

K-Theory and Localization

Recall...

- ∞ -Q. construction of Barwick
- $K: \text{Exact } \infty\text{-Cat} \rightarrow \text{Sp}$
- defined stable ∞ -cat

Goals:

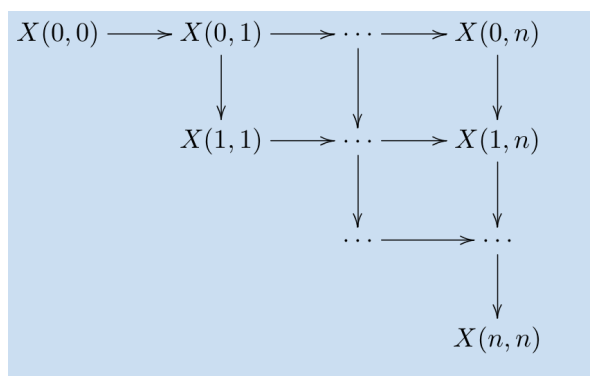
- Motivate K as stable ∞ -invariant
- Additivity Property
- Applications of Additivity, e.g. descent

K of Schemes

Rmk: K in this talk will really be non-connective, as slightly cleaner version of additivity

Waldhausen S

- $K(\mathcal{C}) := |wS(\mathcal{C})|$, $\mathcal{C} = \text{cat w/cofib or } \infty\text{-cat}$ where $S_n \mathcal{C} =$



{see Lurie notes}

- Prop: K via $Q. =$ via $S.$ if both defined
- Prop: $K(\mathcal{C}) \cong K(\text{Stab}(\mathcal{C}))$, so consider $K: \text{Cat}_{\infty}^{\text{str}} \rightarrow \text{Sp}$

• $K(X)$

- recall $K(R) \cong K(\text{Proj}_{f.g.}(R))$, where RHS via $Q.$
- thus, motivates $K(X) := K(\text{Vect}(X))$
- however, Thomason-Trobaugh $\Rightarrow K(X) := K(\text{Perf}(X))$ via $S.$
 - \rightarrow 'deeper base', e.g. G-R-R and pushforward
 - \rightarrow similar motivation for $\infty\text{-cat}$, e.g. Zariski descent

Stable ∞ -Cat

Examples

- $\text{Stab}(\mathcal{C}) = \lim(\dots \rightarrow \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*)$
- $\text{Sp} := \text{Stab}(\text{Anim})$ i.e. $\text{Stab}(S)$
- (most) triangulated cat, e.g. $\mathcal{D}(\mathbb{R})$

Properties

- enriched in Sp ie

$$\begin{array}{ccc} & \xrightarrow{\text{map}(-,-)} & \text{Sp} \\ \mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{\text{Map}(-,-)} & \downarrow \Omega^\infty \\ & & S \end{array}$$

$$\{\text{roughly, } S \rightleftarrows \text{Set}, \text{Sp} \rightleftarrows \text{Ab}\}$$

- pushout \iff pullback, e.g. fiber = cofiber

$\mathcal{D}(X)$

zi.) $\mathcal{D}(R) := \text{Anim}^{\text{nc}}(\text{Mod}_A)$, or model cat, or dg cat etc

zii.) $X = \text{Spec } A$, $\mathcal{D}: (X_{\text{Zar}})^{\text{op}} \rightarrow \text{Cat}_{\infty}$ via $\mathcal{D}(U(f)) := \mathcal{D}(A[f^{-1}])$

Pf - Barr-Beck-Lurie

ziii.) $\exists! \mathcal{D}: (\text{Sch}_{\text{Zar}})^{\text{op}} \rightarrow \text{Cat}_{\infty}$ s.t. extends affine case

i.e. $\mathcal{D}(X) \cong \lim_{S \rightarrow X} \mathcal{D}(S)$, S affine

Rmk: fails for non co-cat version e.g. consider

$$\begin{array}{ccc}
 \text{Spec } \mathbb{Z}[x, x^{-1}] & \rightarrow & \text{Spec } \mathbb{Z}[u] & & \mathcal{D}(\mathbb{P}^1) & \longrightarrow & \mathcal{D}(\mathbb{Z}[u]) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \text{Spec } \mathbb{Z}[t] & \xrightarrow{\quad r \quad} & \mathbb{P}^1 & & \mathcal{D}(\mathbb{Z}[u]) & \longrightarrow & \mathcal{D}(\mathbb{Z}[x, x^{-1}])
 \end{array}$$

but

is not a 2-pullback since \exists nonzero morphism

$\mathcal{O} \rightarrow \mathcal{O}(-2)[1]$ which go to zero in pullback

Prop: $\mathcal{D}(X)^{\heartsuit} = \text{QCoh}(X)$

defn: $F \in \mathcal{D}(R)$ perfect if $F \in \text{thick}(A)$ and

$F \in \mathcal{D}(X)$ perfect if restriction to all affines is

Rmk: $\text{Perf}(-)$ also Zariski sheaf since local condition

Categorical Prereqs

- **Compacts**
 - $x \in \mathcal{C}$ compact if $\text{Map}(x, -)$ commutes w/ filtered colim
 - e.g. finitely presented groups/R-modules etc
 - Rmk: triangulated/stable, STS for coproduct
 - compactly generated = jointly conservative set of compacts
 - Prop: $\mathcal{D}(X)^\omega = \text{Perf}(X)$
- **Idem(\mathcal{C})**
 - ie every idempotent splits
 - 1-Cats: finite lim/colim
 - e.g. $\text{Free}(R) \leftrightarrow \text{Proj}(R)$, $\text{Open} \leftrightarrow \text{Man}$
 - ω -Cats: lim/colim but not finite
 - Prop: K is Morita invariant
- **Ind(\mathcal{C})**
 - freely adjoin filtered colim to \mathcal{C} ; $\text{Fun}^{\text{lex}}(\mathcal{C}^{\text{op}}, S)$ for $\mathcal{C} \in \text{Cat}_{\omega}^{\text{st}}$
 - Prop: $\text{Ind}(\mathcal{C})^\omega = \text{Idem}(\mathcal{C})$
 - Prop: X qcqs, $\text{Ind}(\text{Perf}(X)) \cong \mathcal{D}(X)$

Categories of stable categories

• $\text{Cat}_{\infty}^{\text{st}}$

- small, stable ∞ -cat
- exact functors, i.e. preserve finite lim/colim

• $\text{Cat}_{\infty}^{\text{perf}}$

- small, idempotent-complete stable ∞ -cat
- exact functors, i.e. preserve finite lim/colim
- via Morita invariance, $K: \text{Cat}_{\infty}^{\text{perf}} \rightarrow \text{Sp}$

• $\text{Pr}_{\text{St}}^{\text{L}}$

- presentable stable ∞ -cat w/left adjoint functors

↳ adjoint functor theorem

- Prop: $\text{Cat}_{\infty}^{\text{perf}} \xrightarrow[\sim]{\text{Ind}} \text{Pr}_{\text{St}}^{\text{L}, \omega}$

Karoubi sequences

Verdier quotients

- $\mathcal{D} \subseteq \mathcal{C}$ stable, then $\mathcal{C}/\mathcal{D} = \mathcal{C}[\mathcal{W}^{-1}]$, $\mathcal{W} = \{c \xrightarrow{f} c' \mid \text{cof}(f) \in \mathcal{D}\}$
- for $X, Y \in \mathcal{C}$, $\text{Map}_{\mathcal{C}/\mathcal{D}}(\bar{X}, \bar{Y}) = \text{colim}_{Z \in \mathcal{D}/Y} \text{Map}_{\mathcal{C}}(X, \text{cof}(Z \rightarrow Y))$

defn: A sequence $\mathcal{D} \xrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{E}$ in $\text{Cat}_{\text{co}}^{\text{perf}}$ is

Karoubi if the following hold:

- zi.) $p \circ i = 0$
- zii.) i is fully faithful
- iii.) $\mathcal{E} \cong \text{Idem}(\mathcal{C}/\mathcal{D})$

Prop: let $\mathcal{C} \in \text{Cat}_{\text{co}}^{\text{perf}}$, $\mathcal{D} \subseteq \mathcal{C}$ stable, $p: \mathcal{C} \rightarrow \mathcal{C}/\mathcal{D}$, then
 $\ker L \rightarrow \text{Ind}(\mathcal{C}) \xrightarrow{\text{Ind}(p) = L} \text{Ind}(\mathcal{C}/\mathcal{D})$ is a fiber ie SES

segn in Pr_{st}^L w/ fully faithful right adjoint w/ $\ker L \cong \text{Ind}(\mathcal{D})$

Cor: (Thomason-Neeman localization)

$\mathcal{D} \rightarrow \mathcal{C} \rightarrow \mathcal{C}/\mathcal{D}$ Karoubi $\Leftrightarrow \text{Ind}(-)$ is fiber segn in Pr_{st}^L

Rmk: Bousfield Localizations

Additivity Thm

'Additivity' Thm:

$K: \text{Cat}_{\infty}^{\text{perf}} \rightarrow \text{Sp}$ sends Karoubi segm to fiber segm.

{ie K is a 'localizing invariant'; e.g. THH}

Pf: • Waldhausen 1985, Blumberg-Gepner-Ekueeda 2013

• Barwick 2013

• Hebestreit-Luchmann-Strömle 2023

Universality Thm

K is the universal localizing invariant, e.g. it is

corepresentable in noncommutative motives via

$$\text{map}(\mathcal{U}(\text{Sp}^{\omega}), \mathcal{U}(A)) \cong K(A)$$

Pf: • " "

General Strategy

z.) show $\text{Ind}(\mathcal{C}) \rightarrow \text{Ind}(\mathcal{D})$ is a Bousfield localization

zz.) show kernel is compactly generated

zz.b.) if not, use Efimov K-theory

zzz.) apply additivity

Applications - Descent

I.) Base Case

• consider basic Zariski open i.e.

• let A be comm ring, $f \in A$

• have $\mathcal{D}(A) \xrightarrow[\substack{\text{fully faithful} \\ j = - \otimes_{A[f^{-1}]}^{\perp}}]{\perp} \mathcal{D}(A[f^{-1}])$

Prop: $\text{Ker } j^{\omega} \rightarrow \text{Perf}(A) \rightarrow \text{Perf}(A[f^{-1}])$

Pf - $\mathcal{D}(A), \mathcal{D}(A[f^{-1}])$ compactly generated

- WTS $\text{Ker } j$ compactly generated
- note $\text{Ker } j = \mathcal{D}(A_{\text{on}}(f))$
- $k(f) = \text{cofib}(A \xrightarrow{f} A) \in \mathcal{D}(A_{\text{on}}(f))$ is compact
- STS $\forall M \in \mathcal{D}(A_{\text{on}}(f)), \text{Hom}(k(f), M) = 0 \Rightarrow M = 0$
- $\text{Hom}(k(f), M) = \text{fib}(M \xrightarrow{f} M) = 0 \Rightarrow M \xrightarrow{f} M \Rightarrow M = 0$
- recognize $\text{Perf}(A) = \mathcal{D}(A)^{\omega}$ + apply lemma

Cor: we have a fiber sequence in Sp ,

$$K(\text{Perf}(A_{\text{on}}(f))) \rightarrow K(A) \rightarrow K(A[f^{-1}])$$

Applications - Descent

II.) Zariski Descent

Thm: (Neeman; Bondal - van der Bergh)

let X qcqs scheme, $\mathcal{U} \subseteq X$ qc open, $Z := X \setminus \mathcal{U}$, then
 $\mathcal{D}_{qc}(X \text{ on } Z) \rightarrow \mathcal{D}_{qc}(X) \rightarrow \mathcal{D}_{qc}(\mathcal{U})$ is a SES in Pr_{st}^L and
 each category belongs to $\text{Pr}_{st}^{L,W}$

Cor: $\text{Perf}(X \text{ on } Z) \rightarrow \text{Perf}(X) \rightarrow \text{Perf}(\mathcal{U})$ is Karoubi

Idea - induct on covers since qcqs + direct sum; Koszul complex

Prop: $K: \text{Sch}_{qcqs}^{\text{op}} \rightarrow \text{Sp}$ satisfies Zariski descent

Pf i.) Zariski descent \Leftrightarrow Mayer-Vietoris i.e. WTS

$$X = \mathcal{U} \cup \mathcal{V}, \text{ qc opens} \Rightarrow \begin{array}{ccc} K(X) & \longrightarrow & K(\mathcal{U}) \\ \downarrow \lrcorner & & \downarrow \\ K(\mathcal{V}) & \longrightarrow & K(\mathcal{U} \cap \mathcal{V}) \end{array}$$

ii.) let $Z = X \setminus \mathcal{U}'$, $Z' = \mathcal{V} \setminus \mathcal{U} \cap \mathcal{V}$ then we have

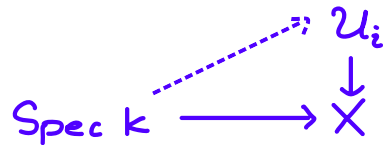
$$\begin{array}{ccccc} \text{Perf}(X \text{ on } Z) & \longrightarrow & \text{Perf}(X) & \longrightarrow & \text{Perf}(\mathcal{U}) & \text{via Zariski descent +} \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow & \text{limits commute} \\ \text{Perf}(\mathcal{V} \text{ on } Z) & \longrightarrow & \text{Perf}(\mathcal{V}) & \longrightarrow & \text{Perf}(\mathcal{U}) & \end{array}$$

$$\begin{array}{ccccc} \text{iii.) } K_Z(X) & \longrightarrow & K(X) & \longrightarrow & K(\mathcal{U}) & \text{iso on fibers} \Rightarrow \\ \downarrow \lrcorner & & \downarrow & & \downarrow & \\ K_{Z'}(\mathcal{V}) & \longrightarrow & K(\mathcal{V}) & \longrightarrow & K(\mathcal{U} \cap \mathcal{V}) & \text{Cartesian \{details next pt\}} \end{array}$$

Applications - Descent

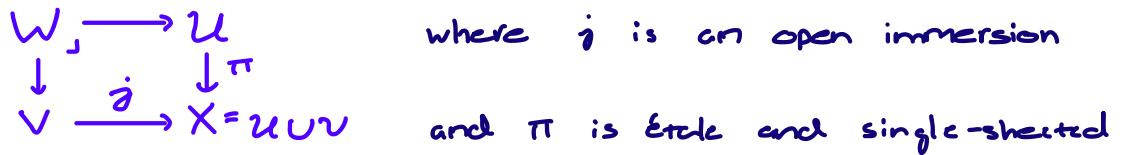
III.) Nisnevich Descent

defn: a Nisnevich cover $\{\mathcal{U}_i \rightarrow X\}$ if each f_i is étale and $\forall \text{Spec } k \rightarrow X, \exists i$ s.t. we have a lift



Prop: K-theory satisfies Nisnevich descent

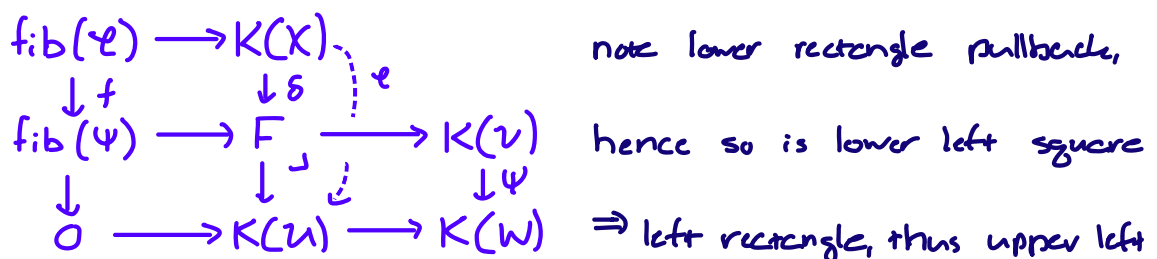
i.) STS that all Nisnevich squares i.e.



on $X \setminus V$, ie $\pi^{-1}(X \setminus V) \cong X \setminus V$ is sent to a homotopy pullback

ii.) Consider $\begin{array}{ccc} K(X) & \longrightarrow & K(U) \\ \downarrow e & & \downarrow \psi \\ K(V) & \longrightarrow & K(W) \end{array}$ and let $Z := X \setminus U$

iii.) $\begin{array}{ccc} \text{Perf}(X \text{ on } Z) & \longrightarrow & \text{Perf}(X) \longrightarrow \text{Perf}(U) \\ \downarrow s & & \\ \text{Perf}(V \text{ on } Z) & \longrightarrow & \text{Perf}(V) \longrightarrow \text{Perf}(W) \end{array}$ and thus via additivity of K ,



\Rightarrow left rectangle, thus upper left pullback \Rightarrow by stability, a pushout, so f equivalence $\Rightarrow \delta$ equivalence

Applications - $K(\mathbb{P}_R^2)$

defn: let \mathcal{C} be a stable ∞ -cat, $\mathcal{D} \subseteq \mathcal{C}$ stable full subcat.

$x \in \mathcal{C}$ is left orthogonal to \mathcal{D} if $\text{Map}(x, d) \simeq * \forall d \in \mathcal{D}$.

let ${}^\perp \mathcal{D}$ and \mathcal{D}^\perp denote subset of left/right orthog, resp.

defn: let $\mathcal{C}(0), \dots, \mathcal{C}(-n)$ be full stable subcat s.t.

i.) $\forall i > j, \mathcal{C}(i) \subseteq {}^\perp \mathcal{C}(j)$

ii.) \mathcal{C} is generated by $\mathcal{C}(0), \dots, \mathcal{C}(-n)$ under finite limits

and colimits. then we say that $\langle \mathcal{C}(0), \dots, \mathcal{C}(-n) \rangle$

is a Semi-orthogonal decomposition of \mathcal{C}

Prop: let $\mathcal{C}_{\leq m}$ be the full stable subcat generated

by $\mathcal{C}(-m) \cup \dots \cup \mathcal{C}(-n)$, $0 \leq m \leq n$. then we have

a split Karoubi seqn $\mathcal{C}_{\leq -m-1} \hookrightarrow \mathcal{C}_{\leq -m} \rightarrow \mathcal{C}(-m)$

Thm: (Beilinson) $D^b(\mathbb{P}_R^2) = \langle \mathcal{O}_X, \mathcal{O}_X(-1) \rangle$

Cor: $K(\mathbb{P}_R^2) \cong K(R) \oplus K(R)$