§2. Constructing Projective Module.

Consider a commutative square of ring homomorphisms

$$\begin{tikzcd} \Lambda \arrow[r, "i_1"] \arrow[d, "i_2"'] & \Lambda_1 \arrow[d, "j_1"] \\ \Lambda_2 \arrow[r, "j_2"'] & \Lambda' \end{tikzcd}$$

Hypothesis 1 Λ is the product of Λ_1 and Λ_2. over Λ'. In other word given element $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$ with common image $j_1(\lambda_1) = j_2(\lambda_2)$?

$\exists! \lambda \in \Lambda$ st $i_1(\lambda) = \lambda_1$ and $i_2(\lambda) = \lambda_2$

Hypothesis 2. One of the map j_1 or j_2 is surjective.

This section is try to build module over Λ once we have P_1 over Λ_1 and P_2. over Λ_2

Note that if $f: R \to S$ is a ring homomorphism And M is a R-module. Then f induced a Left. S module. $S \otimes_{\Lambda} M$. is denoted by $f_\# M$.

$\exists$ a Canonical map $f_*: M \longrightarrow f_\# M$
$m \longrightarrow 1 \otimes m$ which is R-linear map

Basic Construction : Suppose a projective module. P_1 over Λ_1 projective module P_2 over Λ_2 and isomorphism. $h: j_1 \# P_1 \longrightarrow j_2 \# P_2$ over Λ'

Let $M = M(P_1 P_2, h)$ denoted as the subgroup of $P_1 \times P_2$ consisting of all pairs $(p_1 p_2)$ with $h j_{1*}(p_1) = j_{2*}(p_2)$

$$M(P_1, P_2, h) = \{ (p_1, p_2) \mid h\, j_{1*}(p_1) = j_{2*}(p_2) \} \leq P_1 \times P_2$$

$$\begin{tikzcd} M \arrow[r] \arrow[d] & P_1 \arrow[d, "j_{1*}"] \\ P_2 \arrow[r, "j_{2*}"] & j_{2\#} P_2 \end{tikzcd}$$

Analogous of Hypothesse 1 and 2.

M is a Λ-module. $\lambda \cdot (p_1, p_2) = (i_1(\lambda) p_1,\ i_2(\lambda) p_2)$

Thm. M is projective over Λ if P_1 and P_2 f.g then M f.g. ①

Thm Every projective Λ module is isomorphic to $M(P_1 P_2, h)$ for some ②
suitable. P_1 P_2 and h.

Thm. P_1 and P_2 are naturally isomorphic. To $i_{1\#}M$ and $i_{2\#}M$. ③

Lemma 1. Under condition. P_1 P_2 free then. $h = (a_{\alpha\beta})$ for some matrix, denote A, $h(j_{1*} x_\alpha) = \sum a_{\alpha\beta} \cdot j_{2*} y_\beta$
then. $M = M(P_1 P_2, h)$ is free if A is image under j_2 of
an invertible matrix over. Λ_2.

pf: $\{y_\alpha\}$ be a basis of P_2. Note $\{y_\alpha'\}$ $y_\alpha' = \sum c_{\alpha\beta} \cdot y_\beta$ is a new basis
$\{x_\alpha\}$ be a basis of P_1

Then. $Z_\alpha = (x_\alpha, y_\alpha')$ is a basis

Lemma 2.5. if P_1 and P_2 free. j_2 surjective then $M(P_1, P_2, h)$ projective

Let Q_1 free over Λ_1 with same rank as P_2

Let Q_2 --- Λ_2 with same rank as P_1

Then Let $g: j_1\# Q_1 \to j_2\# Q_2$ with corresponding matrix A^{-1}

Then. $M(P_1, P_2, h) \oplus M(Q_1, Q_2, g) \simeq M(P_1 \oplus Q_1, P_2 \oplus Q_2, h\oplus g) \Rightarrow$ free.

free due to Lemma 2.4. $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ $P_i \oplus Q_i$ free.

$$= \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A^{-1} & I \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

since j_2 is surjective so A can be lifted. and all is invertible matrix. $\square$

Logic of Proof Thm 2.1 (Assume P_1 P_2 projective) (j_2 surjective)

Lemma 2.6. $\exists Q_1$ projective over Λ_1 and Q_2 over. Λ_2. st $P_1 \oplus Q_1$ $P_2 \oplus Q_2$ free. $j_1\# Q_1 \simeq j_2 \# Q_2$.

choose Q_1 and Q_2.

$k: j_1\# Q_1 \to j_2\# Q_2$.

Then $M(P_1, P_2, h)$ is projective by Lemma 2.6

P_1, P_2 are f.g. then as the proof of 2.4. 2.5 All. construction preserve f.g

Thus Q_1, Q_2 can be chosen to be f.g. and $M(P_1, P_2, h)$ is the direct summand of f.g. free module. therefore, f.g.

§ Whitehead Group $K_1 A$

1. Def: $GL(R) = \bigcup_{i=1}^{\infty} GL(i, R)$ where $GL(i,R)$ is $i \times i$ general linear group with coefficient in R.

Note that $GL(n, R)$ is injected into $GL(n+1, R)$ by

$$A \longrightarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

Def: $n \times n$ matrix $\boxed{\text{elementary}}$ if it has 1's on its diagonal and at most one nonzero off-diagonal entry.

The subgroup of $GL(n,R)$ generated by elementary matrix is denoted as $E(n,R)$

$E(R) = \bigcup_{i=1}^{\infty} E(i,R)$

Example: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is elementary.

Rmk: $\forall A \in GL(n,R)$ $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ lies in $E(2n, R)$

As. $$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Definition: $GL(R)/E(R)$ is called the Whitehead group $K_1 A$ which is an abelian Group

Prop. (Whitehead's Lemma) For any Ring the Commutator subgroup of $GL(R)$ and of $E(R)$ coincide with $E(R)$ $\quad E(R) \simeq [GL(R), GL(R)]$

$$K_1(R) \simeq GL(R)/E(R) \simeq GL(R)_{ab}$$

pf: $E(R) \subseteq GL(R)$ $\quad [E(R), E(R)] \subseteq [GL(R), GL(R)]$

$$e_{ij}(a) = [e_{ik}(a), e_{kj}(1)] \quad i, j, k \text{ distinct.}$$

Thus each generator of $E(R)$ is commutator of two other generators $[E(R), E(R)] = E(R)$

We need to show $[GL(R), GL(R)] \subseteq E(R)$ $\quad$ Let $A, B \in GL(n, R)$

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}$$

By Rmk: $ABAB^{-1} \in E(R)$

Def: Product $[A] \cdot [B] = [AB]$

Define $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

Thus $[A \oplus B] = [AB \oplus 1] = [AB]$

① Example: Let $R = R_1 \times R_2$. Then. $GL(R) \cong GL(R_1) \times GL(R_2)$

Thus $K_1(R) \simeq K_1(R) \times K_2(R)$

② $\exists$ Trivial. K_1.

Consider $R = End_k(V)$ V is infinite dimensional. Then $K_1(R) = 1$

③ (Morita. Invariance for. K_1)

$K_1(M_n(R)) \cong K_1(R)$ (In analogy to K_0 case)

2. K_1 of Special Rings

2.1. Division Rings and local Rings

Prop. if R. is a Commutative Ring. and. $R^\times = GL(1,R)$ is the group of unit. the determinant. $\det : GL(n,R) \longrightarrow R^\times$ extends to a split surjection $GL(R) \twoheadrightarrow R^\times$ and thus gives a split surjection $K_1(.R) \to R^\times$

Rmk splits defined by $\underset{GL(1,R)}{\overset{R^\times}{\shortparallel}} \longrightarrow GL(R)$ $\boxtimes$

Thus. whe have. $GL(R) = SL(R) \rtimes R^\times$.

And. $K_1(R) = R^\times \oplus SK_1(R)$ $SK_1(R) =$ Kernel. of map. $K_1^* \to R^\times$

(Euclidean domain)

Prop. When. R is a field. then. SK_1 is trivial Thus $K_1(R) \simeq R^\times_{ab}$

(Local ring)

Prop. When R is a Division Ring (local) $R^\times = GL.(1,R) \hookrightarrow GL(R)$ induce surjection $R^\times_{ab} \twoheadrightarrow K_1(R)$

Exercise: Compute $K_1(\mathbb{Z}/m)$

Example: $K_1(\mathbb{Z}) \simeq \{\pm 1\}$ $K_1(\mathbb{Z}[i]) \simeq \{\pm 1, \pm i\}$. $K_1(K[t]) \simeq K^*$

Exact Sequence

Thm. if there is a commutative diagram satisfy and hypothesis 1 & 2.

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \Lambda_2 \arrow[d] \\ \Lambda_1 \arrow[r] & \Lambda' \end{tikzcd}$$

Then. $\exists$ ES

$$K_1\Lambda \to K_1\Lambda_1 \oplus K_1\Lambda_2 \to K_1\Lambda' \xrightarrow{\partial} K_0\Lambda \to K_0\Lambda_1 \oplus K_0\Lambda_2 \to K_0\Lambda'$$

$$x \longrightarrow (i_{1*}x,\ i_{2*}x)$$

$$(y, z) \longrightarrow j_{1*}y - j_{2*}z$$

$$\partial(x) = [M[\Lambda_1^n, \Lambda_2^n, h]] - [\Lambda^n] \in K_0\Lambda.$$

Example. Consider.

$$\begin{tikzcd} \mathbb{Z}\Pi \arrow[r, "i_1"] \arrow[d, "i_2"] & \mathbb{Z}[\zeta] \arrow[d, "j_1"] \\ \mathbb{Z} \arrow[r, "j_2"] & \mathbb{F}_p \end{tikzcd}$$

$\zeta = e^{2\pi i/p}$

$\mathbb{Z}\Pi = \{\sum_{g \in \mathbb{F}_p} \mathbb{Z}g\}$

$i_1(t) = \zeta$ $j_1(\zeta) = 1$ $i_2(t) = 1$ $i_2(t) = 1$

WTS. $i_{1*}: K_0\mathbb{Z}\Pi \to K_0\mathbb{Z}[\zeta]$ is an isomorphism

Note we have exact sequence

$$K_1\mathbb{Z}[\zeta] \oplus K_1\mathbb{Z} \to K_1\mathbb{F}_p \xrightarrow{\partial} K_0\mathbb{Z}\Pi \to K_0\mathbb{Z}[\zeta] \oplus K_0\mathbb{Z} \to K_0\mathbb{F}_p$$