W? K theory of fields

Outline.

I. K theory of finite fields

II. K theory of algebraically closed fields

III. K theory of number fields & $K_*(\mathbb{Z})$

I.

<u>Thm</u> (Quillen):

For a finite field $\mathbb{F}_q$ with $q=p^\nu$ elements,

$$K_n(\mathbb{F}_q) = \begin{cases} 0, & n\geq 2 \text{ even} \\ \mathbb{Z}/(q^i-1), & n=2i-1, \quad i\geq 1 \end{cases}$$

Sketch of proof.

- We will construct a map $BGL\mathbb{F}_q^+ \to BU$, which lifts to $BGL\mathbb{F}_q^+ \to F\Psi^p$ where $F\Psi^p$ is the fiber of $BU \xrightarrow{\Psi^q-1} BU$

$$\begin{tikzcd} BGL\mathbb{F}_q^+ \arrow[d, "f", "\wr"'] \arrow[dr, "g^+"] & & \\ F\Psi^p \arrow[r] & BU \arrow[r, "\Psi^q-1"] & BU \end{tikzcd}$$

And prove that f is a homotopy equivalence

- Representation theory:

Every complex representation over G gives rise to a map $BG \to BU$: (i.e. there is a natural map $R_{\mathbb{C}}(G) \to K^0(BG)$)

For $\rho: G \to GL(V)$ a representation, V complex vector space form the vector bundle $EG \times_G V \to BG$ which corresponds to a map $BG \to BU$.

- Adams operations

Thm: There exists ring homomorphism $\Psi^k : K^0(X) \to K^0(X)$ for all $k \geq 0$, such that

(i) $\Psi^k \circ f^* = f^* \circ \Psi^k$ for $f: X \to Y$

(ii) If L is a line bundle, $\Psi^k(L) = L^k$

(iii) $\Psi^k \circ \Psi^\ell = \Psi^{k\ell}$

(iv) $\Psi^p(\alpha) \equiv \alpha^p \pmod p$ in $K^0(X)$

Pf: We can define $\Psi^k(E) = s_k(\Lambda^1 E, \Lambda^2 E, \cdots, \Lambda^k E)$

s_k is the polynomial defined by

$s_k(\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_k) = x_1^2 + x_2^2 + \cdots + x_k^2$, σ_i elementary sym..

Thm: Ψ^k does not commute with the periodicity iso $K^0(X) \cong K^0(\Sigma^2 X)$

$$\begin{array}{ccc} K^0(X) & \xrightarrow{\beta} & K^0(\Sigma^2 X) \\ \Psi^k \downarrow & & \downarrow \Psi^k \\ K^0(X) & \xrightarrow{\beta} & K^0(\Sigma^2 X) \end{array}$$

but rather, $\Psi^k \beta = k \beta \Psi^k$.

Cor: The action Ψ^k on $K^0(S^{2n})$ is multiplication by k^n.

Therefore from the short exact seq.

$$0 \to \pi_n(BU) \to \pi_n(BU) \to \pi_{n-1}(F\Psi^q) \to 0$$

we see $\pi_{2i}(F\Psi^q) = 0$, $\pi_{2i-1}(F\Psi^q) = \mathbb{Z}/(q^i-1)$.

Rmk: We can also define Adams operations on representations:

For a representation ρ with charactorter χ_ρ,

let $\chi_{\Psi^k(\rho)}(g) := \chi_\rho(g^k)$ for $g \in G$

Then we can prove $\chi_{\Psi^k(\rho)}$ is the character of a virtual representation $\Psi^k(\rho) \in R_{\mathbb{C}}(G)$.

- Ψ^k on $R_{\mathbb{C}}(G)$ and $K(X)$ are compatible:

$$\begin{tikzcd} R_{\mathbb{C}}(G) \arrow[r, "\Psi^k"] \arrow[d] & R_{\mathbb{C}}(G) \arrow[d] \\ K^0(BG) \arrow[r, "\Psi^k"] & K^0(BG) \end{tikzcd}$$

commutes.

- Construction of $g^+ : BGL\mathbb{F}_q^+ \to BU$

Brauer lifting:

Thm: Let $\overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$.

Fix an embeding $\iota : \overline{\mathbb{F}}_p^\times \to \mathbb{C}^\times$ with image the complex roots of unity of order prime to p.

Let G be a finite group, $\rho : G \to GL(n, \overline{\mathbb{F}}_p)$ be a representation over $\overline{\mathbb{F}}_p$.

$\xi_1(g), \cdots, \xi_n(g)$ be the eigenvalues of $\rho(g)$

Then $\chi_{\bar{\rho}}(g) := \iota(\xi_1(g)) + \iota(\xi_2(g)) + \cdots + \iota(\xi_n(g))$

is a complex virtual character, which defines

a complex virtual representation. (called the Brauer lifting of ρ).

Using Brauer lifting, take $G = GL(n, \mathbb{F}_q)$

$\rho = id: GL(n, \mathbb{F}_q) \to GL(n, \mathbb{F}_q)$ the natural representation

$\chi_{\bar{\rho}}$ the Brauer lifting of ρ,

which corresponds to $g: BGL(n, \mathbb{F}_q) \to BU$.

Stabilize to $g: BGL\mathbb{F}_q \to BU$

BU H-space $\Rightarrow$ g lifts to $g^+: BGL\mathbb{F}_q^+ \to BU$.

- Lifting to $F\Psi^q$:

Just note that $\chi_{\bar{\rho}}$ is invariant under Ψ^q

because for $g \in GL(n, \mathbb{F}_q)$

The set of eigenvalues of g is invariant under $x \mapsto x^q$.

So g^+ lifts to

$$f: BGL\mathbb{F}_q^+ \to F\Psi^q$$

It remains to show f is a homotopy equiv.

$\Leftrightarrow$ f induces iso on integral cohomology (H-spaces)

$\Leftrightarrow$ f induces iso on $\mathbb{Q}$-cohomology, $\mathbb{Z}/\ell$-cohomology

(Universal coefficient)

- $\mathbb{Q}$-cohomology: Both sides zero
- $\mathbb{Z}/p$-cohomology: Both sides zero
- $\mathbb{Z}/\ell$-cohomology for $\ell \neq p$ prime: Hard!!

- Quillen's observation from his proof of Adams's conjecture.
  For $\ell \neq p$ prime, $BGL\overline{\mathbb{F}_p} \to BU$ induces iso on $\mathbb{Z}/\ell$-cohomology

  Which is also the origion of Quillen's plus construction.
- Method: similar to $BU(1)^n \to BU(n)$, using maximal tori.

$\square$.

Cor: $K_*(\mathbb{F}_q)$ has trivial product structure.

Using direct limits,

Cor: $K_n(\overline{\mathbb{F}_p}) = \begin{cases} 0, & n \text{ even} \\ \overline{\mathbb{F}_p}^\times \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{p}], & n \text{ odd.} \end{cases}$

II. K theory of algebraically closed fields.

- K theory with finite coefficients:

  $K(R; \mathbb{Z}/m) := \pi_i(KR \wedge H\mathbb{Z}/m)$

  As homology with coefficients, there is a short exact seq:

  $$0 \to K_n(R) \otimes \mathbb{Z}/m \to K_n(R; \mathbb{Z}/m) \to {}_m(K_{n-1}(R)) \to 0$$

  $({}_mA = \{a \in A : ma = 0\})$

Thm: $\exists$ Bott element $\beta \in K_2(\overline{\mathbb{F}_p}; \mathbb{Z}/m)$, $(p \nmid m)$

s.t. $K_*(\bar{\mathbb{F}}_p; \mathbb{Z}/m) = \mathbb{Z}/m[\beta] \cong \pi_*(BU; \mathbb{Z}/m)$

The heart of the computation of algebraically closed fields is the following "Rigidity" theorem:

<u>Thm</u> (Suslin's Rigidity Theorem)

If $k \subseteq F$ is an inclusion of algebraically closed fields then the maps $K_*(k, \mathbb{Z}/m) \to K_*(F; \mathbb{Z}/m)$ are isomorphisms for all m.

<u>Cor</u> (*): Let F be an algebraically closed field of characteristic $p > 0$.

(i) For $p \nmid m$, $K_*(F; \mathbb{Z}/m) = \mathbb{Z}/m[\beta]$ for $\beta \in K_2(F; \mathbb{Z}/m)$

(ii) $K_n(F)$ is uniquely divisible for $n > 0$ even

(iii) For $n > 0$ odd, $K_n(F)$ is the direct sum of a uniquely divisible group and $\mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$

(Uniquely divisible $\Leftrightarrow$ $\forall\, a \in A$, $m \in \mathbb{Z}$, $\exists!\, b \in A$, $a = mb$
$\Leftrightarrow$ A is a $\mathbb{Q}$-module).

For characteristic 0, we have:

<u>Thm</u>: Let F be an alg. closed field of char 0

Then for $\forall\, m > 0$, $\exists\, \beta \in K_2(F; \mathbb{Z}/m)$,

$K_*(F; \mathbb{Z}/m) \cong \mathbb{Z}/m[\beta]$

Pf: Comes from the iso $K_*(\mathbb{C}; \mathbb{Z}/m) \to \pi_*(BU; \mathbb{Z}/m)$ of change-of-topology.

Thm(**): Let F be an alg. closed field of char 0

(i) $K_n(F)$ is uniquely divisible for $n>0$ even.

(ii) For $n>0$ odd, $K_n(F)$ is the direct sum of a uniquely divisible group and $\mathbb{Q}/\mathbb{Z}$.

(*) and (**) give discriptions of K_* of alg. closed fields.
The torsion part is $\mu = \mu(F)$, the group of roots of unity.
For $i>0$, let $\mu(i)$ be an $\mathrm{Aut}(F)$-module,
whose base group is just $\mu = \mu(F)$, while action is given by $g \cdot \xi = g^i(\xi)$ for $g \in \mathrm{Aut}(F)$.

Thm: Let F be an alg. closed field, then the torsion subgroup of $K_{2i-1}(F)$ is isomorphic to $\mu(i)$.

III. K theory of number fields, $K_*(\mathbb{Z})$

For this part, fix F a number field, $\mathcal{O}_F$ the ring of integers.
First we have the localization sequence:

$$\cdots \to \bigoplus_p K_n(\mathcal{O}_F/p) \to K_n(\mathcal{O}_F) \to K_n(F) \to \bigoplus_p K_{n-1}(\mathcal{O}_F/p) \to \cdots$$

Thm: The above sequence breaks up into short exact sequences:

$$0 \to K_n(\mathcal{O}_F) \to K_n(F) \to \bigoplus_p K_{n-1}(\mathcal{O}_F/p) \to 0.$$

Since $\mathcal{O}_F/p$ are finite fields, we have

$K_n(\mathcal{O}_F) \cong K_n(F)$ for $n \geq 3$ odd

$K_n(\mathcal{O}_F) \otimes \mathbb{Q} \cong K_n(F) \otimes \mathbb{Q}$

Thm (Quillen): $K_n(\mathcal{O}_F)$ is finitely generated for all n.

Thm (Borel): The rank of $K_n(\mathcal{O}_F)$ is given by

$n \pmod 4$	0	1	2	3
rk	0	r_1+r_2	0	r_2

except for $\mathrm{rk}\, K_0 = 1$ $\quad \mathrm{rk}\, K_1 = r_1+r_2-1$

where $r_1 :=$ # embeddings $F \to \mathbb{R}$

$r_2 :=$ # conjugate pairs of embeddings $F \to \mathbb{C}$.

Rmk: This is the generalization of Dirchlet's Unit Thm, which asserts that $\mathcal{O}_F^\times \cong \mathbb{Z}^{r_1+r_2-1}$.

Rmk: The period 4 is related to $\pi_*(O) \otimes \mathbb{Q}$

Now for the torsion part of $K_n(F)$, much harder.

There's a certain summand in the torsion called Harris-Segal summand.

Def (e-invariant): For a field F of char$=0$,

$K_*(F) \to K_*(\bar{F})$ is a morphism of $G = \mathrm{Gal}(\bar{F}/F)$ modules with trivial G-action on $K_*(F)$.

Denote the map $K_{2i-1}(F)_{tor} \to K_{2i-1}(\bar{F})_{tor}^G = \mu(i)^G$ by e, called the e-invariant of F.

In case $\mu(i)^G$ is finite, it is cyclic, denote by $w_i(F)$ its order.

Thm: If $\sqrt{-1} \in F$, then $e: K_{2i-1}(F) \to \mathbb{Z}/w_i(F)$ splits. i.e. $K_{2i-1}(F)$ has a torsion direct summand $\mathbb{Z}/w_i(F)$.

Thm: If $F \subseteq \mathbb{R}$, then $\exists$ a direct summand of $K_{2i-1}(F)$, called the Harris–Segal summand, which is isomorphic to

(1) $\mathbb{Z}/w_i(F)$ if $2i-1 \equiv \pm 1 \pmod 8$

(2) $\mathbb{Z}/2w_i(F)$ if $2i-1 \equiv 3 \pmod 8$

(3) $\mathbb{Z}/\frac{1}{2}w_i(F)$ if $2i-1 \equiv 5 \pmod 8$

Now consider the case $F = \mathbb{Q}$. $w_i(\mathbb{Q})$ is given by:

$w_i(\mathbb{Q}) = 2$ for i odd $\qquad B_1 = \frac{1}{6} \quad B_2 = \frac{1}{30} \quad B_3 = \frac{1}{42}$

$w_{2k}(\mathbb{Q}) =$ denominator of $B_k/4k$ Bernoulli numbers

defined by $\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} (-1)^k B_k \cdot \frac{t^{2k}}{(2k)!}$

So the Harris–Segal summand of $K_{2i-1}(\mathbb{Z}) = K_{2i-1}(\mathbb{Q})$ is:

$n \pmod 8$	0	1	2	3	4	5	6	7
H-S summands	/	$\mathbb{Z}/2$	/	$\mathbb{Z}/2w_i$	/	0	/	$\mathbb{Z}/w_i$

And actually H-S summands are all of the torsion in these degrees.

Combined with Borel's result, we've computed $K_n(\mathbb{Z})$ for $n \equiv 1, 3, 5, 7 \pmod 8$

For the remaining degrees, we need more results.

<u>Thm</u>: $\dfrac{|K_{4k-2}(\mathbb{Z})|}{|K_{4k-1}(\mathbb{Z})|} = \dfrac{B_k}{4k} = \dfrac{c_k}{w_{2k}}$

So we can now summarize what we currently know about $K_*(\mathbb{Z})$:

<u>Thm</u>: For $n \not\equiv 0 \pmod 4$, $n > 1$, we have the following:

(1) $n = 8k+1$, $K_n(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$

(2) $n = 8k+2$, $|K_n(\mathbb{Z})| = 2c_{2k+1}$

(3) $n = 8k+3$, $K_n(\mathbb{Z}) = \mathbb{Z}/2w_{4k+2}$

(4) $n = 8k+5$, $K_n(\mathbb{Z}) = \mathbb{Z}$

(5) $n = 8k+6$, $|K_n(\mathbb{Z})| = c_{2k+1}$

(6) $n = 8k+7$, $K_n(\mathbb{Z}) = \mathbb{Z}/w_{4k+4}$.

$K_{22}(\mathbb{Z}) = \mathbb{Z}/691$ (691 prime)

<u>Rmk</u>: Assuming Vandiver's Conjecture:

$p \nmid h_K$, h_K the class number of K = maximal real subfield $\mathbb{Q}(\zeta_p)^+$

or the order of ideal class group of $\mathbb{Z}[\zeta+\zeta^{-1}]$.

(1 counterexample in the first 10^{100} primes).

We have $K_{4n}(\mathbb{Z}) = 0$, and $K_{4n+2}(\mathbb{Z})$ are cyclic.

So we'll get a complete list of K theory of $\mathbb{Z}$:

n	$8k$	$8k+1$	$8k+2$	$8k+3$	$8k+4$
$K_n(\mathbb{Z})$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/2c_{k+1}$	$\mathbb{Z}/2w_{4k+2}$	0

$8k+5$	$8k+6$	$8k+7$
$\mathbb{Z}$	$\mathbb{Z}/c_{2k+1}$	$\mathbb{Z}/w_{4k+4}$.

Rmk: $K_*(\mathbb{Z})$ is closely related to $\mathrm{Im}J$ of π_*^S

Prop: J-homomorphism: $J: \pi_i(SO) \to \pi_i^S$

- $\mathrm{Im}J$ is a direct summand of π_i^S

  For $i=0,1$, $\mathrm{Im}J \cong \mathbb{Z}/2$

  $i=8k+3, 8k+7$, $\mathrm{Im}J \cong \mathbb{Z}/w_{4k+2}, \mathbb{Z}/w_{4k+4}$
- $\mathrm{Im}J$ is detected by Adams e-invariant

  $e: \pi_i^S \to \mathbb{Q}/\mathbb{Z}$, $\pi_i^S = \mathrm{Im}J \oplus \mathrm{Ker}\, e$.
- Quillen's observation: Adams e-invariant is the same as the composition $\pi_{4i-1}^S \to K_{4i-1}(\mathbb{Q}) \xrightarrow{e} \mathbb{Z}/w_{2i}(\mathbb{Q})$
- $\mathrm{Im}J$ injects into $K_{4i-1}(\mathbb{Z})$, maps to 0 in $K_{8k+1}(\mathbb{Z})$

  almost the same as the torsion:

  half of $K_{8k+3}(\mathbb{Z})$, all of $K_{8k+7}(\mathbb{Z})$.
- Adams family μ_{8k+1}, μ_{8k+2} maps to $\mathbb{Z}/2$ in $K_*(\mathbb{Z})$
- $\widetilde{\mathrm{Coker}\, J} = \pi_*^S/(\mathrm{Im}J + \mu_i)$, maps to 0 in $K_*(\mathbb{Z})$

Rmk: The analysis on the torsion of $K_*(\mathcal{O}_F)$

relies heavily on the motivic-to-K-theory SS

$$E_2^{p,q} = H_M^{p-q}(X; \mathbb{Z}/m(-q)) \Rightarrow K_{-p-q}(X; \mathbb{Z}/m)$$

Voevodsky and Rost proved that

$$H_M^n(F, \mathbb{Z}/m(i)) \cong H_{et}^n(F, \mu_m^{\otimes i})$$

This was the celebrated Quillen-Lichtenbaum conjecture.