Lecture 3. 9/22.

1. Definition of K_2

For a Ring Λ, $GL(n,\Lambda) = \{\text{invertible } n\times n \text{ matrix over } \Lambda\}$

$E(n,\Lambda) = \langle e_{ij}^{\lambda} \rangle$

(i,j) entry

$$e_{ij}^{\lambda} = \begin{pmatrix} 1 & & \lambda & \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

$$GL(\Lambda) = \bigcup_n GL(n,\lambda) \qquad E(\Lambda) = \bigcup_n E(n,\lambda)$$

$$E(\Lambda) = [GL(\Lambda), GL(\Lambda)] = [E(\Lambda), E(\Lambda)] \qquad K_1(\Lambda) = GL(\Lambda)/E(\Lambda)$$

Rmk: $E(n,\Lambda)$ satisfying

- $e_{ij}^{\lambda} e_{ij}^{\mu} = e_{ij}^{\lambda+\mu}$ ①
- $[e_{ij}^{\lambda}\ e_{k\ell}^{\mu}] = \begin{cases} 1 & j\neq k,\ \ell\neq i \quad ② \\ e_{i\ell}^{\lambda+\mu} & j=k,\ i\neq \ell \quad ③ \\ e^{-\lambda+\mu} & j\neq k\ i=\ell \quad ④ \end{cases}$

As stated in (Rosenberg. K. theory) Relation ④ can be derived from previous 3 relations

Definition (Steinberg Group)

For $n\geq 3$. $St(n,\Lambda)$ is the group generated by x_{ij}^{λ} ($i\neq j$ $\lambda\in\Lambda$)

and Relations

1) $x_{ij}^{\lambda} x_{ij}^{\mu} = x_{ij}^{\lambda+\mu}$

2) $[x_{ij}^{\lambda}, x_{j\ell}^{\mu}] = x_{i\ell}^{\lambda+\mu}$

3) $[x_{ij}^{\lambda}, x_{k\ell}^{\mu}] = 1$ $j\neq k$ $i\neq\ell$

$$St(\Lambda) = \bigcup_n St(n,\Lambda)$$

$\exists\ \phi: St(\Lambda) \longrightarrow E(\Lambda)$ $\qquad x_{ij}^{\lambda} \longmapsto e_{ij}^{\lambda}$

$\boxed{\text{Def}}$ $K_2(\Lambda) = \operatorname{Ker} \phi$ $\qquad \phi: St(\Lambda) \longrightarrow E(\Lambda)$

Thm 1 $\quad \operatorname{Ker} \phi \subseteq St(\Lambda)$ is the center of $St(\Lambda)$ denote $Z(St(R))$

pf: Let $y \in Z(St(R))$ $\qquad \phi(y) \in Z(E(\Lambda)) = 1$. $\qquad$ Thus $Z(St(R)) \subseteq \operatorname{Ker}\phi$

$y \in \operatorname{Ker} \phi$ $\qquad$ assume. $y \in St(n-1, \lambda)$

Consider $P_n = \langle X_{1n}^{\lambda}, \ldots X_{n-1\,n}^{\lambda} \rangle \subset St(n, \lambda)$

Note: $\phi|_{P_n}$ is injection

Also $y^{-1} P_n y \subseteq P_n$.

Thus y commutes with X_{ij}^{λ} $\quad i \neq j$ $\quad$ Thus $y \in Z(St(R))$

See Proof: Rosenberg Thm 4.2.4

- Universal Central Extensions.

Def: (Central Extension) For a group G, $\phi : X \longrightarrow G$ is a central extension of G if ϕ is surjective and $\operatorname{Ker}\phi \subseteq Z(X)$

Def: (Universal Central Extension)

$\theta : Y \longrightarrow G$ is a universal central extension if

$$\begin{tikzcd} X \arrow[dr, "\varphi"] & \\ Y \arrow[u, "\exists!\, h"] \arrow[r, "\theta"'] & G \end{tikzcd}$$

for X to be central.

Ex. 1. Trivial Extension, A abelian

2. Observation: if G has a universal central extension G is perfect.

Then universal central extension exist denote E, E is perfect

3). $A_5 \cong PSL(2, \mathbb{F}_5)$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow SL(2, \mathbb{F}_5) \longrightarrow PSL(2, \mathbb{F}_5) \quad \boxed{\text{Universal}}$$

(4) $\tilde{G} \xrightarrow{\phi} G$. Hausdorff connected topological groups

and $\operatorname{Ker}\phi$ is discrete.

$0 \to \operatorname{Ker}\phi \to \tilde{G} \twoheadrightarrow G \to 0$ is central extension.

EX:

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow SU(2) \longrightarrow SO(3) \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \widetilde{SL}(n,\mathbb{R}) \longrightarrow SL(n,\mathbb{R}) \longrightarrow 0$$

perfect group.

Thm 2. (Detection Thm)

A group G has a universal central extension $\Leftrightarrow$ G is perfect $G = [G, G]$

And Central Extension $U \xrightarrow{\phi} G$ is universal

$\Leftrightarrow$ (i). U is perfect

(ii) all central extension over U is trivial

Thm 3 if $0 \to A \to E \to G \to 0$ is universal central extension

$A \cong H_2(G, \mathbb{Z})$

Thm 4 $0 \longrightarrow K_2(\Lambda) \longrightarrow St(\Lambda) \longrightarrow E(\Lambda) \longrightarrow 0$ is universal

$K_2(\Lambda) \cong H_2(E(\Lambda))$

Rmk: (1) $E \xrightarrow{\phi} G$ is universal

$\Leftrightarrow H_1(E) = H_2(E) = 0$

(2) $E \xrightarrow{\phi} G$ E is perfect. ϕ : central extension.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\phi \arrow[r] & E \arrow[r] & G \arrow[r] & 0 \\
0 \arrow[r] & H_2 G \arrow[r] \arrow[u] & U \arrow[r] \arrow[u] & G \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

(3) $SL(n, \mathbb{F}_q)$ is perfect except $SL(n, \mathbb{F}_2)$ $SL(n, \mathbb{F}_3)$

$Z(SL(n, \mathbb{F}_q)) = \mu_n(\mathbb{F}_q) = \{x \in \mathbb{F}_q, \ x^n = 1\}$

$PSL(n, \mathbb{F}_q) = SL(n, \mathbb{F}_q) / \mu_n(\mathbb{F}_q)$

$0 \longrightarrow \mu_n(\mathbb{F}_q) \to SL(n, \mathbb{F}_q) \longrightarrow PSL(n, \mathbb{F}_q) \longrightarrow 0 \Rightarrow$

is universal except for finite (n, q)

$\Rightarrow H_2(SL(n, \mathbb{F}_q)) = 0$ except finite $(n, q) \Rightarrow K_2(\mathbb{F}_q) = 0$

Property of K_2.

(1) K_2 is functorial

(2) $K_2(\operatorname{colim} \Lambda) = \operatorname{colim} K_2(\Lambda)$

(3) $K_2(\Lambda)$ is a $K_0\Lambda$-module

(4). $K_2(\Lambda) \triangleq K_2(M_n(\Lambda))$