}$: category. Contractible if $B\mathcal{E}$ is contractible (e.g. $\exists$ terminal/initial obj).

$f: \mathcal{E} \to \mathcal{D}$: functor. For $Y \in \mathcal{D}$, define $Y \backslash f$ be

$\mathrm{Obj}(Y\backslash f) = (X, v)$, $X \in \mathcal{E}$, $v: Y \to f(X)$.

$\mathrm{Mor}((X,v),(X',v')) \longleftrightarrow w: X \to X'$ s.t.
$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[dr, "v'"'] & f(X) \arrow[d, "f(w)"] \\ & f(X') \end{tikzcd}$$
commutes.

"Functorial" $u: Y \to Y' \rightsquigarrow u^*: Y'\backslash f \to Y\backslash f$.

f/X, $X \in \mathcal{E}$ is defined dually.

<u>Thm</u> $\mathcal{E} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ $\rightsquigarrow$ BF, BG a pair of hom. inv btw $B\mathcal{E}$ & $B\mathcal{D}$.

$\mathcal{E} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$. $\eta: F \Rightarrow G$. $\rightsquigarrow$ $BF \simeq BG$ (homotopic)

<u>Thm A</u> If $Y\backslash f$ is contractible, then $Bf: B\mathcal{E} \to B\mathcal{D}$ is a hom. equiv.

$*$: If f/X is " "

<u>Thm B</u> Assume that for every morph $u: Y \to Y'$ in $\mathcal{D}$, $u^*: Y'\backslash f \to Y\backslash f$ is a homotopy equivalence. Then $B\mathcal{E} \to B\mathcal{D}$ is a fibration w\ homotopy fiber $B(Y\backslash f)$.

proof of resol. thm. Let $\mathcal{M}_n :=$ full subcat of objs in $\mathcal{M}$ w/ resol by $\mathcal{P}$ of length $\leq n$.

by defn, $\mathcal{M}_0 = \mathcal{P}$.

Then, $\mathcal{M} = \varinjlim \mathcal{M}_n \rightsquigarrow Q\mathcal{M} = \varinjlim Q\mathcal{M}_n$

$\rightsquigarrow BQ\mathcal{M} = B(\varinjlim Q\mathcal{M}_n) = \varinjlim BQ\mathcal{M}_n$.

Hence, ETS $BQ\mathcal{M}_n \to BQ\mathcal{M}_{n+1}$ is a hom. eq.

<u>Lem</u> $0 \to M' \to M \to M'' \to 0$

(1) $M \in \mathcal{M}_n$, $M'' \in \mathcal{M}_{n+1} \Rightarrow M' \in \mathcal{M}_n$

(2) $M', M'' \in \mathcal{M}_{n+1} \Rightarrow M \in \mathcal{M}_{n+1}$

(3) $M, M'' \in \mathcal{M}_{n+1} \Rightarrow M' \in \mathcal{M}_{n+1}$

) closed under ext & ker.

Hence ETS "2-step" version of the resol. thm.

$\mathcal{P} \subseteq \mathcal{M}$. closed under ext & ker. $\exists$ 2-step resol.

$\rightsquigarrow$
$$\begin{tikzcd} Q\mathcal{P} \arrow[rr] \arrow[dr, "g"'] & & Q\mathcal{M} \\ & \mathcal{E} \arrow[ur, "f"'] & \end{tikzcd}$$
$\mathcal{E} :=$ full subcat of $Q\mathcal{M}$ whose obj is $Q\mathcal{P}$.

ETS g & f are hom. equiv.

(1) g is hom equiv.

Fix $P \in \mathcal{E}$. By Thm A, ETS g/P is contractible.

obj : (P_1, u), $P_1 \twoheadleftarrow P_1' \overset{\mathcal{M}}{\hookrightarrow} P$

Morph :
$$\begin{tikzcd} P_1 & P_1' \arrow[l, two heads] \arrow[r, hook] & P \arrow[dd, equal] \\ P_3 \arrow[u, hook] \arrow[d, two heads] & P_2' \arrow[l, two heads] \arrow[u, hook] \arrow[d, equal] & \\ P_2 & P_2' \arrow[l, two heads] \arrow[r, hook] & P \end{tikzcd}$$

$$\begin{tikzcd} \text{"}P_1''\text{"} = \ker \arrow[r] \arrow[d, phantom, "\cong" description] & P_1' \arrow[r, two heads] & P_1 \\ \ker \arrow[r] & P_2' \arrow[u, hook] \arrow[r, two heads] & P_3 \arrow[u, hook, "\mathcal{P}\text{-adm.}"'] \arrow[r, two heads, "\mathcal{P}\text{-adm.}"'] & P_2 \end{tikzcd}$$
$\dashv P_1'' \subseteq P_2''$

Idea : $\mathrm{Obj}(\mathcal{E}) \longleftrightarrow (P_1, u) \longleftrightarrow P_1'' \underset{\mathcal{P}\text{-adm.}}{\subset} P_1' \underset{\mathcal{M}\text{-adm.}}{\subset} P$.

$(P_2, v) \longrightarrow (P_1, u) \quad \longleftrightarrow \quad \underbrace{P_1'' \subset P_2'' \subset P_2'}_{\mathcal{P}\text{-adm}} \subset P_1' \subset P$.

$\rightsquigarrow$ g/P is a POset!

$$\begin{tikzcd} & (P', 0) & \\ (P', P'') \arrow[ur] & & (0, 0) \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & \text{Projection} & \\ \text{Identity} \arrow[ur] & & \text{Constant} \arrow[ul] \end{tikzcd}$$

$\rightsquigarrow$ Id $\simeq$ Const $\Rightarrow$ Contract.

(2) f is hom. equiv.

Fix $M \in Q\mathcal{M}$. We will show $M \backslash f$ is contractible.

$\mathcal{F} := M\backslash f$. $\quad \mathrm{Ob}(\mathcal{F}) = M \twoheadleftarrow \overline{P} \hookrightarrow P$ $\quad$ adm. mono $\Rightarrow \overline{P} \in \mathcal{P}$

$\cup$

$\mathcal{F}' :=$ Full subcat $M \twoheadleftarrow P = P$.

We have adj ft $\quad \left(M \twoheadleftarrow \overline{P} \hookrightarrow P\right) \longmapsto \left(M \twoheadleftarrow \overline{P} = \overline{P}\right)$.

Since this is adj, $\mathcal{F}' \hookrightarrow \mathcal{F}$ is hom. equiv.

ETS $\mathcal{F}'$ is contractible. $ob(\mathcal{F}') = P \twoheadrightarrow M$. ← adm. epi.

$$Mor(P, P') = \begin{tikzcd} P \arrow[r, two heads] \arrow[dr, two heads] & P' \arrow[d, two heads] \\ & M \end{tikzcd}$$

$\exists$ one such P_0. fix. $P_0 \twoheadrightarrow M$.

$$\begin{tikzcd} P \arrow[r, two heads] & M \\ P \times_M P_0 \arrow[u] \arrow[r, two heads] & P_0 \arrow[u] \end{tikzcd}$$

identity $\Uparrow$ Fib. Prod $\Rightarrow$ Const.

Hence const $\simeq$ id, so $\mathcal{F}'$ contractible.

proof of Devissage: $f : Q\mathcal{B} \to Q\mathcal{A}$. $M \in \mathcal{A}$. ETS f/M contractible.

f/M : (N, u), $N \in \mathcal{B}$, $N \twoheadleftarrow M' \hookrightarrow M$

$\Longleftrightarrow$ $M'' \hookrightarrow M' \hookrightarrow M$ PO set of pairs (M', M'')

$\mathcal{B}$-adm. $\mathcal{A}$-adm. $J(M)$

ETS : $M' \hookrightarrow M$ $\mathcal{B}$-adm $\Rightarrow$ $J(M') \xrightarrow{i} J(M)$ hom. equiv.

homotopy inverse: $(M_1, M_2) \xrightarrow{r} (M_1 \cap M', M_2 \cap M')$

Since $M_1 \cap M' / M_2 \cap M' \hookrightarrow M_1 / M_2 \cap M' \hookrightarrow M_1/M_2 \oplus M/M'$.

$$\begin{tikzcd} & (M_1, M_2 \cap M') \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ i \circ r = (M_1 \cap M', M_2 \cap M') & & (M_1, M_2) = id. \end{tikzcd}$$

$\rightsquigarrow$ $i \circ r \simeq id$. Hence $J(M') \to J(M)$ is hom. equiv.

proof of Localization.

$\mathcal{B} \xrightarrow{e} \mathcal{A} \xrightarrow{s} \mathcal{A}/\mathcal{B}$ $\rightsquigarrow$ $Q\mathcal{B} \xrightarrow{Qe} Q\mathcal{A} \xrightarrow{Qs} Q(\mathcal{A}/\mathcal{B})$.

By the Theorem B, ETS the followings :

(a) For every $u : V' \longrightarrow V$ in $Q(\mathcal{A}/\mathcal{B})$,

$u^* : V \backslash Qs \longrightarrow V' \backslash Qs$ is a hom. equiv.

(b) $Q\mathcal{B} \longrightarrow 0 \backslash Qs$ is hom. equiv.

Note that, any $u : V' \longrightarrow V$ factors to $V' \twoheadleftarrow V_1 \hookrightarrow V$.

which is a composition $\overbrace{V \twoheadleftarrow V_1 \overset{id}{\hookrightarrow} V_1}^{\text{adm. epi}} \overbrace{\overset{id}{\twoheadleftarrow} V_1 \hookrightarrow V}^{\text{adm. mono.}}$

Hence ETS show this for adm. epi / adm mono. Since $QC \simeq QC^{op}$,
ETS show this for adm. mono. For $i: V' \hookrightarrow V$ adm. mono,
$0 \to V' \hookrightarrow V$: ETS this for $0 \to V$. Denote this by i_V.

$\mathcal{F}_V :=$ Full subcat of $V \backslash Q_S$ of pairs (M, u), $u: V \to S(M)$
s.t. u is an isomorphism.
$\mathcal{F}_0 \simeq Q\mathcal{B}$.

<u>Lem</u> $\mathcal{F}_V \hookrightarrow V \backslash Q_S$ is hom. equiv
In particular, this implies (b).

Idea: "approximate" $\mathcal{F}_V$.

From now, we will focus on (a). ETS $\mathcal{F}_V \longrightarrow \underset{\simeq Q\mathcal{B}}{0 \backslash Q_S}$ is hom equiv
For $N \in \mathcal{A}$, define $\mathcal{G}_N$ be
Ob$(\mathcal{G}_N)$ = (M, h), $M \in \mathcal{A}$, $h: M \to N$ a homo in $\mathcal{A}$ s.t. $S(h)$ is iso.
morph: $(M,h) \to (M',h')$: $u: M \to M'$ in $Q\mathcal{A}$ s.t.

$$\begin{tikzcd} M_1 \arrow[r, hook] \arrow[d, two heads] & M' \arrow[d, "h'"] \\ M \arrow[r, "h"] & N \end{tikzcd}$$

commutes.

$g: N \to N'$ a map in $\mathcal{A}$ s.t. $S(g)$ is an iso
$\rightsquigarrow$ $\exists$ a functor $g_*: \mathcal{G}_N \to \mathcal{G}_{N'}$.
<u>Lem</u> g_* is a hom. equiv.
$\exists k_N: \mathcal{G}_N \to Q\mathcal{B}$ s.t. $k_N(M,h) = \ker h$.
<u>Lem</u> k_N is a hom. equiv.

Point: $\mathcal{F}_V$ is "limit" of $\mathcal{G}_N$.
Let $\mathcal{I}_V$ be the cat. of pairs (N, ϕ), $\phi: S(N) \xrightarrow{\sim} V$.
morph: $g: N \to N'$ s.t. $S(N) \to S(N') \to V$ ...
<u>Lem</u> $\mathcal{I}_V$ is a filtering category.
<u>Lem</u> $\varinjlim_{\mathcal{I}_V} \mathcal{G}_N = \mathcal{F}_V$.

· Brown – Gersten – Quillen Spectral Sequence

X: Dedekind Sch

$$\cdots \bigoplus_{x\in X^{(1)}} K_i(k(x)) \to K_i(X) \to K_i(K(X)) \to \cdots \qquad X^{(1)}: \text{codim } 1 \text{ pts.}$$

Reason: $\mathrm{Torsion}(X) \subset \mathrm{Coh}(X) \twoheadrightarrow \mathrm{Coh}(K(X))$

$\quad\ \ \|$
Support codim ≥ 1. Now let X: sep. Noeth. regular sch / smooth var.

$\mathrm{Coh}(X)^p :=$ full subcat of $\mathrm{Coh}(X)$, support codim $\geq p$.

: closed under ext. subobj. quotient ...

$\mathrm{Coh}(X)^1$ = all "torsion" sheaves. So

$$\mathrm{Coh}(X)^1 \subseteq \mathrm{Coh}(X) \to \mathrm{Coh}(K(X)).$$

$$\mathrm{Coh}(X)^2 \subseteq \mathrm{Coh}(X)^1 \to \coprod_{x\in X^{(1)}} \mathrm{Coh}(k(x)) \quad \rightsquigarrow \bigoplus_{x\in X^{(1)}} K_i(k(x)).$$

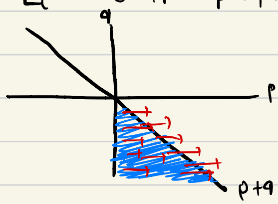
$\mathrm{Coh}(X)^2 = \cdots$?

$\mathrm{Coh}(X)^p$ gives a filtration, take "cohomology"

$\Rightarrow$ BGQ spectral sequence.

Thm [BGQ Spectral Sequence] $E_1^{p,q} = \coprod_{x\in X^{(p)}} K_{-p-q}(x) \Rightarrow K_{-p-q}(X)$

$E_1^{p,q} = 0$ if $p+q>0$ or $p<0$



$$E_1^{0,0} = K_0(K(X)) \simeq \mathbb{Z} \to 0$$

$$E_1^{0,-1} = K_1(K(X)) \simeq K(X)^\times \to E_1^{1,-1} = \bigoplus_{x\in X^{(1)}} K_0(k(x)) \simeq \bigoplus_{x\in X^{(1)}} \mathbb{Z}$$

$f \mapsto \mathrm{div}(f)$.

$$E_1^{0,-2} = K_2(K(X)) \to E_1^{1,-2} = \bigoplus_{x\in X^{(1)}} K_1(k(x)) \simeq \bigoplus_{x\in X^{(1)}} k(x)^\times \to E_1^{2,-2} = \bigoplus_{x\in X^{(2)}} \mathbb{Z}$$

"pointwise div map"

$\rightsquigarrow E_2^{n,-n} = CH^n(X) \qquad (CH \rightsquigarrow K)$

Fact $E_\infty^{n,-n}$ is a quotient of $CH^n(X)$ & kernel is torsion.

$\rightsquigarrow K_0(X)\otimes\mathbb{Q} \simeq CH^*(X)\otimes\mathbb{Q}$.

$\mathcal{K}_p$: Zariski sheafification of $U \mapsto K_p(U)$.

"Gersten's Conj" $\Rightarrow E_2^{p,q} = H^p(X_{Zar}, \mathcal{K}_{-q})$

Thm [Bloch's Formula] $E_2^{p,-p} = CH^p(X) = H^p(X_{Zar}, \mathcal{K}_{-p})$.