

Higher Algebraic K: Second Course

1. Exact categories

Def 1.1 An exact category is a pair $(\mathcal{C}, \mathcal{E})$, \mathcal{C} additive category, \mathcal{E} family of sequences in \mathcal{C} of the form (called admissible exact sequences)

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \quad (*)$$

s.t. \exists an embedding of \mathcal{C} as a full subcategory of an abelian cat \mathcal{A} s.t.

- 1) \mathcal{E} is the class of all sequences $(*)$ in \mathcal{C} which are exact in \mathcal{A} .
- 2) \mathcal{C} closed under extensions in \mathcal{A} in the sense that if $(*)$ exact sequence in \mathcal{A} , $A, C \in \mathcal{C}$, then B is iso to an object in \mathcal{C} .

Terminology f is called admissible monomorphism, g is called admissible epimorphism, if f mono (resp. g epi) in $(*)$.

Def 1.2 \mathcal{C} is closed under kernels of surjections in \mathcal{A} , if whenever a map $f: B \rightarrow C$ surjection in \mathcal{A} , then $\ker f \in \mathcal{C}$.

Def 1.3 $F: \mathcal{B} \rightarrow \mathcal{C}$ functor between exact categories is called exact, if it is additive, and preserves admissible exact sequences.

e.g. 1.4. $P(R)$ = category of finitely generated projective R -mods is an exact sequence. What's more, $K_0(R) = K_0 P(R)$, where the latter means an abelian gp w/ generators $[B]$, $B \in P(R)$, and $[B] = [A] + [C]$ for \forall s.e.s. $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $P(R)$.

e.g. 1.5. If \mathcal{C} is exact, then \mathcal{C}^{op} is exact.

e.g. 1.6. (Ko). Formalize $Ko(\mathcal{C})$ appearing in e.g. 1.4.:

Let \mathcal{C} = exact, small cat. Then we define $Ko\mathcal{C}$ to be an abelian gp w/ generators $[B]$, one for each object $B \in \mathcal{C}$, and subject to the relation $[B] = [A] + [C]$ for every admissible exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

in \mathcal{C} . Here "+" is a formal operation.

Def 1.7 Let \mathcal{P} be an additive subcat of an abelian cat \mathcal{A} . A \mathcal{P} -resolution

$P_0 \rightarrow B$ of an object $B \in \mathcal{A}$ is an exact sequence in \mathcal{A} :

$$\dots \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow B \rightarrow 0,$$

where all $P_i \in \mathcal{P}$. The number $\min\{n \in \mathbb{N} : P_i = 0, i > 0\}$ is called

\mathcal{P} -dimension of B .

Thm 1.8 (Resolution Theorem)

Let $\mathcal{P} \subset \mathcal{C} \subset \mathcal{A}$ be an inclusion of additive cats. \mathcal{A} abelian (hence gives the notion of admissible exact sequences to \mathcal{P} and \mathcal{C}). Suppose

① $\forall \text{ obj } C \text{ of } \mathcal{C}, C \text{ has a finite } \mathcal{P}\text{-dimension}$

② \mathcal{C} closed under kernels of surjections in \mathcal{A} .

Then, $\mathcal{P} \subset \mathcal{C}$ induces an iso $Ko(\mathcal{P}) \cong Ko\mathcal{C}$.

Pf Sketch. $P_0 \rightarrow C$ \mathcal{P} -resolution. Then

$$0 \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow C \rightarrow 0.$$

is exact, $n < \infty$. $[C] = \sum (-1)^i [P_i] \in Ko\mathcal{C}$. So $Ko\mathcal{P} \rightarrow Ko\mathcal{C}$

is surjective. To prove the injectivity, use the following comparison Lemma.

Lem 1.9 (Comparison)

Given $f: C \rightarrow C'$ in \mathcal{C} and a finite \mathcal{P} -resolution $P'_0 \rightarrow C'$, then

\exists a finite \mathcal{P} -resolution $P_0 \rightarrow C$ s.t. the following diagram commutes:

$$\begin{array}{ccccccccccccccc}
 0 & \rightarrow & P_m & \rightarrow & \dots & \rightarrow & P_n & \rightarrow & P_{n-1} & \rightarrow & \dots & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & C & \rightarrow & 0 \\
 & & & & & & \downarrow \varphi_n & & \downarrow \varphi_{n-1} & & & & \downarrow \varphi_1 & & \downarrow \varphi_0 & & \downarrow f & & \\
 & & & & & & 0 & \rightarrow & P'_n & \rightarrow & P'_{n-1} & \rightarrow & \dots & \rightarrow & P'_1 & \rightarrow & P'_0 & \rightarrow & C' & \rightarrow & 0
 \end{array}$$

2. Q-construction

Let \mathcal{C} be an exact, small category. We define a new category $Q\mathcal{C}$ as follows:

- Obj $Q\mathcal{C} = \text{Obj } \mathcal{C}$.

- Mor: a morphism from A to B is an equiv. class of diagrams

$$A \xleftarrow{p} C \xrightarrow{i} B$$

where $p =$ admissible epi, $i =$ admissible mono, in \mathcal{C} .

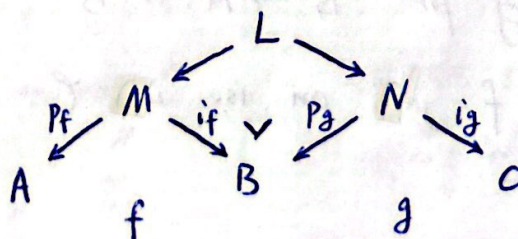
▲ The equivalent relation is given by:

$$A \xleftarrow{p} C \xrightarrow{i} B \sim A \xleftarrow{p'} C' \xrightarrow{i'} B$$

if \exists iso $\eta: C \rightarrow C'$ s.t. diagram commutes:

$$\begin{array}{ccccc}
 A & \xleftarrow{p} & C & \xrightarrow{i} & B \\
 & \swarrow p' & \downarrow \eta & \searrow i' & \\
 & & C' & &
 \end{array}$$

- Composition of two morphisms $f: A \rightarrow B$, $g: B \rightarrow C$ is the pullback



Then $g \circ f : A \leftarrow L \rightarrow C$

- (Exercise) draw diagram indicating associativity.

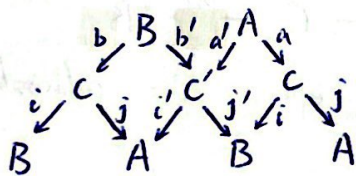
Def 2.1 Let $C \in \mathcal{C}$ be an object. An admissible subobject of C is an equivalence class of admissible monos $C' \rightarrow C$ in \mathcal{C} .

Rk 2.2 $\forall f: A \rightarrow B$ in QC , f determines a unique admissible subobject of B in \mathcal{C} . In particular, if \mathcal{C} has an zero object 0 , then $0 \rightarrow B$ in QC are 1-1 correspondent to admissible subobjects of B .

Prp 2.3 \forall iso in QC are 1-1 correspondence to iso in \mathcal{C} .

pf. $\forall f: A \cong B$ in \mathcal{C} , $f \mapsto A \xleftarrow{f^{-1}} B = B$ and $B \xleftarrow{f} A = A$.

$\forall f: A \xrightarrow{\cong} B$ in QC , f rep by $A \xleftarrow{j} C \xrightarrow{i} B$. It has an inverse $f^{-1}: B \xleftarrow{j'} C' \xrightarrow{i'} A$ s.t. $f \circ f^{-1} = id_B$. Consider the diagram $f^{-1} \circ f = id_A$



Then $j' \circ a' = i \circ a : A \rightarrow B$
 $j \circ b = i' \circ b' : B \rightarrow A$.

Check they give an iso in \mathcal{C} :

$$\begin{aligned} A \xrightarrow{i \circ a} B \xrightarrow{i' \circ b'} A &: (i' \circ b') \circ (i \circ a) = (j \circ b) \circ (i \circ a) \\ &= j \circ (b \circ i) \circ a \\ &= j \circ id_B \circ a \\ &= j \circ a \\ &= id_A. \end{aligned}$$

Similarly for $B \rightarrow A \rightarrow B$.

Thus, f gives an iso in \mathcal{C} .

Def 2.4 (Q-construction)

Consider \mathcal{C} . Taking its nerve $N\mathcal{C}$ gives a simplicial set. Denote $B\mathcal{C} = |N\mathcal{C}|$, its geometric realization. This is called the Quillen's Q-construction.

Thm 2.5 The geometric realization $B\mathcal{C}$ is a CW cpx, connected, w/ $\pi_1 B\mathcal{C} \cong K_0\mathcal{C}$. Element of $\pi_1 B\mathcal{C}$ corresponds to $[C] \in K_0\mathcal{C}$, rep. by the based loop composed of the two edges from 0 to C:

$$0 \rightarrow C \rightarrow 0$$

Pf. [Weibel, Proposition 6.2, Ch. 4].

Def 2.6 Let \mathcal{C} be small, exact category. Then $K\mathcal{C} = \text{space } \Omega B\mathcal{C}$, and we set $K_n(\mathcal{C}) = \pi_n K\mathcal{C} = \pi_{n+1}(B\mathcal{C})$, $n \geq 0$.

- By Theorem 2.5, $K_0\mathcal{C}$ is the same as the classical one.

Thm 2.7 ($+ = Q$)

For every ring R , $\Omega BQP(R) \cong K_0 R \times BGL(R)^+$.

Thus $K_n P(R) \cong K_n R$, $n \geq 0$.

- See [Weibel, Section 7, Ch. 4].

3. ∞ -categorical generalization.

Upshot An ∞ -cat \mathcal{C} is a simplicial set satisfying the inner horn extension property, i.e. $\forall n \geq 2, 0 < i < n$.

$$\begin{array}{ccc} \Lambda_j^n & \xrightarrow{s} & \mathcal{C} \\ \downarrow & \nearrow \tilde{s} & \\ \Delta^n & & \end{array}$$

\tilde{s} lifts s , $\Lambda_j^n = \Delta^n$ w/ the opposite edge of vertex j removed.

FACT 3.1 Nerve functor $N: \underline{\text{Cat}} \rightarrow \underline{\text{qCat}} \in \text{sSet}$
 category of categories category of ∞ -cats

$X \in \text{sSet}$, then $X \cong N\mathcal{C}$ for some category \mathcal{C} , iff X satisfies unique inner horn extension property.

Def 3.2 \mathcal{C} ∞ -cat, then \mathcal{C} is additive, if its homy cat $h\mathcal{C}$ is additive.

Note 3.3 If \mathcal{C}_0 ordinary (1-cat), \mathcal{C}_0 additive, then $N\mathcal{C}_0$ is additive ∞ -cat.

Def 3.4 \mathcal{C} ∞ -cat, $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{C}$ subcats containing all equivalences, morphisms in $\mathcal{C}_1, \mathcal{C}_2$ are called ingressive and egressive, respectively. A pullback square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

is ambigressive, if $X' \rightarrow Y'$ ingressive (i.e. in \mathcal{C}_1)
 $X \rightarrow Y$ egressive (i.e. in \mathcal{C}_2)

Dually, a pushout square $\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X' & \longrightarrow & Y' \end{array}$ is ambigressive,

if $X \rightarrow Y$ ingressive, $X' \rightarrow Y'$ egressive.

Call $(\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2)$ an exact ∞ -cat if

- 1) \mathcal{C} additive
- 2) ambigressive pullback = ambigressive pushout.
- 3) $\forall X \in \mathcal{C}$, $0 \rightarrow X$ is ingressive, $X \rightarrow 0$ egressive, pushouts (resp. pullbacks) of ingressive (resp. egressive) morphisms

exist, and are ingressive (resp. egressive). This condition is equivalent to say $(\mathcal{C}, \mathcal{C}_1)$ Waldenhansen ∞ -cat, $(\mathcal{C}, \mathcal{C}_2)$ co Waldenhansen ∞ -cat.

Note 3.5 If \mathcal{C} exact, then can find two subcats of NE s.t. it becomes an exact ∞ -cat. Now admissible mono = ingressive, admissible epi = egressive.

Def 3.6 \mathcal{C} ∞ -cat. \mathcal{C} is stable, if \mathcal{C} has 0, and

- ① \forall morphism has fibers (form pullback) and cofibers (form pushout)
- ② fiber sequence = cofiber sequence.

Note 3.7 \mathcal{C} stable ∞ -cat. Then $h\mathcal{C}$ is a triangulated cat. \mathcal{C} is also an exact ∞ -cat, \forall morphisms are both ingressive & egressive.

• Q-construction of exact ∞ -cat.

Goal To define $Q\mathcal{C}$. \mathcal{C} exact ∞ -cat, s.t. every morphism from X to Y is a span

$$X \xleftarrow{f} Z \xrightarrow{g} Y, \quad \text{w/ } f \text{ egressive, } g \text{ ingressive.}$$

composition = take pullback, as in the classical setting. To do that,

we need the following:

Def 3.8 Let $\varepsilon: [n] \mapsto [n]^{\text{op}} * [n] = [2n+1]$ be a functor from Δ to itself, where $*$ = join. This is called edgewise subdivision functor.

Def 3.9 $X \in \text{sSet}$: The edgewise subdivision of X is the simplicial set $\text{sd } X := \varepsilon^* X$, ε^* = endofunctor on sSet induced by ε .

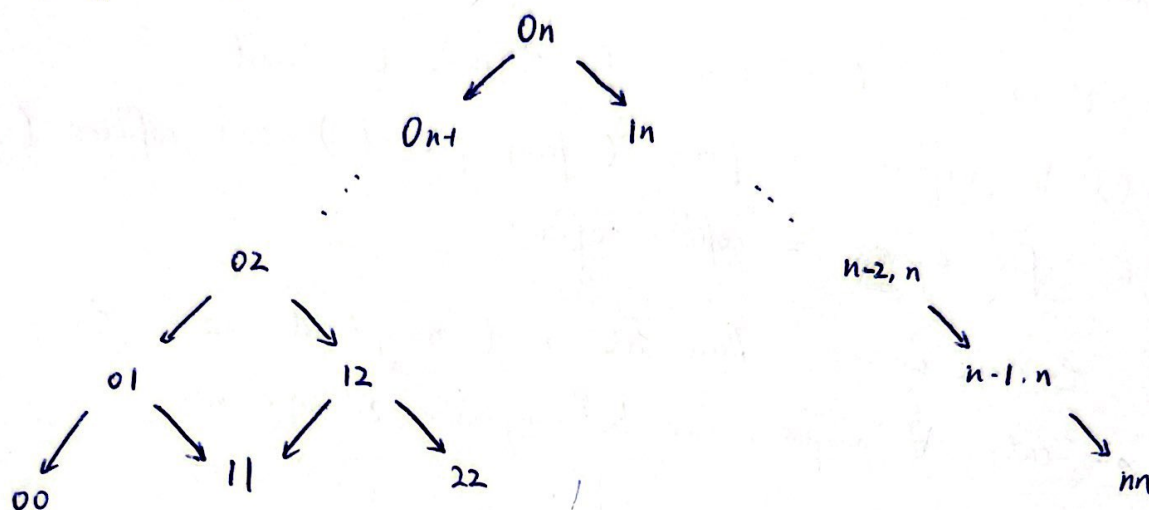
$$\text{So } (\text{sd } X)_n = (\varepsilon^* X)_n = X_{2n+1}$$

Prop 3.10 If X ∞ -cat, then $\text{sd } X$ is an ∞ -cat.

Terminology 3.11 $\text{sd } X$ is called twisted arrow ∞ -cat, if X ∞ -cat.

Construction

Let $\mathcal{Q}_n = \text{cat}$ of obj (i, j) w/ $0 \leq i \leq j \leq n$, morphism \exists only when $i' \geq i, j' \leq j, (i, j) \rightarrow (i', j')$. Then can draw the cat as $(ij \text{ short for } (i, j))$



Claim 3.12 $\text{sd } \Delta[n] = N\mathcal{Q}_n$, $\Delta[n]$ standard n -simplex

► I don't know why. One possible reason is that: \mathcal{Q}_n has a natural Segal map $\mathcal{Q}_n \mathcal{E} \rightarrow \prod_{Q_0 \mathcal{E}} Q_1 \mathcal{E}$ and forms a Segal space, which is another model for ∞ -cat that essentially the same as quasi-cat.

Def 3.13 \mathcal{C} exact ∞ -cat, denote by $\mathcal{Q}\mathcal{C}$ the ∞ -cat whose n -simplices are

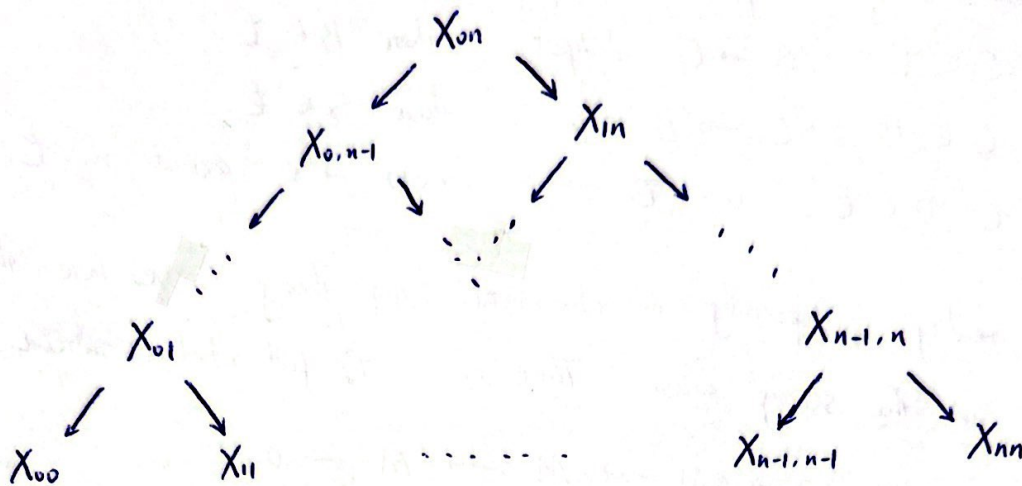
vertices of $\mathcal{Q}_n \mathcal{C}$, where $\mathcal{Q}_n \mathcal{C} \subset \text{Fun}(N\mathcal{Q}_n, \mathcal{C})$ consisting of ambigressive functors, that is, $\forall F \in \mathcal{Q}_n \mathcal{C}$, F sends squares

$$\begin{array}{ccc} (i, j) & \longrightarrow & (k, j) \\ \downarrow & & \downarrow \\ (i, \ell) & \longrightarrow & (k, \ell) \end{array}$$

$0 \leq i \leq \ell \leq j \leq n$, to an ambigressive pullback.

Prop 3.14 Q is functorial, i.e. $Q: \text{Exact}_\infty \rightarrow \text{qCat}$.
↑ cut of exact ∞ -cats.

Rk 3.15 \mathcal{C} exact ∞ -cat. Then $Q\mathcal{C}$ has n -simplex of the form:



s.t. \forall square is ambigrressive.

Thm 3.16 Let \mathcal{C} be an ordinary exact category (i.e. exact 1-category). Then

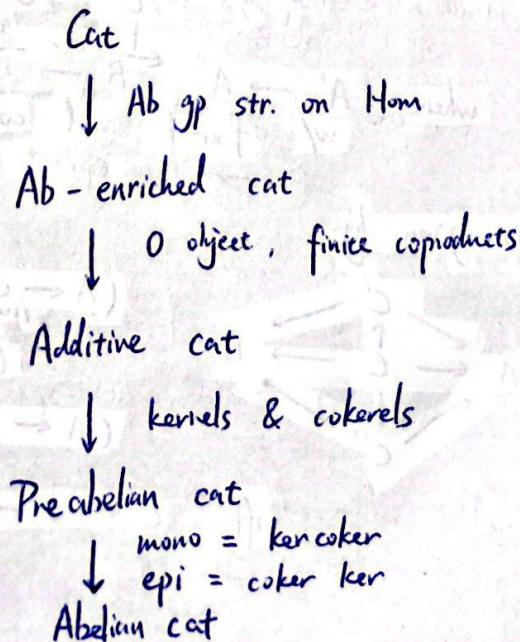
$$Q(N\mathcal{C}) \cong N(Q\mathcal{C})$$

That is, the classical Q -construction coincides w/ ∞ -categorical Q -construction.

pf. See [Barnick & Rognes, Prop 3.11].

4. Localization, Dévissage. (No Proof)

Let \mathcal{A} be abelian cat. Recall that we have a flow chart:



Def 4.1 Let $\mathcal{C} \subset \mathcal{A}$ be full abelian subcat of \mathcal{A} . \mathcal{C} is thick, if it is closed under subobjects, quotients and extensions, i.e.

- 1) $C \in \mathcal{C}$, $B \rightarrow C$ subobject, then $B \in \mathcal{C}$.
- 2) $C \in \mathcal{C}$, $C \rightarrow B$ epi, then $B \in \mathcal{C}$.
- 3) $C, D \in \mathcal{C}$, $0 \rightarrow C \rightarrow X \rightarrow D \rightarrow 0$ exact in \mathcal{C} , then $X \in \mathcal{C}$.

Rk 4.2 In reality, especially in chromatic htpy theory, we use the thick subcategory in the strong sense. That is, \mathcal{C} full abelian subcat of \mathcal{A} , and

$$0 \rightarrow C \rightarrow X \rightarrow D \rightarrow 0$$

exact in \mathcal{C} , then $X \in \mathcal{C} \Leftrightarrow C, D \in \mathcal{C}$.

* Off-topic 4.3 (Algebraic thick subcat theorem)

Let $\mathcal{C}_n \subset \mathcal{C}_L^{(p)}$ be the full subcat of $\mathcal{C}_L^{(p)}$ = finitely presented graded L -modules with a power series action (where $L = \mathbb{Z}[v_1, v_2, \dots]$, $|v_i| = 2i$ is the Lazard ring) satisfying $v_n^{-1}M = 0$ for $M \in \mathcal{C}_L^{(p)}$. Then any thick subcat of $\mathcal{C}_L^{(p)}$ equals to \mathcal{C}_n for some $n \geq 0$. (p-localized)

Def 4.4 Let $\mathcal{C} \subset \mathcal{A}$ be thick subcat. The quotient cat \mathcal{A}/\mathcal{C} is

- obj: Obj \mathcal{A} .

- mor: $\text{Hom}_{\mathcal{A}/\mathcal{C}}(A, B) = \text{colim}_{A', B'} \text{Hom}_{\mathcal{A}}(A', B')$

where $A' \rightarrow A$, $A' \subset \mathcal{A}$ s.t. $A/A' \in \mathcal{C}$.

$B \rightarrow B'$ epi, s.t. $\ker \in \mathcal{C}$.

Theorem 4.5 (Localization)

If $\mathcal{C} \subset \mathcal{A}$ be thick, abelian, full subcat of \mathcal{A} , \mathcal{C} small. Then

$BQA \xrightarrow{\text{loc}} BQ(\mathcal{A}/\mathcal{C})$ is a homotopy fibration

w/ htpy fiber BQC , where loc is induced by the natural

functor on quotient cat: $\text{loc}: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$.

Theorem 4.6 (Dévissage)

If $\mathcal{C} \subset \mathcal{A}$ full, abelian, subcat of \mathcal{A} , closed under subobjects, quotients, and finite products. We also assume \mathcal{C} is exact. If \forall object $C \in \mathcal{A}$ has a finite filtration

$$0 = C_r \subset \dots \subset C_1 \subset C_0 = C.$$

where $C_i \in \mathcal{A}$, $C_i/C_{i-1} \in \mathcal{C}$. Then $KA \cong K\mathcal{C}$.

pf idea: Quillen Theorem A. See [Weibel, K-book, Theorem 4.1, Ch. 5]

• Application

Let $R =$ Noetherian ring.

$\text{f.g. Mod}_R =$ full subcat of $R\text{Mod}$ of finitely generated R -modules

Write $G_i(R) := K_i(\text{f.g. Mod}_R)$.

Prop 4.7 If I nilpotent ideal in R , then $G_i(R/I) \cong G_i(R)$.

pf Sketch $\forall M \in \text{f.g. Mod}_R$, suppose $I^n = 0$,

$$0 = MI^n \subset MI^{n-1} \subset \dots \subset MI \subset M$$

$MI^i \in \text{f.g. Mod}_R$, $MI^i/MI^{i-1} \in \text{f.g. Mod}_{R/I}$. Use dévissage.

• Connection to algebraic geometry & (quasi-)coherent sheaf?