

§2. Constructing Projective Module.

Consider a commutative square of ring homomorphisms

$$\begin{array}{ccc} \Lambda & \xrightarrow{i_1} & \Lambda_1 \\ i_2 \downarrow & & \downarrow j_1 \\ \Lambda_2 & \xrightarrow{j_2} & \Lambda' \end{array}$$

Hypothesis 1. Λ is the product of Λ_1 and Λ_2 over Λ' . In other words given element $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$ with common image $j_1(\lambda_1) = j_2(\lambda_2)$
 $\exists! \lambda \in \Lambda$ st $i_1(\lambda) = \lambda_1$ and $i_2(\lambda) = \lambda_2$

Hypothesis 2. One of the map j_1 or j_2 is surjective.

This section is try to build module over Λ once we have P_1 over Λ_1 and P_2 over Λ_2

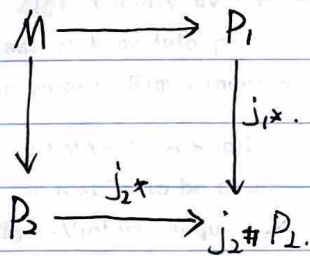
Note that if $f: R \rightarrow S$ is a ring homomorphism and M is a R -module. Then f induced a left S module $S \otimes_R M$ is denoted by $f_{\#} M$.

\exists a canonical map $f_{\#}: M \rightarrow f_{\#} M$
 $m \rightarrow 1 \otimes m$ which is R -linear map

Basic Construction: Suppose a projective module P_1 over Λ_1 projective module P_2 over Λ_2 and isomorphism $h: j_1 \# P_1 \rightarrow j_2 \# P_2$ over Λ'

Let $M = M(P_1, P_2, h)$ defined as the subgroup of $P_1 \times P_2$ consisting of all pairs (p_1, p_2) with $h(j_1 \# p_1) = j_2 \# p_2$

$$M(P_1, P_2, h) = \{ (P_1, P_2) \mid h(j_{1*}(P_1) = j_{2*}(P_2)) \} \leq P_1 \times P_2$$



Analogous of Hypotheses 1 and 2.

M is a Λ -module. $\lambda(P_1, P_2) = (i_1(\lambda)P_1, i_2(\lambda)P_2)$

Thm. M is projective over Λ if P_1 and P_2 f.g. then M f.g. ①

Thm. Every projective Λ -module is isomorphic to $M(P_1, P_2, h)$ for some suitable P_1, P_2 and h . ②

Thm. P_1 and P_2 are naturally isomorphic to $i_{1*}M$ and $i_{2*}M$. ③

Lemma 1. Under condition P_1, P_2 free then $h = (A_{\alpha\beta})$ for some matrix A then $M = M(P_1, P_2, h)$ is free if A is image under j_{2*} of an invertible matrix over Λ_2 .

pf: $\{y_\alpha\}$ be a basis of P_2 . Note $\{y'_\alpha\}$ $y'_\alpha = \sum C_{\alpha\beta} y_\beta$ is a new basis $\{x_\alpha\}$ be a basis of P_1

Then $Z_\alpha = (x_\alpha, y'_\alpha)$ is a basis

Lemma 2.5. if P_1 and P_2 free. j_2 surjective. then $M(P_1, P_2, h)$ projective

Let Q_1 free over Λ_1 with same rank as P_2

Let Q_2 free over Λ_2 with same rank as P_1

Then let $g: j_1 \# Q_1 \rightarrow j_2 \# Q_2$ with corresponding matrix A^{-1}

Then $M(P_1, P_2, h) \oplus M(Q_1, Q_2, g) \simeq M(P_1 \oplus Q_1, P_2 \oplus Q_2, h \oplus g) \Rightarrow$ free.

free due to Lemma 2.4. $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ $P_i \oplus Q_i$ free.

$$= \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A^{-1} & I \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

Since j_2 is surjective so A can be lifted. and all is invertible. matrix. \square

Logic of Proof Thm 2.1 (Assume P_1, P_2 projective) (j_2 surjective)

Lemma 2.6. $\exists Q_1$ projective over Λ_1 and Q_2 over Λ_2 so $P_1 \oplus Q_1$
 $P_2 \oplus Q_2$ free. $j_1 \# Q_1 \simeq j_2 \# Q_2$.

choose Q_1 and Q_2 .

$$h: j_1 \# Q_1 \rightarrow j_2 \# Q_2$$

Then $M(P_1, P_2, h)$ is projective by Lemma 2.6

P_1, P_2 are f.g. then as the proof of 2.4, 2.5 All. Construct preserve f.g.

Thus Q_1, Q_2 can be chosen to be f.g. and MCP, P, h is the direct sum of f.g. free module. therefore f.g.

$$\begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix} =$$

§. Whitehead. Group K_1A

1. Def: $GL(R) = \bigcup_{i=1}^{\infty} GL(i, R)$ where $GL(i, R)$ is $i \times i$ general linear group with coefficients in R .

Note that $GL(n, R)$ is injected into $GL(2n, R)$ by

$$A \longrightarrow \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

Def: $n \times n$ matrix **elementary** if it has 1's on its diagonal and at most one nonzero off-diagonal entry.

The subgroup of $GL(n, R)$ generated by elementary matrix is denoted as $E(n, R)$

$$E(R) = \bigcup_{i=1}^{\infty} E(i, R)$$

Example: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is elementary.

Rmk: $\forall A \in GL(n, R)$ $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ lies in $E(2n, R)$

$$\text{As } \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A^{-1} & I \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

Definition: $GL(R)/E(R)$ is called the Whitehead group $K_1(R)$ which is an abelian group

Prop. (Whitehead's Lemma) For any Ring the commutator subgroup of $GL(R)$ and of $E(R)$ coincide with $E(R)$

$$E(R) \cong [GL(R), GL(R)]$$

$$K_1(R) \cong GL(R)/E(R) \cong GL(R)_{ab} \longleftarrow A$$

pf: $E(R) \subseteq [GL(R), GL(R)]$ $[E(R), E(R)] \subseteq [GL(R), GL(R)]$

$$e_{ij}(a) = [e_{ik}(a), e_{kj}(1)] \quad i, j, k \text{ distinct.}$$

Thus each generator of $E(R)$ is commutator of two other generators $[E(R), E(R)] = E(R)$

We need to show: $[GL(R), GL(R)] \subseteq E(R)$ Let $A, B \in GL(n, R)$

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & B^{-1}A^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}$$

By Amk: $ABAB^{-1} \in E(R)$

Def: Product $[A] \cdot [B] = [AB]$

Define $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

Thus $[A \oplus B] = [AB \oplus I] = [AB]$

① Example: Let $R = R_1 \times R_2$. Then $GL(R) \cong GL(R_1) \times GL(R_2)$

Thus $K_1(R) \cong K_1(R_1) \times K_1(R_2)$

② \exists Trivial K_1 .

Consider $R = \text{End}_k(V)$ V is infinite dimensional. Then $K_1(R) = 1$

③ (Morita Invariance for K_1)

$K_1(M_n(R)) \cong K_1(R)$ (In analogy to K_0 case)

2. K_1 of Special Rings

2.1. Division Rings and local Rings

Prop. if R is a commutative Ring and $R^\times = GL(1, R)$ is the group of units the determinant $\det: GL(n, R) \rightarrow R^\times$ extends to a split surjection $GL(R) \rightarrow R^\times$ and thus gives a split surjection $K_1(R) \rightarrow R^\times$

Rank splits defined by
$$\begin{array}{ccc} R^\times & \longrightarrow & GL(R) \\ \parallel & & \square \\ GL(1, R) & & \end{array}$$

Thus we have $GL(R) = SL(R) \times R^\times$

And $K_1(R) = R^\times \oplus SK_1(R)$ $SK_1(R) = \text{Kernel of map } K_1^* \rightarrow R^\times$
(Euler domain)

Prop. When R is a field then SK_1 is trivial Thus $K_1(R) \cong R^\times$
(Local Ring)

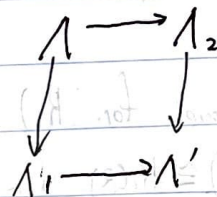
Prop. When R is a Division Ring $R^\times = GL(1, R) \hookrightarrow GL(R)$ induce surjection $R^\times \twoheadrightarrow K_1(R)$ (Local)

Exercise: Compute $K_1(\mathbb{Z}/m)$

Ex: $K_1(\mathbb{Z}) \cong \{\pm 1\}$ $K_1(\mathbb{Z}[i]) \cong \{\pm 1, \pm i\}$ $K_1(K[t]) \cong K^*$

Exact Sequence

Thm. if there is a commutative diagram satisfy
and hypothesis 1 & 2.



Then \exists ES

$$K_1 \Lambda \rightarrow K_1 \Lambda_1 \oplus K_1 \Lambda_2 \rightarrow K_1 \Lambda'_1 \xrightarrow{\partial} K_0 \Lambda \rightarrow K_0 \Lambda_1 \oplus K_0 \Lambda_2 \rightarrow K_0 \Lambda'$$

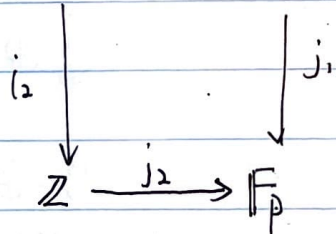
$$X \rightarrow (i_{1*} X, i_{2*} X)$$

$$(y, z) \rightarrow j_{1*} y - j_{2*} z$$

$$\partial(x) = [M[\Lambda_1^n, \Lambda_2^n, h]] - [\Lambda^n] \in K_0 \Lambda$$

Example. Consider $\mathbb{Z} \xrightarrow{i_1} \mathbb{Z}[\gamma]$

$$\gamma = e^{2\pi i/p}$$



$$\mathbb{Z}[\gamma] = \left\{ \sum_{g \in \mathbb{F}_p} z_g \right\}$$

$$i_1(\pm 1) = \gamma$$

$$j_1(\gamma) = 1$$

$$i_2(\pm 1) = 1$$

$$j_2(\gamma) = 1$$

WTS. $i_{1*}: K_0 \mathbb{Z}[\gamma] \rightarrow K_0 \mathbb{Z}[\gamma]$ is an isomorphism

Note we have exact sequence

$$K_1 \mathbb{Z}[\gamma] \oplus K_1 \mathbb{Z} \xrightarrow{\partial} K_0 \mathbb{Z}[\gamma] \rightarrow K_0 \mathbb{Z}[\gamma] \oplus K_0 \mathbb{Z} \rightarrow K_0 \mathbb{F}_p$$