K_2 of a Field (Matsumoto)

§ 1 Introduction

§ 1.1 Recap of K_2

Thm Let Λ be a ring. Then $E(\Lambda)$ admits a universal central extension $0 \to K_2(\Lambda) \to St(\Lambda) \to E(\Lambda) \to 1$ defining $St(\Lambda)$ and $K_2(\Lambda)$.

Thm The Steinberg group may be realized as the free group on symbols x_E indexed by $E \in E(\Lambda)$ modulo relations:

$$x_{ij}^{\lambda} x_{ij}^{\mu} = x_{ij}^{\lambda+\mu} \qquad [x_{ij}^{\lambda}, x_{jk}^{\mu}] = x_{ik}^{\lambda\mu} \qquad [x_{ij}^{\lambda}, x_{kl}^{\mu}] = 1$$

$i \neq k$ $\qquad$ $j \neq k,\ i \neq l$

Rmk. The Steinberg group only remembers the trivial relations among the elementary matrices, and $K_2(\Lambda)$ remembers the rest. But there is another interesting subgroup of $St(\Lambda)$.

Thm We defined symbols $w_{ij}(u) \in St_n(\Lambda)$, defined $W \subseteq St_n(\Lambda)$ to be the subgroup generated by these symbols, defined

$$C_n = W \cap K_2\Lambda \qquad \{u,v\} = [h_{ij}(u), h_{ik}(v)] \in C_n$$

and showed that

(1) The abelian group $C_n \subseteq Z(St_n(\Lambda))$ is generated by the Steinberg symbols $\{u,v\}$.

(2) when Λ is a field, Thm 9.12 said the kernel of $St_n(\Lambda) \to GL_n(\Lambda)$ is contained in W.

It follows that $K_2\Lambda = \varinjlim C_n$ is generated by the Steinberg symbols when Λ is a field.

(3) If u, v, w, and $1-u$ are units, the the Steinberg symbols satisfy

$$\{uv,w\} = \{u,w\}\{v,w\}, \quad \{u,vw\} = \{u,v\}\{u,w\}, \quad \{u,1-u\} = 1$$

Q: Are there any relations in $K_2\Lambda$ in addition to those of item (3)?

A: Not for Λ a field. (See § 2.1)

§ 1.2 Discretely Valued Fields

Def Let R be a commutative ring. A valuation on R is a totally ordered abelian group Γ ← value group together with a map $R \xrightarrow{v} \Gamma \cup \{\infty\}$ such that the following hold: (Take $\infty + x = \infty$)

$$v(0) = \infty \qquad v(xy) = v(x) + v(y)$$
$$v(1) = 0 \qquad v(x+y) \geq \min\{v(x), v(y)\}.$$

Two valuations $v_1 : R \to \Gamma_1 \cup \{\infty\}$ and $v_2 : R \to \Gamma_2 \cup \{\infty\}$ are called equivalent if there is an isomorphism of totally ordered abelian groups $\Gamma_1 \xrightarrow{\sim} \Gamma_2$ commuting with v_1 and v_2.

Prop. There is a map $\{\text{Valuations on } R\}/\sim \;\to \operatorname{Spec} R$ sending $v \mapsto v^{-1}(\infty)$ which admits a section sending

$$\mathfrak{p} \mapsto \left(x \mapsto \begin{cases} \infty & \text{if } x \in \mathfrak{p} \\ 0 & \text{if } x \notin \mathfrak{p} \end{cases}\right)$$

Hence we may view valuations as generalizations of prime ideals.

Pf. Follows from definitions. ▨

Def A valuation ring is a domain $\mathcal{O}$ such that for every $x \in \operatorname{Frac}(\mathcal{O})$ either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

Prop. A domain $\mathcal{O}$ is a valuation ring if and only if

$$\mathcal{O} = \{x \in \operatorname{Frac}(\mathcal{O}) : v(x) \geq 0\}$$

for some valuation v on $\operatorname{Frac}(\mathcal{O})$.

Pf. If $\mathcal{O}$ is a valuation ring, then $\Gamma := \operatorname{Frac}(\mathcal{O})^\times/\mathcal{O}^\times$ is totally ordered by $x \leq y$ iff $\frac{y}{x} \in \mathcal{O}$, and the projection $\operatorname{Frac}(\mathcal{O})^\times \to \operatorname{Frac}(\mathcal{O})^\times/\mathcal{O}^\times$ defines a valuation. Conversely, since $0 \leq v(x)$ or $\underset{\Updownarrow\ v(x^{-1}) \geq 0}{v(x) \leq 0}$ for all $x \in \operatorname{Frac}(\mathcal{O})$, $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. ▨

Prop. Valuation rings are local with maximal ideal

$$\mathfrak{m} = \{x \in \operatorname{Frac}(\mathcal{O}) : v(x) > 0\}.$$

← strict inequality

Pf. Since $\mathfrak{m}$ is an ideal, we just need to show

$$\mathcal{O}^\times = \{x \in \operatorname{Frac}(\mathcal{O}) : v(x) = 0\}$$

is the set of units. If $x, y \in \mathcal{O}$ and $xy = 1$ then $v(x) + v(y) = 0$ which forces $v(x) = v(y) = 0$. If $v(x) = 0$ then $v(x^{-1}) = -v(x) = 0$ so $x^{-1} \in \mathcal{O}$ too. ▨

§ 2 Theorem Statement

§ 2.1 Steinberg Symbols on Fields

Thm (Matsumoto) Let F be a field. Then

$$K_2(F) = (F^\times \otimes_{\mathbb{Z}} F^\times)/\langle x \otimes (1-x)\rangle_x.$$

Unpacking universal properties, an equivalent way to say this would be that the pair $(K_2F, \{-,-\})$ represents the functor

$$\mathrm{Ab} \to \mathrm{Set}$$
$$A \mapsto \left\{\begin{matrix}\text{Steinberg symbols} \\ \text{on } F \text{ valued in } A\end{matrix}\right\}$$

Def A Steinberg symbol on F valued in A is a map of abelian groups

$$(F^\times \otimes_{\mathbb{Z}} F^\times)/\langle x \otimes (1-x)\rangle_x \to A$$

or equivalently a function $F^\times \times F^\times \xrightarrow{c} A$ satisfying

(1) $c(xy, z) = c(x,z)\,c(y,z)$

(2) $c(x, yz) = c(x,y)\,c(x,z)$

(3) $c(x, 1-x) = 1$

Rmk. Our examples of Steinberg symbols on fields will mostly come from valuation theory.

Prop. Let $v : F \to \mathbb{Z} \cup \{\infty\}$ be a discrete valuation on a field F. Then the formula below defines a Steinberg symbol on F valued in the group of units $k_v^\times$ of the residue field.

$$d_v : F^\times \times F^\times \to k_v^\times$$
$$(x,y) \mapsto (-1)^{v(x)v(y)} \cdot x^{v(y)} y^{-v(x)}$$

Pf. Computational, but at least note that this well-defined since

$$v(x^{v(y)} y^{-v(x)}) = v(x^{v(y)}) + v(y^{-v(x)}) = v(y)v(x) - v(x)v(y) = 0$$

implies $x^{v(y)} y^{-v(x)} \in \mathcal{O}_v^\times$. ▨

§ 2.2 Examples

Ex. The p-adic valuation $v_p(p^\alpha \frac{a}{b}) = \alpha$ on $\mathbb{Q}$ gives rise to a Steinberg symbol given by the formula in the previous proposition. For $p \neq 2$ let us denote this Steinberg symbol $(-,-)_p$.

Ex. This does not give information when $p = 2$. For $p = 2$, we define a Steinberg symbol on $\mathbb{Q}$ valued in $\{\pm\}$ by

$$(x,y)_2 = (-1)^{i(x)i(y) + j(x)k(y) + k(x)j(y)}$$

where $i(x), j(x), k(x)$ are defined (uniquely) by writing

$$x = (-1)^{i(x)} 2^{j(x)} 5^{k(x)} \frac{a}{b}$$

where $a \equiv_8 b \equiv_8 1$.

Ex. $3 = (-1) \cdot 5 \cdot \frac{3^2}{-3 \cdot 5}$, and in general you use the fact that

$$\frac{a}{b} = 2^j \frac{a'}{b'} = 2^j \frac{(a')^2}{a'b'}$$
$$= (-1)\, 2^j\, 5 \frac{(a')^2}{(-5)a'b'}$$
$$= 2^j\, 5 \frac{(a')^2}{5a'b'}$$
$$= (-1)\, 2^j \frac{(a')^2}{-a'b'}$$

and choose which line to use according to whether $a'b' \equiv_8 1, 3, 5$, or 7. (Note that odd squares are $\equiv_8 1$.)

Rmk. If p and q are odd primes, then

$$(p,q)_2 = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)} = \begin{cases} 1 & \text{if } p \equiv_4 1 \text{ or } q \equiv_4 1 \\ -1 & \text{if } p, q \equiv_4 3 \end{cases}$$

depends only on the residues of p and q modulo 4.

Ex. Although $\mathbb{Q}$ does not have a valuation corresponding to the standard absolute value $|-|_\infty$, it does have a Steinberg symbol associated to this norm:

$$(x,y)_\infty = (-1)^{\left(\frac{\operatorname{sgn}(x)-1}{2}\right)\left(\frac{\operatorname{sgn}(y)-1}{2}\right)}$$

$$= \begin{cases} 1 & \text{if } x > 0 \text{ or } y > 0 \\ -1 & \text{if } x, y < 0 \end{cases}$$

which depends only on the signs of x and y.

§ 3 Application

§ 3.1 Calculation of $K_2\mathbb{Q}$

Thm There is an isomorphism defined by the indicated mapping of Steinberg symbols below:

$$K_2\mathbb{Q} \xrightarrow{\sim} \{\pm\} \oplus \mathbb{F}_3^\times \oplus \mathbb{F}_5^\times \oplus \cdots$$
$$\{x,y\} \mapsto ((x,y)_p)_{p \text{ prime}}$$

Pf. The idea is to define the isomorphism inductively, using the following filtration of $K_2\mathbb{Q}$.

$$L_1 = L_2 \subsetneq L_3 = L_4 \subsetneq L_5 = L_6 \subsetneq L_7 = \cdots \subseteq K_2\mathbb{Q}$$
$$L_i = \langle \{x,y\} : |x|_\infty, |y|_\infty \leq i,\ x, y \in \mathbb{Z}\rangle \qquad \varinjlim_i L_i = K_2\mathbb{Q}$$

(Since the Steinberg symbol is bimultiplicative, the filtration only jumps at primes.) We need to see inductively that there is an isomorphism

$$L_p \xrightarrow{\sim} \{\pm\} \oplus \mathbb{F}_3^\times \oplus \cdots \oplus \mathbb{F}_p^\times$$
$$\{x,y\} \mapsto ((x,y)_2, (x,y)_3, \ldots, (x,y)_p)$$

Base case follows from

$$\{1,1\} = \{-1,1\} = \{1,-1\} = \{-1,-1\}^2 = 1$$

(See Lemma 9.8 in Milnor.) So assume claim holds for all primes less than $p > 2$. Then we need to see that

$$1 \to L_{p-1} \xrightarrow{\iota} L_p \xrightarrow{\pi} \mathbb{F}_p^\times \to 1$$

is split exact. Indeed, the desired section is defined by

$$\begin{tikzcd} \{x,y\} \arrow[d, mapsto] & L_p \arrow[d, "\pi"'] & \{x,p\} \\ (x,y)_p & \mathbb{F}_p^\times \arrow[u, bend right, "s"'] & x \arrow[u, mapsto] \end{tikzcd}$$

Here π is well-defined by Matsumoto, s is well-defined by a computation, $\pi\iota$ is constant because $(x,y)_p = 1$ as soon as $v_p(x) = v_p(y) = 0$, and exactness at L_p is also a computation (see Lemma 11.7 in Milnor). Finally $\pi s = \mathrm{id}$ because

$$(x,p)_p = (-1)^{v_p(x)v_p(p)} x^{v_p(p)} p^{-v_p(x)} = (-1)^{1 \cdot 0} x^1 y^{-0} = x.$$

So we get the desired splitting and it is not hard to see that the map

$$L_p \to \{\pm\} \oplus \mathbb{F}_3^\times \oplus \cdots \oplus \mathbb{F}_p^\times$$

takes $\{x,y\}$ to $((x,y)_2, \ldots, (x,y)_p)$. ▨

Rmk. One can show along the same lines that if x is an indeterminate, then

$$K_2(F(x)) = K_2F \oplus \bigoplus_{p \text{ irred}} (F[x]/(p))^\times.$$

§ 3.2 Quadratic Reciprocity

Prop. Let $p \neq 2$ be a prime. Then there are exactly two maps $\mathbb{F}_p^\times \to \{\pm\}$. The nontrivial one is given by

$$x \mapsto x^{\frac{p-1}{2}} = \left(\frac{x}{p}\right) \leftarrow \text{Legendre Symbol}$$

Pf. Both statements follow from $\mathbb{F}_p^\times$ being cyclic. ▨

Rmk. Recall that we had a Steinberg symbol $(-,-)_\infty$ on $\mathbb{Q}$ valued in $\{\pm\}$ not appearing in the direct sum decomposition of $K_2\mathbb{Q}$. It is natural to ask what the map

$$K_2\mathbb{Q} \to \{\pm\}$$
$$\{x,y\} \mapsto (x,y)_\infty$$

looks like when factored through $\{\pm\} \oplus \mathbb{F}_3^\times \oplus \mathbb{F}_5^\times \oplus \cdots$. By the previous proposition, this factorization sends

$$K_2\mathbb{Q} \to \{\pm\} \oplus \mathbb{F}_3^\times \oplus \mathbb{F}_5^\times \oplus \cdots \to \{\pm\}$$
$$\{x,y\} \mapsto ((x,y)_2, (x,y)_3, (x,y)_5, \ldots) \mapsto (x,y)_2^{\varepsilon_2} \cdot \prod_{p \neq 2} \left(\frac{(x,y)_p}{p}\right)^{\varepsilon_p}$$

where ε_p is either 0 or 1.

Thm $(x,y)_\infty = \prod_p \left(\frac{(x,y)_p}{p}\right)$

Pf. By the remark, we only need to show each $\varepsilon_p = 1$. The argument depends on the residue of p mod 8:

(a) $p = 2$. Plug in $x = y = -1$. Then

$$-1 = (x,y)_\infty = (x,y)_2^{\varepsilon_2} \prod_{p \neq 2} \left(\frac{(x,y)_p}{p}\right)^{\varepsilon_p} = (x,y)_2^{\varepsilon_2}$$

forces $\varepsilon_2 = 1$.

(b) $p \equiv_8 3$ or 5. Plug in $x = 2$ and $y = p$. Then

$$1 = (x,y)_\infty = (x,y)_2 \prod_{p \neq 2} \left(\frac{(x,y)_p}{p}\right)^{\varepsilon_p} = (x,y)_2 \left(\frac{(x,y)_p}{p}\right)^{\varepsilon_p}$$
$$= (-1)\left(\frac{(x,y)_p}{p}\right)^{\varepsilon_p}$$

forces $\varepsilon_p = 1$.

(c) $p \equiv_8 7$. Plug in $x = -1$ and $y = p$. Then

$$1 = (x,y)_\infty = \text{same as case (b)} = (x,y)_2 \left(\frac{(x,y)_p}{p}\right)^{\varepsilon_p} = (-1)\left(\frac{(x,y)_p}{p}\right)^{\varepsilon_p}$$

forces $\varepsilon_p = 1$.

(d) $p \equiv_8 1$. This is the difficult case. Suppose that some $\varepsilon_p = 0$, and that p is minimal with this property. By Lemma 11.9, there exists a prime $q < \sqrt{p}$ such that $\left(\frac{p}{q}\right) = -1$ (i.e. p is not a quadratic residue mod q). Then $\varepsilon_q = 1$ because if $q \equiv_8 1$ this would contradict the minimality of p and otherwise it would contradict the conclusions of one of the previous cases. Now plug in $x = p$ and $y = q$. Then

$$1 = (x,y)_\infty = \cancel{(x,y)_2}^{\,1} \prod_{p > 2} \left(\frac{(x,y)_p}{p}\right)^{\varepsilon_p} = \left(\frac{(x,y)_q}{q}\right)\left(\frac{(x,y)_p}{p}\right)^{\varepsilon_p}$$
$$= (-1)\left(\frac{(x,y)_p}{p}\right)^{\varepsilon_p}$$

forces $\varepsilon_p = 1$. ▨

Cor (Quadratic Reciprocity). Let p and q be distinct odd primes. Then plugging $x = p$ and $y = q$ into the theorem gives

$$1 = (x,y)_\infty = (x,y)_2 \prod_{p > 2} \left(\frac{(x,y)_p}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)} \left(\frac{(p,q)_p}{p}\right)\left(\frac{(p,q)_q}{q}\right)$$

$$\Rightarrow (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)} = \left(\frac{q}{p}\right)\left(\frac{p}{q}\right).$$

Thm (Weil) There is an analogous formula for rational function fields, namely

$$(f,g)_\infty^{-1} = \prod_{p \text{ irred}} \operatorname{Norm}_{(F[x]/(p))/F}((f,g)_{(p)})$$

where $(f,g)_{(p)}$ corresponds the p-adic valuation and $(f,g)_\infty$ corresponds the valuation at ∞.

§ 4 Proof

(Since F is a field, $E(F) = SL(F)$.) The idea is to construct a central extension

$$0 \to (F^\times \otimes_{\mathbb{Z}} F^\times)/(x \otimes (1-x)) \to G \to SL_n(F) \to 1.$$

(It is not so easy to describe G.) Given this, there is a unique map

$$\begin{tikzcd} & K_2F \arrow[r] \arrow[d, dashed, "f"'] & St(F) \arrow[d, dashed, "\exists!"] \arrow[dr] & \\ 0 \arrow[ur] \arrow[r] & (F^\times \otimes F^\times)/(x \otimes (1-x)) \arrow[r] \arrow[u, bend right, dashed, "g"'] & G \arrow[r] & SL_n(F) \to 1 \end{tikzcd}$$

coming from the definition of $St(F)$ and $K_2(F)$ given in § 1.1. There is also a map $(F^\times \otimes F^\times)/(x \otimes (1-x)) \xrightarrow{g} K_2F$ corresponding to the symbol $\{-,-\}$ constructed on K_2F. These maps are mutually inverse to each other. One can see that $fg = \mathrm{id}$ as soon as f is shown to send $\{x,y\} \mapsto x \otimes y$ (follows from Lemmas 12.1 & 8.3.). In particular g is injective. And g is surjective because K_2F is generated by the symbols $\{x,y\}$.