9/29.

Lecture 4.

Let R be a Commutative Ring.

$\varphi: St(R) \longrightarrow E(R)$

Let $X, Y \in E(R)$

$\exists\ x, y$ st. $\varphi(x) = X$ $\varphi(y) = Y$.

$[X, Y]$ $a, b \in Ker\ \varphi$.

$[Xa, Yb] = [X, Y]$

$\varphi:([x, y]) = [\varphi(x), \varphi(y)] = [X, Y] = 1$

$[x, y] \in Ker\ \varphi = K_2 R$

Let $u, v \in R^\times$

$X = \begin{pmatrix} u & & 0 \\ & u^{-1} & \\ 0 & & 1 \end{pmatrix}$ $Y = \begin{pmatrix} V & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & V^{-1} \end{pmatrix}$

Def: $\forall\ u, v \in R^\times$, the Steinberg symbol of u & v is the element

$\{u, v\} = [x, y] \in K_2(R)$

Lemma: Properties of Steinberg symbol.

$u, v, w \in R^\times$

(1). $\{u, v\} = \{v, u\}^{-1}$

(2) $\{uv, w\} = \{u, w\}\{v, w\}$

(3) $\{u, vw\} = \{u, v\}\{u, w\}$

(4) $\{u, -u\} = -1$.

(5) $\{u, 1-u\} = 1$. when $u \in R^\times$

Example. $R = \mathbb{Z}$. $R^\times = \{\pm 1\}$

$\{1, 1\}, \{1, -1\}, \{-1, 1\} = 1$

$1 = \{-1, 1\} = \{-1, (-1)(-1)\} = \{-1, -1\} \cdot \{-1, -1\}$

$\mathrm{Ord}(\{-1, -1\}) \le 2$.

$K_2 \mathbb{Z} \cong \mathbb{Z}/2$

Fact: $\varphi(X_{12}(1) X_{21}(-1) X_{12}(1)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in E(R)$

$\Downarrow$

$\varphi(\qquad)^4 = 1$

$\underbrace{(X_{12}(1) X_{21}(-1) X_{12}(1))^4}_{\{-1,-1\}} \in \mathrm{Ker}\,\varphi = K_2 R$

The subgroup W & H.

R ring. $a, b, c, d \in R$. $i \ne j \in \mathbb{Z}^+$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{(i,j)}$: obtained from $Id = I \in GL(R)$ by replacing

(i, i) coordinate by a

(i, j) by b

(j, i) by c

(j, j) by d

$\varphi(X_{ij}(a)) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{(ij)} = e_{ij}(a)$

$\exists$ invertible $P \in GL(R)$ depending only on i, j st.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{(i,j)} = P \begin{bmatrix} a & b \\ c & d \end{bmatrix} P^{-1}$$

$$\Rightarrow A^{(ij)} B^{(ij)} = (AB)^{ij}$$

For $i \neq j \in \mathbb{Z}^+$, $a, b \in R^\times$

$$W_{ij}(a) = X_{ij}(a) X_{ji}(-a^{-1}) X_{ij}(a)$$

$$h_{ij}(a) = W_{ij}(a) \, W_{ij}(-1)$$

$$\varphi(w_{ij}(a)) = \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}^{ij}$$

$$\varphi(h_{ij}(a)) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{ij}$$

Left multiplication by $\varphi(X_{ij}(a))$: $\begin{pmatrix} \text{ith row} \to \text{ith row} \\ + a(\text{jth row}) \end{pmatrix}$

$$\varphi(w_{ij}(1)) = \begin{pmatrix} \text{ith row} \to (-1)\, i \text{ row} \\ i \text{ row} \to j \text{ row} \end{pmatrix}$$

$$\varphi(h_{ij}(a)) = \begin{pmatrix} \text{ith row} \to a \text{ ith row} \\ \text{jth row} \to a^{-1} \text{ th row} \end{pmatrix}$$

$W = \langle w_{ij}(a) \rangle \leq St(R)$

$H =: \langle h_{ij}(a) \rangle \trianglelefteq W \leq St(R)$

$\left(\begin{array}{l} P_n(R) \subseteq Gl_n(R) \quad \text{permutation matrix} \\ D_n(R) \subseteq Gl_n(R) \quad \text{diagonal matrix} \end{array} \right.$

$\underbrace{P_n(R) \cdot D_n(R)}_{\Downarrow} \leq GL_n(R)$ $\qquad P_n \hookrightarrow P_{n+1}$

monomial matrix $\qquad D_n \hookrightarrow D_{n+1}$

$P(R) \cap D(R) = \{I\}$

$\forall M \in P(R)D(R) \quad \exists!$ decomposition

$M = AB$, $A \in P(R)$, $B \in D(R)$

$P(R) \ni A = \sigma \in S_\infty = \bigcup_n S_n$.

underlying permutation of M.

$\forall w \in W$

$\varphi(w)$ is monomial

$\varphi(W) \subseteq P(R)D(R) \cap E(R)$

Def. : $\varphi : W \longrightarrow S_\infty$

$w \longmapsto \sigma$

where σ is the underlying permutation of $\varphi(w)$

ψ is surjective as $\varphi.(w_{ij}(1))$ is the transposition (i,j)

$$\varphi.(h_{ij}(a)) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{(i,j)}$$

$$\varphi(H) \subseteq D(R) \cap E(R)$$

Thus $H \subset \operatorname{Ker}\psi$

Prop: $H = \operatorname{Ker}\psi$

Cor: $K_2(R) \cap W = K_2(R) \cap H$.

Cor. $S_\infty = W/H$

$w \in W$

$\varphi(w) = 1 \in E(R) \Rightarrow w \in \operatorname{Ker}\psi = H$

Rmk: $n \in \mathbb{N}$ $(i,j \leq n)$

$H_n \subseteq W_n \leq St(R)$

$n \geq 3$. $W_n/H_n \cong S_n$.

if R is a field: W_n/H_n = Weyl group associated $SL_n(R)$

SYM: subgroup generated by Steinberg symbol $\{u,v\}$

Thm: $R^\times$ abelian $\Rightarrow W \cap K_2(R) = $ Sym.

$K_2(R) = $ SYM iff $K_2(R) \subset W$

$\mathbb{Z}[i]=R.$

$R^{\times}=\{\pm 1,\pm i\}=\langle i\rangle$

$\{i,i\}^3=\{i,i^3\}=\{i,-i\}=1$

$\{i,i\}^4=\{i,i^4\}=\{i,1\}=1.$

$\{i,i\}=1$ $\qquad$ $K_2(R)=1$