Higher Algebraic K : Second Course

1. Exact categories

Def 1.1 An exact category is a pair $(\mathcal{C}, \mathcal{E})$, $\mathcal{C}$ additive category, $\mathcal{E}$ family of sequences in $\mathcal{C}$ of the form (called admissible exact sequenes)

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \qquad (*)$$

s.t. $\exists$ an embedding of $\mathcal{C}$ as a full subcategory of an abelian cat $\mathcal{A}$ s.t.

1) $\mathcal{E}$ is the class of all sequences $(*)$ in $\mathcal{C}$ which are exact in $\mathcal{A}$.

2) $\mathcal{C}$ closed under extensions in $\mathcal{A}$ in the sense that if $(*)$ exact sequence in $\mathcal{A}$, $A, C \in \mathcal{C}$, then B is iso to an object in $\mathcal{C}$.

Terminology f is called admissible monomorphism, g is called admissible epimorphism, if f mono (resp. g epi) in $(*)$.

Def 1.2 $\mathcal{C}$ is closed under kernels of surjections in $\mathcal{A}$, if whenever a map $f: B \to C$ surjection in $\mathcal{A}$, then $\ker f \in \mathcal{C}$.

Def 1.3 $F: \mathcal{B} \to \mathcal{C}$ functor between exact categories is called exact, if it is additive, and preserves admissible exact sequences.

e.g. 1.4. $P(R)$ = category of finitely generated projective R-mods is an exact sequence. What's more, $K_0(R) = K_0 P(R)$, where the latter means an abelian gp w/ generators $[B]$, $B \in P(R)$, and $[B] = [A] + [C]$ for $\forall$ s.e.s. $0 \to A \to B \to C \to 0$ in $P(R)$.

e.g. 1.5. If $\mathcal{C}$ is exact, then $\mathcal{C}^{op}$ is exact.

e.g. 1.6. (K_0). Formalize $K_0(\mathcal{C})$ appearing in e.g. 1.4. :

Let $\mathcal{C}$ = exact, small cat. Then we define $K_0\mathcal{C}$ to be an abelian gp w/ generators $[B]$, one for each object $B \in \mathcal{C}$, and suject to the relation $[B] = [A] + [C]$ for every admissible exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{C}$. Here "+" is a formal operation.

<u>Def 1.7</u> Let $\mathcal{P}$ be an additive subcat of an abelian cat $\mathcal{A}$. A $\mathcal{P}$-resolution $P_\bullet \to B$ of an object $B \in \mathcal{A}$ is an exact sequence in $\mathcal{A}$:

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to B \to 0,$$

where all $P_i \in \mathcal{P}$. The number $\min\{n \in \mathbb{N} : P_i = 0,\ i > 0\}$ is called $\mathcal{P}$-dimension of B.

<u>Thm 1.8</u> (Resolution Theorem)

Let $\mathcal{P} \subset \mathcal{C} \subset \mathcal{A}$ be an inclusion of additive cats. $\mathcal{A}$ abelian (hence gives the notion of admissible exact sequences to $\mathcal{P}$ and $\mathcal{C}$). Suppose

① $\forall$ obj C of $\mathcal{C}$, C has a finite $\mathcal{P}$-dimension

② $\mathcal{C}$ closed under kernels of surjections in $\mathcal{A}$.

Then, $\mathcal{P} \subset \mathcal{C}$ induces an iso $K_0(\mathcal{P}) \cong K_0\mathcal{C}$.

<u>pf Sketch</u>. $P_\bullet \to C$ $\mathcal{P}$-resolution. Then

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to C \to 0.$$

is exact. $n < \infty$. $[C] = \sum (-1)^i [P_i] \in K_0\mathcal{C}$. So $K_0\mathcal{P} \to K_0\mathcal{C}$ is surjective. To prove the injectivity, use the following comparison Lemma.

<u>Lem 1.9</u> (Comparison)

Given $f: C \to C'$ in $\mathcal{C}$ and a finite $\mathcal{P}$-resolution $P'_\bullet \to C'$, then $\exists$ a finite $\mathcal{P}$-resolution $P_\bullet \to C$ s.t. the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & P_m \arrow[r] & \cdots \arrow[r] & P_n \arrow[r] \arrow[d, "\varphi_n"] & P_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\varphi_1"] & P_0 \arrow[r] \arrow[d, "\varphi_0"] & C \arrow[r] \arrow[d, "f"] & 0 \\
 & & 0 \arrow[r] & P'_n \arrow[r] & P'_{n-1} \arrow[r] & \cdots \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

2. Q-construction

Let $\mathcal{C}$ be an exact, small category. We define a new category $Q\mathcal{C}$ as follows:

- Obj $Q\mathcal{C}$ = Obj $\mathcal{C}$.
- Mor: a morphism from A to B is an equiv. class of diagrams

$$A \xleftarrow{p} C \xrightarrow{i} B$$

where p = admissible epi, i = admissible mono in $\mathcal{C}$.

▲ The equivalent relation is given by:

$$A \xleftarrow{p} C \xrightarrow{i} B \quad \sim \quad A \xleftarrow{p'} C' \xrightarrow{i'} B$$

if $\exists$ iso $\eta: C \to C'$ s.t. diagram commutes:

$$\begin{tikzcd}
A & C \arrow[l, "p"'] \arrow[r, "i"] \arrow[d, "\eta"] & B \\
 & C' \arrow[ul, "p'"] \arrow[ur, "i'"'] &
\end{tikzcd}$$

- Composition of two morphisms $f: A \to B$, $g: B \to C$ is the pullback

$$\begin{tikzcd}
 & & L \arrow[dl] \arrow[dr] & & \\
 & M \arrow[dl, "p_f"'] \arrow[dr, "i_f"] \arrow[rr, phantom, "\vee"] & & N \arrow[dl, "p_g"'] \arrow[dr, "i_g"] & \\
A & & B & & C
\end{tikzcd}$$

$$f \qquad\qquad g$$

Then $g \circ f : A \leftarrow L \rightarrow C$

- (Exercise) draw diagram indicating associativity.

<u>Def 2.1</u> Let $C \in \mathcal{C}$ be an object. An admissible subobject of C is an equivalence class of admissible monos $C' \rightarrow C$ in $\mathcal{C}$.

<u>Rk 2.2</u> $\forall f: A \rightarrow B$ in $Q\mathcal{C}$, f determines a unique admissible subobject of B in $\mathcal{C}$. In particular, if $\mathcal{C}$ has an zero object 0, then $0 \rightarrow B$ in $Q\mathcal{C}$ are 1-1 correpondent to admissible subobjects of B.

<u>Prp 2.3</u> $\forall$ iso in $Q\mathcal{C}$ are 1-1 correspondence to iso in $\mathcal{C}$.

pf. $\forall f: A \cong B$ in $\mathcal{C}$. $f \rightsquigarrow A \xleftarrow{f^{-1}} B = B$ and $B \xleftarrow{f} A = A$.

$\forall f: A \xrightarrow{\cong} B$ in $Q\mathcal{C}$, f rep by $A \xleftarrow{j} C \xrightarrow{i} B$. It has an inverse

$f^{-1}: B \xleftarrow{j'} C' \xrightarrow{i'} A$ s.t. $f \circ f^{-1} = id_B$. Consider the diagram

$f^{-1} \circ f = id_A$

$$\begin{tikzcd}
 & & B \arrow[dl, "b"'] \arrow[dr, "b'"] & & A \arrow[dl, "a'"'] \arrow[dr, "a"] & & \\
 & C \arrow[dl, "i"'] \arrow[dr, "j"] & & C' \arrow[dl, "i'"'] \arrow[dr, "j'"] & & C \arrow[dl, "i"'] \arrow[dr, "j"] & \\
B & & A & & B & & A
\end{tikzcd}$$

Then $j' \circ a' = i \circ a : A \rightarrow B$

$j \circ b = i' \circ b' : B \rightarrow A$.

Check they give an iso in $\mathcal{C}$:

$$A \xrightarrow{i \circ a} B \xrightarrow{i' \circ b'} A : \quad \begin{aligned} (i' \circ b') \circ (i \circ a) &= (j \circ b) \circ (i \circ a) \\ &= j \circ (b \circ i) \circ a \\ &= j \circ id_B \circ a \\ &= j \circ a \\ &= id_A. \end{aligned}$$

Samilarly for $B \rightarrow A \rightarrow B$.

Thus, f gives an iso in $\mathcal{C}$.

Def 2.4 (Q-construction)

Consider $Q\mathcal{C}$. Taking its nerve $NQ\mathcal{C}$ gives a simplicial set. Denote $BQ\mathcal{C} = |NQ\mathcal{C}|$, its geometric realization. This is called the Quillen's Q-construction.

Thm 2.5 The geometric realization $BQ\mathcal{C}$ is a CW cpx, connected, w/ $\pi_1 BQ\mathcal{C} \cong K_0\mathcal{C}$. Element of $\pi_1 BQ\mathcal{C}$ corresponds to $[C] \in K_0\mathcal{C}$, rep. by the based loop composed of the two edges from 0 to C:

$$0 \to C \to 0$$

pf. [Weibel, Proposition 6.2, Ch. 4].

Def 2.6 Let $\mathcal{C}$ be small, exact category. Then $K\mathcal{C}$ = space $\Omega BQ\mathcal{C}$, and we set $K_n(\mathcal{C}) = \pi_n K\mathcal{C} = \pi_{n+1}(BQ\mathcal{C})$, $n \geq 0$.

- By Theorem 2.5, $K_0\mathcal{C}$ is the same as the classical one.

Thm 2.7 (+ = Q)

For every ring R, $\Omega BQ\,P(R) \simeq K_0R \times BGL(R)^+$.

Thus $K_n P(R) \cong K_nR$, $n \geq 0$.

- See [Weibel, Section 7, Ch. 4].

3. ∞-categorical generalization.

Upshot An ∞-cat $\mathcal{C}$ is a simplicial set satisfying the inner horn extension property, i.e. $\forall\ n \geq 2$, $0 < i < n$.

$$\begin{tikzcd} \Lambda^n_j \arrow[r, "s"] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, "\tilde{s}"'] & \end{tikzcd}$$

$\tilde{s}$ lifts s, $\Lambda^n_j = \Delta^n$ w/ the opposite edge of vertex j removed.

FACT 3.1 Nerve functor $N: \underset{\text{category of categories}}{\underbrace{Cat}} \longrightarrow \underset{\text{category of } \infty\text{-cats}}{\underbrace{qCat}} \subseteq sSet$

$X \in sSet$, then $X \cong N\mathcal{C}$ for some category $\mathcal{C}$, iff X satisfies unique inner horn extension property.

Def 3.2 $\mathcal{C}$ ∞-cat, then $\mathcal{C}$ is additive, if its htpy cat $h\mathcal{C}$ is additive.

Note 3.3 If $\mathcal{C}_0$ ordinary (1-cat), $\mathcal{C}_0$ additive, then $N\mathcal{C}_0$ is additive ∞-cat.

Def 3.4 $\mathcal{C}$ ∞-cat, $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{C}$ subcats containing all equivalences, morphisms in $\mathcal{C}_1$, $\mathcal{C}_2$ are called ingressive and egressive, respectively. A pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

is ambigressive, if $X' \to Y'$ ingressive (i.e. in $\mathcal{C}_1$)

$X \to Y$ egressive (i.e. in $\mathcal{C}_2$)

Dually, a pushout square
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$
is ambigressive,

if $X \to Y$ ingressive, $X' \to Y'$ egressive.

Call $(\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2)$ an exact ∞-cat if

1) $\mathcal{C}$ additive

2) ambigressive pullback = ambigressive pushout.

3) $\forall X \in \mathcal{C}$, $0 \to X$ is ingressive. $X \to 0$ egressive.

pushouts (resp. pullbacks) of ingressive (resp. egressive) morphisms exist, and are ingressive (resp. egressive). This condition is equivalent to say $(\mathcal{C}, \mathcal{C}_1)$ Waldenhausen ∞-cat, $(\mathcal{C}, \mathcal{C}_2)$ co Waldenhausen ∞-cat.

<u>Note 3.5</u> If $\mathcal{C}$ exact, then can find two subcats of $N\mathcal{C}$ s.t. it becomes an exact ∞-cat. Now admissible mono = ingressive, admissible epi = egressive.

<u>Def 3.6</u> $\mathcal{C}$ ∞-cat. $\mathcal{C}$ is stable, if $\mathcal{C}$ has 0, and

① $\forall$ morphism has fibers (form pullback) and cofibers (form pushout)

② fiber sequence = cofiber sequence.

<u>Note 3.7</u> $\mathcal{C}$ stable ∞-cat. Then $h\mathcal{C}$ is a triangulated cat. $\mathcal{C}$ is also an exact ∞-cat, $\forall$ morphisms are both ingressive & egressive.

- Q-construction of exact ∞-cat.

<u>Goal</u> To define $Q\mathcal{C}$. $\mathcal{C}$ exact ∞-cat, s.t. every morphism from X to Y is a span $X \xleftarrow{f} Z \xrightarrow{g} Y$, w/ f egressive, g ingressive. composition = take pullback, as in the classical setting. To do that, we need the following:

<u>Def 3.8</u> Let $\varepsilon : [n] \longmapsto [n]^{op} * [n] = [2n+1]$ be a functor from Δ to itself, where $*$ = join. This is called edgewise subdivision functor.

<u>Def 3.9</u> $X \in sSet$. The edgewise subdivision of X is the simplicial set

$$sd\,X := \varepsilon^* X, \quad \varepsilon^* = \text{endofunctor on } sSet \text{ induced by } \varepsilon.$$

So $(sd\,X)_n = (\varepsilon^* X)_n = X_{2n+1}$

<u>Prop 3.10</u> If X ∞-cat, then $sd\,X$ is an ∞-cat.

<u>Terminology 3.11</u> $sd\,X$ is called twisted arrow ∞-cat, if X ∞-cat.

Construction

Let Q_n = cat of obj (i,j) w/ $0 \le i \le j \le n$, morphism $\exists$ only when $i' \ge i$, $j' \le j$, $(i,j) \to (i',j')$. Then can draw the cat as (ij short for (i,j))

$$\begin{tikzcd}[column sep=small]
 & & & & 0n \arrow[dl] \arrow[dr] & & & & & \\
 & & & 0,n\text{-}1 & & 1n & & & & \\
 & & \iddots & & & & \ddots & & & \\
 & & 02 \arrow[dl] \arrow[dr] & & & & & n\text{-}2,n \arrow[dr] & & \\
 & 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & & & & n\text{-}1,n \arrow[dr] & \\
00 & & 11 & & 22 & & & & & nn
\end{tikzcd}$$

<u>Claim 3.12</u> $\mathrm{sd}\,\Delta[n] = NQ_n$, $\Delta[n]$ standard n-simplex

► I don't know why. One possible reason is that: Q_n has a natural Segal map $Q_n\mathcal{C} \to \prod_{Q_0\mathcal{C}} Q_1\mathcal{C}$ and forms a Segal space, which is another model for ∞-cat that essentially the same as quasi-cat.

<u>Def 3.13</u> $\mathcal{C}$ exact ∞-cat, denote by $Q\mathcal{C}$ the ∞-cat whose n-simplices are vertices of $Q_n\mathcal{C}$, where $Q_n\mathcal{C} \subset \mathrm{Fun}(NQ_n, \mathcal{C})$ consisting of ambigressive functors, that is, $\forall F \in Q_n\mathcal{C}$, F sends squares

$$\begin{tikzcd}
(i,j) \arrow[r] \arrow[d] & (k,j) \arrow[d] \\
(i,l) \arrow[r] & (k,l)
\end{tikzcd}$$

$0 \le i \le l \le j \le n$, to an ambigressive pullback.

<u>Prp 3.14</u> Q is functorial, i.e. $Q : \mathrm{Exact}_\infty \longrightarrow q\mathcal{C}at$.

cat of exact ∞-cats.

Rk 3.15 $\mathcal{C}$ exact ∞-cat. Then $Q\mathcal{C}$ has n-simplex of the form:

$$\begin{tikzcd}
 & & & X_{0n} \arrow[dl] \arrow[dr] & & & \\
 & & X_{0,n-1} \arrow[dl] \arrow[dr] & & X_{1n} \arrow[dl] \arrow[dr] & & \\
 & \iddots & & \ddots \; \iddots & & \ddots & \\
 & X_{01} \arrow[dl] \arrow[dr] & & & & X_{n-1,n} \arrow[dl] \arrow[dr] & \\
X_{00} & & X_{11} & \cdots\cdots & X_{n-1,n-1} & & X_{nn}
\end{tikzcd}$$

s.t. $\forall$ square is ambigressive.

Thm 3.16 Let $\mathcal{C}$ be an ordinary exact category (i.e. exact 1-category). Then

$$Q(N\mathcal{C}) \cong N(Q\mathcal{C})$$

That is, the classical Q-construction coincides w/ ∞-categorical Q-construction.

pf. See [Barwick & Rognes, Prop 3.11].

4. Localization, Dévissage. (No Proof)

Let A be abelian cat. Recall that we have a flow chart:

$$\begin{tikzcd}
\text{Cat} \arrow[d, "\text{Ab gp str. on Hom}"] \\
\text{Ab-enriched cat} \arrow[d, "\text{0 object, finite coproducts}"] \\
\text{Additive cat} \arrow[d, "\text{kernels \& cokernels}"] \\
\text{Preabelian cat} \arrow[d, "\substack{\text{mono} = \text{ker coker} \\ \text{epi} = \text{coker ker}}"] \\
\text{Abelian cat}
\end{tikzcd}$$

Def 4.1 Let $\mathcal{C} \subset \mathcal{A}$ be full abelian subcat of $\mathcal{A}$. $\mathcal{C}$ is thick, if it is closed under subobjects, quotients and extensions, i.e.

1) $C \in \mathcal{C}$, $B \to C$ subobject, then $B \in \mathcal{C}$.
2) $C \in \mathcal{C}$, $C \to B$ epi, then $B \in \mathcal{C}$.
3) $C, D \in \mathcal{C}$, $0 \to C \to X \to D \to 0$ exact in $\mathcal{C}$, then $X \in \mathcal{C}$.

Rk 4.2 In reality, especially in chromatic htpy theory, we use the thick subcategory in the strong sense. That is, $\mathcal{C}$ full abelian subcat of $\mathcal{A}$, and

$$0 \to C \to X \to D \to 0$$

exact in $\mathcal{C}$, then $X \in \mathcal{C} \Leftrightarrow C, D \in \mathcal{C}$.

***Off-topic 4.3** (Algebraic thick subcat theorem)

Let $\mathcal{C}_n \subset \mathcal{C}_L^{(p)}$ be the full subcat of $\mathcal{C}_L^{(p)} =$ finitely presented graded L-modules (p-localized) with a power series action (where $L = \mathbb{Z}[v_1, v_2, \cdots]$, $|v_i| = 2i$ is the Lazard ring) satisfying $v_n^{-1}M = 0$ for $M \in \mathcal{C}_L^{(p)}$. Then any thick subcat of $\mathcal{C}_L^{(p)}$ equals to $\mathcal{C}_n$ for some $n \geq 0$.

Def 4.4 Let $\mathcal{C} \subset \mathcal{A}$ be thick subcat. The quotient cat $\mathcal{A}/\mathcal{C}$ is

- obj: $\mathrm{Obj}\,\mathcal{A}$.
- mor: $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(A, B) = \operatorname*{colim}_{A', B'} \mathrm{Hom}_{\mathcal{A}}(A', B')$

where $A' \to A$, $A' \subset A$ s.t. $A/A' \in \mathcal{C}$.

$B \to B'$ epi, s.t. $\ker \in \mathcal{C}$.

Theorem 4.5 (Localization)

If $\mathcal{C} \subset \mathcal{A}$ be thick, abelian, full subcat of $\mathcal{A}$, $\mathcal{C}$ small. Then

$$BQ\mathcal{A} \xrightarrow{loc} BQ(\mathcal{A}/\mathcal{C}) \text{ is a homotopy fibration}$$

w/ htpy fiber $BQ\mathcal{C}$, where loc is induced by the natural functor on quotient cat: $loc : \mathcal{A} \to \mathcal{A}/\mathcal{C}$.

Theorem 4.6 (Dévissage)

If $\mathcal{C} \subset \mathcal{A}$ full, abelian, subcat of $\mathcal{A}$, closed under subobjects, quotients, and finite products. We also assume $\mathcal{C}$ is exact. If $\forall$ object $C \in \mathcal{A}$ has a finite filtration

$$0 = C_r \subset \cdots \subset C_1 \subset C_0 = C.$$

where $C_i \in \mathcal{A}$, $C_i / C_{i-1} \in \mathcal{C}$. Then $K\mathcal{A} \simeq K\mathcal{C}$.

Pf idea : Quillen Theorem A. See [Weibel, K-book, Theorem 4.1, Ch. 5]

- Application

Let R = Noetherian ring.

$f.g.Mod_R$ = full subcat of $RMod$ of finitely generated R-modules

Write $G_i(R) := K_i(f.g.Mod_R)$.

Prop 4.7 If I nilpotent ideal in R, then $G_i(R/I) \cong G_i(R)$.

Pf Sketch $\forall\ M \in f.g.Mod_R$, suppose $I^n = 0$,

$$0 = MI^n \subset MI^{n-1} \subset \cdots \subset MI \subset M$$

$MI^i \in f.g.Mod_R$, $MI^i / MI^{i-1} \in f.g.Mod_{R/I}$. Use dévissage.

- Connection to algebraic geometry & (quasi-)coherent sheaf ?