K-Theory and Localization

Recall...

- ∞-Q. construction of Barwick
- $K :$ Exact ∞-Cat $\longrightarrow$ Sp
- defined stable ∞-cat

Goals:

- Motivate K as stable ∞-invariant
- Additivity Property
- Applications of Additivity, e.g. descent

K of Schemes

Rmk: K in this talk will really be non-connective, as slightly cleaner version of additivity

Waldhausen $S_\bullet$

- $K(\mathcal{C}) := |wS_\bullet(\mathcal{C})|$, $\mathcal{C}$ = cat w/cofib or ∞-cat where $S_n\mathcal{C}$ =

$$\begin{tikzcd}
X(0,0) \arrow[r] & X(0,1) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & X(0,n) \arrow[d] \\
 & X(1,1) \arrow[r] & \cdots \arrow[r] \arrow[d] & X(1,n) \arrow[d] \\
 & & \cdots \arrow[r] & \cdots \arrow[d] \\
 & & & X(n,n)
\end{tikzcd}$$

{see Lurie notes}

- Prop: K via $Q_\bullet$ = via $S_\bullet$ if both defined
- Prop: $K(\mathcal{C}) \cong K(\mathrm{Stab}(\mathcal{C}))$, so consider $K : \mathrm{Cat}_\infty^{st} \to \mathrm{Sp}$

- $K(X)$
  - recall $K(R) \cong K(\mathrm{Proj}_{f.g.}(R))$, where RHS via $Q_\bullet$
  - thus, motivates $K(X) := K(\mathrm{Vect}(X))$
  - however, Thomason-Trobaugh $\Rightarrow$ $K(X) := K(\mathrm{Perf}(X))$ via $S_\bullet$
    - $\to$ 'deeper base', e.g. G-R-R and pushforward
    - $\to$ similar motivation for ∞-cat, e.g. Zariski descent

Stable ∞-Cat

Examples

- $\mathrm{Stab}(\mathcal{C}) = \lim(\cdots \to \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*)$
- $\mathrm{Sp} := \mathrm{Stab}(\mathrm{Anim})$ i.e. $\mathrm{Stab}(\mathcal{S})$
- (most) triangulated cat, e.g. $\mathcal{D}(R)$

Properties

- enriched in Sp ie

$$\begin{tikzcd} & & \mathrm{Sp} \arrow[d, "\Omega^\infty"] \\ \mathcal{C}^{op} \times \mathcal{C} \arrow[urr, dashed, "\mathrm{map}(-,-)"] \arrow[rr, "\mathrm{Map}(-,-)"'] & & \mathcal{S} \end{tikzcd}$$

{ roughly, $\mathcal{S} \leftrightsquigarrow \mathrm{Set}$, $\mathrm{Sp} \leftrightsquigarrow \mathrm{Ab}$ }

- pushout $\Longleftrightarrow$ pullback, e.g. fiber = cofiber

$\mathcal{D}(X)$

i.) $\mathcal{D}(R) := \mathrm{Anim}^{nc}(\mathrm{Mod}_A)$, or model cat, or dg cat etc

ii.) $X = \mathrm{Spec}\, A$, $\mathcal{D} : (X_{zar})^{op} \to \mathrm{Cat}_\infty$ via $\mathcal{D}(\mathcal{U}(f)) := \mathcal{D}(A[f^{-1}])$

Pf - Barr-Beck-Lurie

iii.) $\exists!\ \mathcal{D} : (\mathrm{Sch}_{zar})^{op} \to \mathrm{Cat}_\infty$ s.t. extends affine case

i.e. $\mathcal{D}(X) \simeq \lim_{S \to X} \mathcal{D}(S)$, S affine

Rmk: fails for non ∞-cat version e.g. consider

$$\begin{tikzcd} \mathrm{Spec}\, \mathbb{Z}[x,x^{-1}] \arrow[r] \arrow[d] & \mathrm{Spec}\, \mathbb{Z}[u] \arrow[d] \\ \mathrm{Spec}\, \mathbb{Z}[t] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathbb{P}^1 \end{tikzcd} \quad \text{but} \quad \begin{tikzcd} \mathcal{D}(\mathbb{P}^1) \arrow[r] \arrow[d] & \mathcal{D}(\mathbb{Z}[u]) \arrow[d] \\ \mathcal{D}(\mathbb{Z}[u]) \arrow[r] & \mathcal{D}(\mathbb{Z}[x,x^{-1}]) \end{tikzcd}$$

is not a 2-pullback since $\exists$ nonzero morphism

$\mathcal{O} \to \mathcal{O}(-2)[1]$ which go to zero in pullback

Prop: $\mathcal{D}(X)^{\heartsuit} = \mathrm{QCoh}(X)$

defn: $\mathcal{F} \in \mathcal{D}(R)$ <u>perfect</u> if $\mathcal{F} \in \mathrm{thick}(A)$ and

$\mathcal{F} \in \mathcal{D}(X)$ perfect if restriction to all affines is

Rmk: $\mathrm{Perf}(-)$ also Zariski sheaf since local condition

Categorical Prereqs

- Compacts
  - $x \in \mathcal{C}$ compact if $Map(x,-)$ commutes w/ filtered colim
  - e.g. finitely presented groups/R-modules etc
  - Rmk: triangulated/stable, STS for coproduct
  - compactly generated = jointly conservative set of compacts
  - Prop: $\mathcal{D}(X)^{\omega} = Perf(X)$

- $Idem(\mathcal{C})$
  - ie every idempotent splits
  - 1-Cats: finite lim/colim
    - e.g. $Free(R) \hookrightarrow Proj(R)$, $Open \hookrightarrow Man$
  - ∞-Cats: lim/colim but not finite
  - Prop: K is Morita invariant

- $Ind(\mathcal{C})$
  - freely adjoin filtered colim to $\mathcal{C}$; $Fun^{lex}(\mathcal{C}^{op}, S)$ for $\mathcal{C} \in Cat_{\infty}^{st}$
  - Prop: $Ind(\mathcal{C})^{\omega} = Idem(\mathcal{C})$
  - Prop: X qcqs, $Ind(Perf(X)) \cong \mathcal{D}(X)$

Categories of stable categories

- Cat_∞^{st}
  - <u>small</u>, stable ∞-cat
  - exact functors, ie preserve finite lim/colim

- Cat_∞^{perf}
  - <u>small</u>, idempotent-complete stable ∞-cat
  - exact functors, ie preserve finite lim/colim
  - via Morita invariance, $K : Cat_\infty^{perf} \longrightarrow Sp$

- Pr_{St}^{L}
  - presentable stable ∞-cat w/ left adjoint functors

    ↳ adjoint functor theorem
  - Prop: $Cat_\infty^{perf} \xrightarrow[\sim]{Ind} Pr_{St}^{L,\omega}$

Karoubi sequences

Verdier quotients

- $\mathcal{D} \subseteq \mathcal{C}$ stable, then $\mathcal{C}/\mathcal{D} = \mathcal{C}[W^{-1}]$, $W = \{c \xrightarrow{f} c' \mid \mathrm{cof}(f) \in \mathcal{D}\}$
- for $X, Y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}/\mathcal{D}}(\overline{X}, \overline{Y}) = \mathrm{colim}_{z \in \mathcal{D}/Y} \mathrm{Map}_{\mathcal{C}}(X, \mathrm{cof}(z \to Y))$

defn: A sequence $\mathcal{D} \xrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{E}$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is

<u>Karoubi</u> if the following hold:

i.) $p \circ i \simeq 0$

ii.) i is fully faithful

iii.) $\mathcal{E} \simeq \mathrm{Idem}(\mathcal{C}/\mathcal{D})$

Prop: let $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$, $\mathcal{D} \subseteq \mathcal{C}$ stable, $p : \mathcal{C} \to \mathcal{C}/\mathcal{D}$, then

$\ker L \to \mathrm{Ind}(\mathcal{C}) \xrightarrow{\mathrm{Ind}(p) = L} \mathrm{Ind}(\mathcal{C}/\mathcal{D})$ is a fiber ie SES

seqn in Pr^L_{st} w/ fully faithful right adjoint w/ $\ker L \cong \mathrm{Ind}(\mathcal{D})$

Cor: (Thomason-Neeman localization)

$\mathcal{D} \to \mathcal{C} \to \mathcal{C}/\mathcal{D}$ Karoubi $\Leftrightarrow$ $\mathrm{Ind}(-)$ is fiber seqn in Pr^L_{st}

Rmk: Bousfield Localizations

Additivity Thm

'Additivity' Thm:

$K : \mathrm{Cat}_\infty^{perf} \to Sp$ sends Karoubi seqn to fiber seqn.

{ie K is a 'localizing invariant'; e.g. THH}

Pf: · Waldhausen 1985, Blumberg-Gepner-Tabuada 2013

· Barwick 2013

· Hebestreit-Lachmann-Steimle 2023

Universality Thm

K is the universal localizing invariant, e.g. it is corepresentable in noncommutative motives via

$\mathrm{map}(\mathcal{U}(Sp^\omega), \mathcal{U}(A)) \simeq K(A)$

Pf: · " "

General Strategy

i.) show $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ is a Bousfield localization

ii.) show kernel is compactly generated

iii.b.) if not, use Efimov K-theory

iii.) apply additivity

Applications - Descent

I.) Base Case

- consider basic Zariski open i.e.
- let A be comm ring, $f \in A$
- have $\mathcal{D}(A) \underset{j = - \otimes A[f^{-1}]}{\overset{\text{fully faithful}}{\rightleftarrows}} \mathcal{D}(A[f^{-1}])$

Prop: $\ker j^{\omega} \to \mathrm{Perf}(A) \to \mathrm{Perf}(A[f^{-1}])$

Pf - $\mathcal{D}(A), \mathcal{D}(A[f^{-1}])$ compactly generated

- WTS $\ker j$ compactly generated
- note $\ker j = \mathcal{D}(A \text{ on } (f))$
- $k(f) = \mathrm{cofib}(A \xrightarrow{f} A) \in \mathcal{D}(A \text{ on } (f))$ is compact
- STS $\forall M \in \mathcal{D}(A \text{ on } (f))$, $\mathrm{Hom}(k(f), M) = 0 \Rightarrow M = 0$
- $\mathrm{Hom}(k(f), M) = \mathrm{fib}(M \xrightarrow{f} M) = 0 \Rightarrow M \underset{\sim}{\xrightarrow{f}} M \Rightarrow M = 0$
- recognize $\mathrm{Perf}(A) = \mathcal{D}(A)^{\omega}$ + apply lemma

Cor: we have a fiber sequence in Sp,

$$K(\mathrm{Perf}(A \text{ on } (f)) \longrightarrow K(A) \longrightarrow K(A[f^{-1}])$$

Applications – Descent

II.) Zariski Descent

Thm: (Neeman; Bondal – van der Bergh)

let X qcqs scheme, $\mathcal{U} \subseteq X$ qc open, $Z := X \setminus \mathcal{U}$, then

$\mathcal{D}_{qc}(X \text{ on } Z) \to \mathcal{D}_{qc}(X) \to \mathcal{D}_{qc}(\mathcal{U})$ is a SES in Pr^{L}_{st} and

each category belongs to $\mathrm{Pr}^{L,\omega}_{st}$

Cor: $\mathrm{Perf}(X \text{ on } Z) \to \mathrm{Perf}(X) \to \mathrm{Perf}(\mathcal{U})$ is Karoubi

Idea – induct on covers since qcqs + direct sum; Koszul complex

Prop: $K : \mathrm{Sch}_{qcqs}^{op} \to \mathrm{Sp}$ satisfies Zariski descent

Pf i.) Zariski descent $\Leftrightarrow$ Mayer-Vietoris i.e. WTS

$X = \mathcal{U} \cup \mathcal{V}$, qc opens $\Rightarrow$

$$\begin{tikzcd} K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & K(\mathcal{U}) \arrow[d] \\ K(\mathcal{V}) \arrow[r] & K(\mathcal{U} \cap \mathcal{V}) \end{tikzcd}$$

ii.) let $Z = X \setminus \mathcal{U}'$, $Z' = V \setminus \mathcal{U} \cap \mathcal{V}$ then we have

$$\begin{tikzcd} \mathrm{Perf}(X \text{ on } Z) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Perf}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Perf}(\mathcal{U}) \arrow[d] \\ \mathrm{Perf}(\mathcal{V} \text{ on } Z) \arrow[r] & \mathrm{Perf}(\mathcal{V}) \arrow[r] & \mathrm{Perf}(\mathcal{W}) \end{tikzcd}$$

via Zariski descent + limits commute

iii.)

$$\begin{tikzcd} K_Z(X) \arrow[r] \arrow[d, "\wr"'] & K(X) \arrow[r] \arrow[d] & K(\mathcal{U}) \arrow[d] \\ K_{Z'}(V) \arrow[r] & K(\mathcal{V}) \arrow[r] & K(\mathcal{U} \cap \mathcal{V}) \end{tikzcd}$$

iso on fibers $\Rightarrow$ Cartesian {details next pt}

Applications - Descent

III.) Nisnevich Descent

defn: a Nisnevich cover $\{\mathcal{U}_i \xrightarrow{f_i} X\}$ if each f_i is étale and $\forall \operatorname{Spec} k \to X$, $\exists i$ s.t. we have a lift

$$\begin{tikzcd} & \mathcal{U}_i \arrow[d] \\ \operatorname{Spec} k \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

Prop: K-theory satisfies Nisnevich descent

i.) STS that all Nisnevich squares i.e.

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{U} \arrow[d, "\pi"] \\ V \arrow[r, "j"] & X = \mathcal{U} \cup V \end{tikzcd}$$

where j is an open immersion and π is étale and single-sheeted on $X \setminus V$, ie $\pi^{-1}(X/V) \cong X/V$ is sent to a homotopy pullback

ii.) Consider

$$\begin{tikzcd} K(X) \arrow[r] \arrow[d, "e"] & K(V) \arrow[d, "\psi"] \\ K(\mathcal{U}) \arrow[r] & K(W) \end{tikzcd}$$

and $\delta : K(X) \to F$, F the pullback and let $Z := X \setminus \mathcal{U}$

iii.) $\operatorname{Perf}(X \text{ on } Z) \to \operatorname{Perf}(X) \to \operatorname{Perf}(\mathcal{U})$
$\quad \wr\downarrow$
$\operatorname{Perf}(V \text{ on } Z) \to \operatorname{Perf}(V) \to \operatorname{Perf}(W)$

and thus via additivity of K,

$$\begin{tikzcd} \operatorname{fib}(\varphi) \arrow[r] \arrow[d, "f"] & K(X) \arrow[d, "\delta"] \arrow[dr, dashed, bend left, "\varphi"] & \\ \operatorname{fib}(\psi) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & K(V) \arrow[d, "\psi"] \\ 0 \arrow[r] & K(\mathcal{U}) \arrow[r] & K(W) \end{tikzcd}$$

note lower rectangle pullback, hence so is lower left square $\Rightarrow$ left rectangle, thus upper left pullback $\Rightarrow$ by stability, a pushout, so f equivalence $\Rightarrow \delta$ equivalence

Applications - $K(\mathbb{P}^2_R)$

defn: let $\mathcal{C}$ be a stable ∞-cat, $\mathcal{D} \subseteq \mathcal{C}$ stable full subcat.

$x \in \mathcal{C}$ is left orthogonal to $\mathcal{D}$ if $\mathrm{Map}(x, d) \simeq * \;\; \forall d \in \mathcal{D}$.

let ${}^\perp\mathcal{D}$ and $\mathcal{D}^\perp$ denote subcat of left/right orthog, resp.

defn: let $\mathcal{C}(0), \cdots, \mathcal{C}(-n)$ be full stable subcat s.t.

i.) $\forall i > j$, $\mathcal{C}(i) \subseteq {}^\perp \mathcal{C}(j)$

ii.) $\mathcal{C}$ is generated by $\mathcal{C}(0), \cdots, \mathcal{C}(-n)$ under finite limits and colimits. then we say that $\langle \mathcal{C}(0), \cdots, \mathcal{C}(-n) \rangle$ is a <u>Semi-orthogonal decomposition</u> of $\mathcal{C}$

Prop: let $\mathcal{C}_{\leq m}$ be the full stable subcat generated by $\mathcal{C}(-m) \cup \cdots \cup \mathcal{C}(-n)$, $0 \leq m \leq n$. then we have a split Karoubi seqn $\mathcal{C}_{\leq -m-1} \hookrightarrow \mathcal{C}_{\leq -m} \rightarrow \mathcal{C}(-m)$

Thm: (Beilinson) $D^b(\mathbb{P}^2_R) = \langle \mathcal{O}_x, \mathcal{O}_x(-1) \rangle$

Cor: $K(\mathbb{P}^2_R) \cong K(R) \oplus K(R)$