Higher Algebraic K : First course

I. Quillen's "+"-construction.

<u>Def 1.1.</u> X CW cpx. X is called acyclic if it has the homology of a pt. i.e. $\tilde{H}_*(X) = 0$. Unless otherwise stated, we'll always assume that H_*, H^* takes coeff. in $\mathbb{Z}$.

<u>Lem 1.1.</u> Suppose X is acyclic. Then

1) X is connected.

2) $\pi_1 X$ is perfect (i.e. $(\pi_1 X)^{ab} = \{1\}$).

3) $H_2(\pi_1 X, \mathbb{Z}) = 0$.

▶ Sketch of proof. 1), 2): trivial by definition.

3) Consider the Postnikov filtration of X: Write $G = \pi_1 X$.

$$\begin{tikzcd} \vdots \arrow[d] & \\ X_2 \arrow[d] & \\ X_1 = K(\pi_1 X, 1) & X \arrow[l, "f"'] \arrow[ul] \end{tikzcd}$$

Claim $K(\pi_1 X, 1) \cong B\pi_1 X$. This is b/c considering universal G-principle bundle $G \to EG \to BG$ and associated l.e.s. in htpy gps gives $\pi_{*+1} BG \cong \pi_* G$, and the result follows from Whitehead Thm.

Now consider the htpy fiber seq $Ff \to X \xrightarrow{f} K(\pi_1 X, 1)$.

Use Serre spectral sequence $E^2_{p,q} = H_p(BG, H_q(Ff, \mathbb{Z})) \Rightarrow H_{p+q}(X, \mathbb{Z})$

Finally check that $Ff \simeq \tilde{X}$, universal covering of X, and $E^2_{p,q} \cong H_p(G, H_q(\tilde{X}, \mathbb{Z}))$ fits into the s.e.s.

$$H_2(X, \mathbb{Z}) \to H_2(G, H_0(\tilde{X}, \mathbb{Z})) \to H_0(G, H_1(\tilde{X}, \mathbb{Z})) \to H_1(X, \mathbb{Z})$$
$$\to H_1(G, H_0(\tilde{X}, \mathbb{Z})).$$

Def 1.2 $f: X \to Y$ is acyclic if Ff is acyclic, where Ff is the homotopy fiber of f. This implies Ff connected & $\pi_1 Ff$ perfect.

Cor 1.2 $\ker f_* \trianglelefteq \pi_1 X$ is perfect.

Prp 1.3 1) $f: X \to Y$ is acyclic and $\pi_1 f$ iso $\Leftrightarrow$ $\pi_i f$ iso. $\forall i \geq 0$.

2) $f: X \to Y$ acyclic iff $f_*: H_*(X, f^*\mathcal{L}) \to H_*(Y, \mathcal{L})$ iso for any local system $\mathcal{L}$. (locally constant sheaf).

Rk 1.1 1) is easy.

2) needs Comparison Theorem in spectral sequence and involves a lot of technical details.

Now we're ready to define the "+" construction.

Def 1.3 Let $P \trianglelefteq \pi_1 X$ be perfect. X CW cpx, based. An acyclic map $f: X \to Y$ is called a "+" construction on X rel P if $P = \ker f_*$, where $f_*: \pi_1 X \to \pi_1 Y$.

e.g. 1.1 X acyclic, $f: X \to pt$ acyclic. It's a "+" construction on X rel $\pi_1 X$.

Thm 1.4 (Quillen)

Let $P \trianglelefteq \pi_1 X$ be perfect. Then a "+" construction on X rel. P, namely $f: X \to Y$, exists, and it is unique up to htpy

Def 1.4 The perfect radical of G, denoted $P(G)$, is the unique largest perfect normal subgp of G. (Need Zorn's lem to prove this).

$X^+ :=$ "+" construction on X rel $P(\pi_1 X)$.

Let R = associative, unital ring. $0_R \neq 1_R$. $BGL(R)$ is the classifying space of $GL(R)$. $GL(R) = \mathrm{colim}_n GL_n(R)$. $GL_n R \hookrightarrow GL_{n+1} R$
$A \longmapsto \begin{bmatrix} A & \\ & 1 \end{bmatrix}$

<u>Def 1.5</u> The higher algebraic K-theory $K_n(R) = \pi_n(BGL(R)^+)$, $\forall\, n \geq 1$.
Write $K(R) = K_0(R) \times BGL(R)^+ = \coprod_{K_0(R)} BGL(R)^+$. An alternative definition of higher K is $K_n(R) = \pi_n K(R)$. $\forall\, n \geq 0$.

<u>Cor 1.5</u> $K_1(R) = \pi_1(BGL(R)^+) = \pi_1(BGL(R)) / P(\pi_1(BGL(R)))$
$= GL(R) / [GL(R), GL(R)]$

<u>Prop 1.6</u> If $f: BG \longrightarrow BG^+$ is a "+" construction, Ff = htpy fiber of f, then $\pi_1 F(f)$ is the universal central extension of $P(\pi_1 BG)$ and $\pi_2 BG^+ \cong H_2(P, \mathbb{Z})$. $P = P(\pi_1 BG)$.

<u>Cor 1.7</u> $K_2 R = \pi_2(BGL(R)^+) \cong H_2(E(R); \mathbb{Z})$ = classical K_2.

We postpone the proof of Prop 1.6 after introducing the construction of "+" construction.

<u>Prop 1.8</u> $K: \mathrm{Ring} \longrightarrow \mathrm{ho}(\mathrm{Top})$ are functorial.
$K_n: \mathrm{Ring} \longrightarrow \mathrm{Ab}$

▲ Construction of "+" construction:

Step 1: $P = \pi_1 X$.

$\pi_1 X = P = [S^1, X]$. $\forall\, \varphi_\alpha: S^1 \to X \in \pi_1 X$, one can attach e^2_α (2-cell) to kill $[\varphi_\alpha]$. Now suppose we killed all $[\varphi_\alpha]$, and get X' simply connected, then from the relative homology

$$0 \to H_2 X \to H_2 X' \to H_2(X', X) \to H_1 X$$
$$\hphantom{0 \to H_2 X \to H_2 X' \to H_2(X', X) \to{}} \overset{\|}{0}$$

$H_1 X = 0$ since $\pi_1 X = P(\pi_1 X) = P$, and $H_2(X', X)$ is free w/ basis e^2_α. Now $H_2 X' \cong H_2 X \oplus H_2(X', X)$

$H_2 X' \cong \pi_2 X'$ by Hurewicz since $\pi_1 X' = 0$.

Set $X^+ =$ attaching e^3_β to X' along $\psi_\beta : S^2 \to X'$, where ψ_β corresponds to a basis of $H_2(X', X)$.

Now X^+ is simply connected. To check $H_* X \cong H_* X^+$:

$H_* X'$ differs from $H_* X$ by $H_2(X', X) = \bigoplus_{\alpha \in I} \mathbb{Z}\alpha$, $\alpha = [e^2_\alpha]$

$H_* X^+$ differs from $H_* X'$ by $H_3(X^+, X) = \bigoplus_{\beta \in I} \mathbb{Z}\beta$, $\beta = [e^3_\beta]$

but $\partial [e^3_\beta] = e^2_\beta$, $\forall \beta \in I$. $\Rightarrow$ $H_*(X^+, X)$ vanishes.

Step 2 : $P \trianglelefteq \pi_1 X$:

Let $\tilde{X} \to X$ covering corresponding to P. $\pi_1 \tilde{X} = P$. By step 1, $\exists$ $\tilde{X}^+$ for $\tilde{X}$. Let $X^+ = \operatorname{colim} \left(\begin{array}{ccc} \tilde{X} & \hookrightarrow & \tilde{X}^+ \\ \downarrow & & \\ X & & \end{array} \right)$

Check this indeed is the one desired:

van Kampen $\Rightarrow$ $\pi_1 X^+ \cong \pi_1 X / P$

also $X^+/X \underset{\text{homo}}{\cong} \tilde{X}^+/\tilde{X}$ $\Rightarrow$ $H_*(X^+, X) = H_*(\tilde{X}^+, \tilde{X}) = 0$.

<u>pf of Prop 1.6</u>. Similar to Step 2 of construction. Consider

$$\begin{array}{ccc} BP & \longrightarrow & BP^+ \\ \downarrow & & \downarrow \\ BG & \longrightarrow & BG^+ \end{array}$$

$P \trianglelefteq \pi_1 BG = G$ acts freely & properly discontinuously on EG. So $EG/P \to BG = EG/G$ is a normal covering of BG w/ covering gp G/P.
Attaching 2-, 3-cells to BG lifts to BP $\Rightarrow$ BP^+ normal covering of BG^+

Thus $\pi_n(BP^+) \cong \pi_n(BG^+)$, $n \geq 2$. But BP^+ simply connected, by Hurewicz $\Rightarrow$ $\pi_2 BP^+ \cong H_2(BP^+) \cong H_2(BP) \cong H_2(P, \mathbb{Z})$.

The last isomorphism follows from the following theorem:

<u>Thm 1.9</u> [Rosenberg 5.1.27] $H_*(G, A) \cong H_*(BG; A)$, for $A \in Ab$ w/ trivial G-action.

<u>Rk 1.2</u>. Moreover, consider the following map of fiber seqs:

$$\begin{tikzcd}
Ff \arrow[r] \arrow[d, equal] & BP \arrow[r, "f|_{BP}"] \arrow[d] & BP^+ \arrow[d] \\
Ff \arrow[r] & BG \arrow[r, "f"] & BG^+
\end{tikzcd}$$

Passing to π_*:

$$\begin{tikzcd}
\cdots \arrow[r] & \underset{\overset{\parallel}{0}}{\pi_2 BP} \arrow[r] \arrow[d] & \pi_2 BP^+ \arrow[r] \arrow[d, "\cong"] & \pi_1 Ff \arrow[r] \arrow[d, equal] & \overset{P}{\overset{\parallel}{\pi_1 BP}} \arrow[r] \arrow[d] & \overset{0}{\overset{\parallel}{\pi_1 BP^+}} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \underset{\underset{0}{\parallel}}{\pi_2 BG} \arrow[r] & \pi_2 BG^+ \arrow[r] & \pi_1 Ff \arrow[r] & \underset{\underset{G}{\parallel}}{\pi_1 BG} \arrow[r] & \pi_1 BG^+ \arrow[r] & \cdots
\end{tikzcd}$$

Take $G = GL(R)$, $P = E(R)$, we get $\pi_2 BP^+ \cong \pi_2 BG^+ \cong K_2(R)$
$= H_2(E(R), \mathbb{Z})$

It's now clear that $\pi_1 Ff = St\, R$, the Steinberg gp. by the universal central extension of $E(R)$.

II. Properties of K_n.

- Background in Top: $X \in Top$ is a H-space if there $\exists$ cts map $\mu: X \times X \to X$ w/ neutral elet $e \in X$ s.t.

  $x \mapsto \mu(x, e)$, $x \mapsto \mu(e, x)$

  are homotopic to id.

e.g. top gps, $SO(n)$, $SU(n)$, $O(n)$, $GL_n(\mathbb{R})$, ...

Claim $BGL(R)^+$ is a H-space.

Consider $\mu: GL(R) \times GL(R) \longrightarrow GL(R)$

$$((a_{ij}), (b_{ij})) \mapsto (c_{ij}) = \begin{cases} a_{\frac{i+1}{2}\frac{j+1}{2}}, & i \equiv j \equiv 1 \bmod 2 \\ b_{\frac{i}{2}\frac{j}{2}}, & i \equiv j \equiv 0 \bmod 2 \\ 0, & \text{else.} \end{cases}$$

μ induces $B\mu: B(GL(R) \times GL(R)) \cong BGL(R) \times BGL(R) \longrightarrow BGL(R)$

$$\rightsquigarrow \mu_*: BGL(R)^+ \times BGL(R)^+ \longrightarrow BGL(R)^+.$$

μ_* gives a htpy-commutative H-space structure on $BGL(R)^+$.

[Theorem 5.2.12., Rosenberg]

- Moreover, for R, S associative, unital rings, one defines similarly

$$GL_p(R) \times GL_q(S) \longrightarrow GL_{pq}(R \otimes S) \quad \text{by } R^p \otimes S^q \cong (R \otimes S)^{pq}$$

$$\rightsquigarrow \mu_{p,q}: BGL_p(R)^+ \times BGL_q(S)^+ \longrightarrow BGL_{pq}(R \otimes S)^+ \longrightarrow BGL(R \otimes S)^+$$

$\mu_{p,q}$ compatible w/ stablization.

Note that $BGL(R \otimes S)^+$ is a H-space. can define a "balanced product" $\gamma_{p,q}: BGL_p(R)^+ \times BGL_q(S)^+ \longrightarrow BGL(R \otimes S)^+$

$$\gamma_{p,q}(x,y) = \mu_{p,q}(x,y) - \mu_{p,q}(x,*) - \mu_{p,q}(*,y)$$

$$\rightsquigarrow \gamma: BGL(R)^+ \wedge BGL(S)^+ \longrightarrow BGL(R \otimes S)^+$$

Passing to $\pi_{p+q}(-)$:

$$\gamma_*: \pi_{p+q}(BGL(R)^+ \wedge BGL(S)^+) \overset{①}{\cong} \pi_p BGL(R)^+ \otimes \pi_q BGL(S)^+ \longrightarrow \pi_{p+q} BGL(R \otimes S)^+$$

where ① is obtained by using the fact $BGL(-)^+$ is a H-space, and [Theorem 8.7, Whitehead, Elements in Homotopy Theory].

<u>Thm 2.1</u> (Loday) $\gamma_* : K_p R \otimes K_q S \longrightarrow K_{p+q}(R \otimes S)$

is natural in R, S, bilinear, associative. If R is commutative, then

$$K_p(R) \otimes K_q(R) \longrightarrow K_{p+q}(R)$$

is graded commutative.

<u>Rk 2.1</u> If $r_1, r_2, \dots, r_n$ are units of comm. ring R, then the product of $r_i \in K_1(R)$ is an element $\{r_1, \dots, r_n\}$ of $K_n(R)$. When $n=2$, this is the Steinberg symbol.

III. Something I like*

*: <u>not quite relevant to the topic</u>.

1. Connection to π_n^s, stable homotopy theory.

Let Σ_∞ = symmetric gp on ∞ many elements. Consider $X = B\Sigma_\infty$, and the "+" construction on X rel. A_∞ (alternating gp). Then

<u>Thm 3.1</u> (Barratt - Priddy - Quillen - Segal)

$$X^+ \simeq \Omega^\infty S^\infty = \operatorname*{colim}_n \Omega^n S^n \simeq \Omega^\infty \Sigma^\infty S^0$$

In particular, $\pi_i(B\Sigma_\infty^+) = \pi_i^s$. We can do the following:

a) X^+ is an infinite loop space. By May's recognition principle, it's an E_∞-space. But $\pi_0 X^+$ not a gp but monoid $\Rightarrow$ cannot relate it to some connective spectrum. (Better: consider $\mathbb{Z} \times X^+$, ...)

b) Is there any way to resolve (1) ?

Yes! Consider $KR = BGL(R)^+ \times K_0 R$. This is a group-like E_∞-space Hence, it defines a connective spectrum, denoted KR. Moreover, this is a ring spectrum. The reason, I believe, due to the "Group Completion

Theorem'' [Theorem 3.2.1, Adams, Infinite loop space].

c) Some computations of stable stems.

$\pi_1^s = \pi_1(B\Sigma_\infty^+) = (\Sigma_\infty)^{ab} \cong \mathbb{Z}/2$, generated by the equivalence class of odd permutations.

Consider
$$\begin{tikzcd} BA_\infty \arrow[r] \arrow[d] & BA_\infty^+ \arrow[d] \\ B\Sigma_\infty \arrow[r] & B\Sigma_\infty^+ \end{tikzcd}$$

By the same pf as in Prop 1.6, $\pi_2^s = \pi_2(B\Sigma_\infty^+) \cong H_2(A_\infty, \mathbb{Z})$.

d) Further question motivating Q-construction:

i) K is defined for rings. How about for others?

ii) exact sequences relating each K_i's ?

2. $K_n(\mathbb{F}_q)$, operations, Adams conjecture.

Consider the real / complex K-theory KR / KU. There exists a unique collection of cohomology operations on them: $\psi^k: K_{\mathbb{C}}(-) \to K_{\mathbb{C}}(-)$ s.t.

(note that $KU_n = \begin{cases} \mathbb{Z}\times BU, & n \text{ even} \\ \Omega BU, & n \text{ odd}\end{cases}$, $K_{\mathbb{C}}^n = [-, KU_n]$)

Def 3.1 (Adams operations) 1) $\psi^k\psi^l = \psi^l\psi^k = \psi^{kl}$, $k, l \geq 0$

2) $\psi^k(L) = L^k$, L line bundle.

This is a kind of cohomological operation.

- Construction of Adams operations: Consider $\lambda_t: \mathrm{Vect}(X) \to K_{\mathbb{C}}^0 X[[t]]$
$$E \mapsto \sum_{k\geq 0} \Lambda^k(E)t^k$$

and $\psi_t: K_{\mathbb{C}}^0 X \to K_{\mathbb{C}}^0 X[[t]]$, w/ $\varepsilon^{|E|}$ = trivial bundle w/ rk = rk E.
$$E \mapsto \varepsilon^{|E|} - t\frac{d}{dt}\ln(\lambda_{-t}(E))$$

then $\psi_t(E) = \sum_{k\geq 0} \psi^k(E)\, t^k$.

An important concept useful is Brauer Lifting, lifts k-reps of a finite gp G to complex ones. Here k is $\mathbb{F}_q$ in mind. Let $\rho: G \to GL_n k$ be a finite-dim rep of G. $S_\rho(g) \subset k^\times$ is the set of eigenvalues of $\rho(g)$ w/ multiplicity.

Def 3.2 Brauer character of ρ, denoted $\chi_\rho^{br}: G \to \mathbb{C}$, is

$$\chi_\rho^{br}(g) = \sum_{\lambda \in S_\rho(g)} \iota(\lambda),$$

where $\iota: k^\times \hookrightarrow \mathbb{C}^\times$ is an embedding.

Thm 3.2 (Green) $\forall\, \rho \in R_k(G)$, $\exists!\ \rho^{br} \in R_{\mathbb{C}}(G)$ s.t. $\chi_{\rho^{br}} = \chi_\rho^{br}$.

Thus $\rho \mapsto \rho^{br}$ extends to

Def 3.3 $(-)^{br}: R_k(G) \to R_{\mathbb{C}}(G)$ is called the Brauer lifting of k-reps of G.

Now $\forall\, \rho_{\mathbb{C}} \in R_{\mathbb{C}}(G)$, it gives $\mathbb{C}^n$ (for some n) a left action of G by

$$g \cdot v := (\rho(g))(v)$$

Consider it's associated bundle $EG \times_G \mathbb{C}^n \to BG$. This process encodes into a map $\beta: R_{\mathbb{C}}(G) \longrightarrow K^0_{\mathbb{C}}(BG) = [BG, \mathbb{Z}\times BU]$

$$= \mathrm{Vect}_{\mathbb{C}}(BG).$$

Now $\beta \circ (-)^{br}: R_k(G) \to K^0_{\mathbb{C}}(BG)$. Adams operations behave well w.r.t. Brauer lifting:

Prop 3.3 $\chi_{\psi^k\rho}(g) = \chi_\rho(g^k)$, $\forall\, g \in G$.

$$\chi^{br}_{\psi^k\rho} = \chi^{br}_\rho.$$

Thus we have $\beta\circ(-)^{br}: R_k(G) \to K^0_{\mathbb{C}}(BG)^{\psi^k}$. Quillen's work on $K_n(\mathbb{F}_q)$ arises from his proof of Adams Conjecture. Namely,

<u>Thm 3.4</u> (Adams Conjecture) Let $J: K_{\mathbb{C}}^{*}(X) \longrightarrow Sph(X)$ be the J-homomorphism, where $Sph(X)$ = gp of stable spherical fibrations on X, finite CW cpx. J is induced by taking one pt compactification of v.b. / X. Let $k \in \mathbb{Z}$, $x \in K_{\mathbb{C}}^{*}(X)$, then $k^n J(\psi^k x - x) = 0$, for some $n \geq 0$ integer.

<u>Key idea</u> $G = GL_n(\mathbb{F}_q)$ in previous setting. Construct $\alpha: BGL(\mathbb{F}_q) \to BU$, by taking colim $(\alpha_n : BGL_n(\mathbb{F}_q) \to BU)$, each α_n is the Brauer lifting obtained from identity map $1: GL_n(\mathbb{F}_q) \to GL_n(\mathbb{F}_q)$, viewed as a rep. of G.

Now for the diagram

$$\begin{tikzcd} BGL(\mathbb{F}_q) \arrow[r, "\alpha"] & BU \arrow[r] \arrow[d, "\mu"'] & C \arrow[dl, "\beta"] \\ & BF[p^{-1}] & \end{tikzcd}$$

where C = mapping cone of α, μ is induced from $x \mapsto J(\psi^k x - x)$

$BF = \mathrm{colim}_n (F(n)$: $F(n)$ is monoid of based htpy equivalences $S^n \to S^n$),

Quillen showed μ, β nullhtpic $\Rightarrow$ conjecture is proved.

▶ Ref: [Quillen (1970). The Adams Conjecture. Topology Vol. 10. pp. 67-80 Pergamon Press 1971]

Now given the construction of $\alpha: BGL(\mathbb{F}_q) \longrightarrow BU$, it induces $\tilde{\alpha}: BGL(\mathbb{F}_q)^{+} \longrightarrow BU$,

and hence $\tilde{\alpha}_*: K_n(\mathbb{F}_q) \longrightarrow \pi_n BU$. Consider $\psi^q - 1: BU \to BU$.

<u>Thm 3.5</u> (Quillen) The homotopy fiber of $\psi^q - 1$ is $BGL(\mathbb{F}_q)^{+}$.

<u>Cor 3.6</u> For every finite field $\mathbb{F}_q$, $n \geq 1$, we have

$$K_n(\mathbb{F}_q) = \pi_n(BGL(\mathbb{F}_q)^{+}) \cong \begin{cases} \mathbb{Z}/(q^i - 1), & n = 2i-1 \\ 0, & \text{else} \end{cases}$$

Note $\pi_{2i} BU = K_{\mathbb{C}}^{*}(S^{2i})$. The result follows from Bott Periodicity.