12/12. Hochschild Homology & Cyclic Homology

§1 Hochschild Homology.

§1.1. Hochschild complex and HH group

Consider $C_n(A,M) = M \otimes A^{\otimes n}$ (M is a bi-module. A is K-algebra)

The HH boundary is a K-linear map

$b: M \otimes A^{\otimes n} \longrightarrow M \otimes A^{\otimes n-1}$ given by

$b(m, a_1, \dots a_n) = (ma_1, \dots a_n) + \sum_{i=1}^{n-1} (-1)^i (m, a_1, \dots a_n) + (-1)^n (a_n m, \dots a_{n-1})$

Note if we define $d_i = (m, a_1, \dots a_i a_{i+1} \dots a_n)$ $1 \le i \le n$

$d_0 = (ma_1, \dots \dots a_n)$

$d_n = (a_n m, \dots a_{n-1})$

$$b = \sum_{i=0}^{n} (-1)^i d_i$$

Easy to check $b \circ b = 0$

Hochschild Complex.

$C(A,M) \quad \longrightarrow M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \cdots \xrightarrow{b} M \otimes A \xrightarrow{b} M.$

When $M = A$

$C(A) = C_*(A) \quad \longrightarrow A^{\otimes n+1} \xrightarrow{b} A^{\otimes n} \longrightarrow \cdots \longrightarrow A^{\otimes 2} \xrightarrow{b} A$

is sometime called cyclic bar complex

Elementary calculation.

$H_0(A,M) = M / \{am - ma \mid a \in A,\ m \in M\}$

$H_0(A) = A/[A,A]$ if A is commutative $[A,A]$ is trivial.

if $A = K$ Then $HH_0(K) = K$. $HH_n(K) = 0$ for $n > 0$

Prop. if the unital algebra A is projective as a module over K then for any A-bimodule M there is an isomorphism

$$H_n(A,M) \cong \mathrm{Tor}_n^{A_e}(M,A) \qquad A_e = A \otimes A^{op}$$

Prop. if A is unital and commutative then $HH_1(A) \cong \Omega^1_{A/k}$.
if M is a symmetric bimodule then $H_1(A,M) \cong M \otimes \Omega^1_{A/k}$

Prop. For any commutative k-algebra A.
the antisymmetrization map induces a canonical map

$$\varepsilon_n : \Omega^n_{A/k} \longrightarrow HH_n(A)$$

if A is smooth Algebra then ε_n is actually an isomorphism by HKR thm.

Bar construction.

$$A^{\otimes n+1} \xrightarrow{b'} A^{\otimes n} \longrightarrow \cdots \xrightarrow{b'} A^{\otimes 2} \xrightarrow{\mu} A$$

$$b' = \Sigma(-1)^i d_i$$

§. Cyclic Homology.

Recall the hochschild complex

$$A \longleftarrow A\otimes A \longleftarrow A\otimes A\otimes A \longleftarrow$$

$\mathbb{Z}/_{n+1}$ acts on $A^{\otimes n+1}$

$$t_n(a_0; a_1 \cdots a_n) = (-1)^n (a_n, a_0 \cdots a_{n-1})$$

define $N = 1 + t + \cdots t^n$

One could check. $(1-t)b' = b(1-t)$ $\quad b'N = Nb$

Thus we have a double complex denote $CC(A)$

$$\begin{tikzcd}
\downarrow & \downarrow & \downarrow & \\
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & \arrow[l] \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b"] & A^{\otimes 2} \arrow[l] \arrow[d, "b"] & \arrow[l] \\
A & A' \arrow[l, "1-t"'] & A \arrow[l] & \arrow[l]
\end{tikzcd}$$

Def: The cyclic homology groups $HC_n(A)$ of the associated K-algebra A are the homology group of the $Tot.CC(A)$

$$HC_n(A) = H_n(Tot\, CC(A))$$

· Conne's Complex

$$C^{\lambda}_{*}(A) \qquad \xrightarrow{b} C^{\lambda}_{n}(A) \xrightarrow{b} C^{\lambda}_{n-1}(A) \longrightarrow \cdots$$

$$C^{\lambda}_{n}(A) = \frac{A^{\otimes n+1}}{(1-t)}$$

$$p: \mathrm{Tot}\, CC(A) \longrightarrow C^{\lambda}(A)$$

$A^{\otimes n+1} \longrightarrow A^{\otimes n+1}/(1-t)$ on the first colum.

Thm: if A is an algebra over a ring K contains $\mathbb{Q}$

$HC_*(A) \longrightarrow H^{\lambda}_{*}(A)$ is an isomorphism

Lemma (Killing Contractible Complex) Let

$$A_n \oplus A_n' \xrightarrow{d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} A_{n-1} \oplus A_{n-1}' \longrightarrow$$

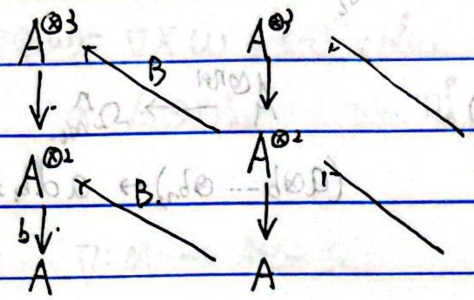
be a complex of K-module such that (A_*', δ) is a complex and is contractible with contracting homotopy $h: A_n' \longrightarrow A_{n+1}'$ Then following inclusion of complex

$$(id, -h\gamma) \quad (A_*, \alpha - \beta h \gamma) \hookrightarrow (A_* \oplus A_*', d)$$

By Applying the Lemma to $CC(A)$ where A_*' is the odd column successively

$\alpha = b \quad \beta = (1-t) \quad \gamma = N \quad \delta = -b' \quad h = -s$

Thus denote. $B = (1-t)sN$



Thus we Rearrange to get a new Bicomplex $B(A)$

$$\begin{tikzcd}
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
A^{\otimes 2} \arrow[d, "b"'] & A \arrow[l, "B"'] & \\
A & &
\end{tikzcd}$$

$B(A) = A^{\otimes p-q+1}$ if $q \geq p$ and $= 0$ otherwise.

Thus $H_n(Tot(B(A)) = HC_n(A)$

Elementary Computation. $HC_0(A) = HH_0(A) = A/[A,A]$

Thm. (Connes' Periodicity Exact Sequence)

$$\longrightarrow HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \longrightarrow HH_{n-1}(A)$$

pf: Consider the exact sequence of bicomplex

$$0 \to CC(A)^{\{2\}} \longrightarrow CC(A) \longrightarrow CC(A)[2,0] \longrightarrow 0$$

$\Downarrow$ two column bicomplexes

$\Downarrow$ shift by 2 column

Remark: HC, HH and de Rham cohomology

$$\begin{array}{ccc} HH_n(A) & \xrightarrow{B} & HH_{n+1}(A) \\ \downarrow \pi & & \downarrow \pi \\ \Omega^n_{A/k} & \longrightarrow & \Omega^{n+1}_{A/k} \end{array}$$

$A^{\otimes n+1} \longrightarrow \Omega^n_{A/k}$

$(a \otimes b_1 \cdots \otimes b_n) \to a\, db_1\, db_2 \cdots db_n$

Thm: $HC(A/k) \cong \Omega^n_A / d\Omega^{n-1}_A \oplus H^{n-2}_{DR}(A) \oplus \cdots \oplus$ when A is smooth.

pf: By SS

By similar construction we could have

The $HP = HC^{per}$ $\quad HC^-$

$$HC^{per}_0(A) = \prod_{i \geq 0} H^{2i}_{DR}(A)$$
$$HC^{per}_1(A) = \prod_{i \geq 0} H^{2i+1}_{DR}(A)$$

§3. HH, HC and K theory

Def: A connection on A-module M is a K-linear map

$$\nabla: M\otimes_A \Omega^n_{A/k} \longrightarrow M\otimes_A \Omega^{n+1}_{A/k}$$

$$\nabla(x\omega) = \nabla x\, \omega + (-1)^n x\, d\omega$$

$x \in M\otimes\Omega^n_{A/k}$ $\quad \omega\in\Omega^p_{A/k}$

Note. $\nabla: M \longrightarrow M\otimes_A \Omega^1_{A/k}$

$\nabla(ma) = (\nabla m)a + m\,da \qquad m\in M \quad a\in A$

Prop. : $\nabla^2 : M\otimes_A \Omega^*_{A/k} \longrightarrow M\otimes_A \Omega^{*+2}_{A/k}$ is Ω-linear.

particularly $\nabla^2: M \longrightarrow M\otimes \Omega^2_{A/k}$ is A-linear.

As Suppose M is free over A φ - A-linear

Then. $\varphi: M \longrightarrow M\otimes_A \Omega^*_{A/k}$ is a matrix with coefficient in $\Omega^*_{A/k}$

Thus $\varphi \in \mathrm{End}_A(M)\otimes_A \Omega^*_{A/k}$,

Thus we could define a map: $\mathrm{End}_A(M)\otimes_A \Omega^*_{A/k} \xrightarrow{tr\otimes id} \Omega^*_{A/k}$.

One could check we have commutative diagram

$$\begin{array}{ccc}
\mathrm{End}_A(M)\otimes_A \Omega^*_{A/k} & \xrightarrow{[\nabla,-]} & \mathrm{End}_A(M)\otimes \Omega^{*+1}_{A/k} \\
\downarrow{\scriptstyle tr\otimes id} & & \downarrow{\scriptstyle tr\otimes id} \\
\Omega_{A/k} & \xrightarrow{d} & \Omega^{*+1}_{A/k}
\end{array}$$

Prop: the homogeneous component of degree $2n$ of $ch(M, \nabla) = tr(\operatorname{Exp}(R))$ is a cycle in $\Omega^{2n}_{A/k}$.

Note $\exp(R) = id + R + R^2/2 + \cdots \in \prod \operatorname{End}_A(M) \otimes \Omega^{2n}_{A/k}$.

Note. $d\, tr(R) = tr([\nabla, R]) = 0$

Thus $\frac{R^n}{n!}$ is a cohomology cycle in deRham cohomology.

Rmk: cohomology class $ch(M, \nabla)$ is independent of the choice of ∇

The existence of the connection is garanteed by Levi-Civita Connection.

Note if M is free of dimension 1. then exterior differential is a connection, more generally if M is free of dimension r.

$$M \otimes \Omega^*_A \cong (\Omega_A)^r \qquad (d \cdots d) : (\Omega^*_A)^r \longrightarrow (\Omega^{*+1}_A)^r$$

Note: for M f.g.p. Then M is identified with one of the idempotent in $M_r(A)$ $\quad M = \operatorname{Im} e \quad A^r = \operatorname{Im} e \oplus \operatorname{Im}(1-e)$

$$M \otimes_A \Omega^*_A \hookrightarrow A^r \otimes_A \Omega^*_A \xrightarrow{(d, \ldots, d)} A^r \otimes_A \Omega^{*+1}_A \xrightarrow{e \otimes id} M \otimes \Omega^{*+1}$$

Thus $ch(M, \nabla) = ch(\operatorname{Im} e, \nabla_e) = \frac{1}{n!} \text{ class } tr(e\,de\,de \cdots de) \in \Omega^{2n}_{A/k}$.

Thus By previous construction we have.

$$ch_0^{\cdot}: K_0(A) \longrightarrow H_{\cdot DR}^{ev}(A)$$

One could check. $ch(M) = ch(M')$ if $M \simeq M'$ are isomorphic

$$ch(M_1 \oplus M_2) = ch(M_1) + ch(M_2)$$

$$ch(M_1 \otimes M_2) = ch(M_1) \cdot ch(M_2)$$

We want to extend ch_0 To HC, H^{λ}, HH.

First we want to extend to H^{λ}.

Note let $e \in M_r(A)$

Consider $e^{\otimes n+1} \in C_n(R)$

Suppose we are in. $C_n^{\lambda}(R) = C_n(R)/(1-t)$

Thus $e^{\otimes n} = (-1)^{n-1} e^{\otimes n}$. so $e^{\otimes n} = 0$ if n is odd

and $e^{\otimes n+1}$ is a cycle In. $C_n^{\lambda}(R)$

Thus. $ch_{0,n}^{\lambda}: K_0(A) \longrightarrow H_{2n}^{\lambda}(A)$

$$[e] \longrightarrow tr((-1)^n e^{\otimes 2n+1})$$

$\Downarrow$

generalized trace map.

$$tr: M_r(M) \otimes M_r(A)^n \longrightarrow M \otimes A^n.$$

$$tr(\alpha \otimes \beta \otimes \cdots \gamma) = \sum \alpha_{i_0 i_1} \otimes \beta_{i_1 i_2} \otimes \cdots \otimes \gamma_{i_n i_0}$$

for all possible index $(i_0, \cdots i_n)$

Lemma: For any idempotent $e \in M(A)$ let.

$$y_i = (-1)^i \cdot \frac{(2i)!}{i!} e^{\otimes 2i+1} \in M(A)^{\otimes 2i+1}$$

$$z_i = (-1)^{i+1} \frac{(2i)!}{2(i!)} e^{\otimes 2i} \in M(A)^{\otimes 2i}$$

$C(e) = (y_n, z_n, y_{n-1} \cdots y_1) \in M(A)^{\otimes 2n+1} \cdots \oplus M(A)$

is an $2n$-cycle in $\mathrm{Tot}\, CC(M(A))$

Image of $c(e)$ in $\mathrm{Tot}\, BC(M(A))$ is $(y_n \cdots y_1)$

Thus we have a lift.

$$ch_{0,n} \quad K_0(A) \to HC_{2n}(A)$$

$$ch_{0,n}([e]) = tr(c(e))$$

$$S \circ ch_{0,n}([e]) = ch_{0,n-1}([e])$$

Prop. (Dennis Trace map)

Let. $G = GL_r(A)$ $\quad f : K[GL_r(A)] \longrightarrow M_r(A)$ $\quad f$ is fusion map extend. to K algebra

Then. $K[G^n] \longrightarrow K[G^{n+1}] \cong K[G]^{\otimes n+1} \xrightarrow{f^{\otimes n+1}} M_r(A)^{\otimes n+1} \xrightarrow{tr} A^{\otimes n+1}$

After applying H_n to the sequence we have a induced map

$$Dtr : H_n(GL_r(A), K) \longrightarrow HH_n(A)$$

Remark. similarly we have map ch^-, $ch_n^{per.}$ ch_{n-2i}

That. makes all map compatible.

$$\begin{tikzcd}
 & HC_n^-(A) \arrow[d, "h"] \\
H_n(GL(A)) \arrow[ur, "ch^-"] \arrow[r, "Der."'] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
 & {} \arrow[dl, "ch^-"'] \arrow[d, "ch_n^{per.}"] \arrow[dr, "ch_{n-2i}"] & & \\
HC_n^-(A) \arrow[r] & HC_n^{per.}(A) \arrow[r] & HC_{n-2i}(A) \arrow[r] & {}
\end{tikzcd}$$

The Der can be used to define ~~at~~ absolute chern character.

$$ch_n^- : K_n(A) \longrightarrow HC_n^-(A) \quad \text{defined as}$$

$$K_n(A) = \pi_n(BGL(A)^+) \longrightarrow H_n(BGL(A)^+) \cong H_n(BGL(A)) = H_n(GL(A))$$

Thus $ch_n^- : K_n(A) \longrightarrow H_n(GL(A)) \longrightarrow HC_n^-(A) \quad n \geq 1.$