Lecture 5: 10/6.

§4. $K_1(a) \longrightarrow K_1(\Lambda) \longrightarrow K_1(\Lambda/a) \longrightarrow K_0(a) \longrightarrow K_0(\Lambda) \longrightarrow K_0(\Lambda/a)$

Def:
$$\begin{tikzcd}
\Lambda \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& D \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \Lambda \arrow[d] \\
& \Lambda \arrow[r] & \Lambda/a
\end{tikzcd}$$

D is called the double of Λ along a.

Rmk: By Functoriality of K_0 & K_1 the maps $K_i(D) \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} K_i(\Lambda)$ are split epis

$$0 \longrightarrow K_i(a) \longrightarrow K_i(D) \longrightarrow K_i(\Lambda) \longrightarrow 0 \quad \text{is split exact.}$$

Pluging into the MV Sequence in §3

$$K_1(a) \oplus K_1(\Lambda) \longrightarrow K_1(\Lambda) \oplus K_1(\Lambda) \longrightarrow K_1(\Lambda/a) \oplus 0 \longrightarrow K_0(a) \oplus K_0(\Lambda)$$
$$\longrightarrow K_0(a) \oplus K_0(\Lambda) \longrightarrow K_0(\Lambda) \oplus K_0(\Lambda) \longrightarrow K_0(\Lambda) \oplus 0$$

A more familiar definition of $K_1(a)$

$$\begin{tikzcd}
1 \arrow[r] & E(a) \arrow[r] \arrow[d] & GL(a) \arrow[r] \arrow[d] & K_1(a) \arrow[d] & \\
1 \arrow[r] & E(D) \arrow[r] \arrow[d] & GL(D) \arrow[r] \arrow[d] & K_1(D) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & E(\Lambda) \arrow[r] \arrow[d] & GL(\Lambda) \arrow[r] \arrow[d] & K_1(\Lambda) \arrow[r] \arrow[d] & 1 \\
& 1 & 1 & 1 &
\end{tikzcd}$$

$$
\begin{aligned}
GL(a) &= \operatorname{Ker}(GL(D) \longrightarrow GL(\Lambda)) = \operatorname{Ker}(\varinjlim GL_n(D) \to \varinjlim GL(\Lambda)) \\
&= \varinjlim \operatorname{Ker}(GL_n(D) \to G_n(\Lambda)) \\
&= \varinjlim \operatorname{Ker}(GL_n(\Lambda) \to GL(\Lambda/a)) \\
&= (I_n + M_n(a)) \cap GL_n(\Lambda)
\end{aligned}
$$

$$
\begin{tikzcd}
\text{Rings} \arrow[r, "GL_n"] \arrow[d, "M_n(-)"'] & \text{Gp.} \\
\text{Ring.} \arrow[ur, "(-)^\times"'] &
\end{tikzcd}
$$