CHROMATIC HOMOTOPY THEORY

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Dec 29, 2022

Contents

1. Lecture 1 (Shuhan Zheng): 09/27/2022

Let $\xi = \{p : E \to B\}$ be a complex vector bundle with a Hermitian metric over B. The disk bundle $D(\xi)$ consists of all vectors v with $|v| \leq 1$, and the sphere bundle $S(\xi)$ consists of of all vectors v with $|v| = 1$. The **Thom space** of ξ is defined to be $Th(\xi) \coloneqq D(\xi)/S(\xi)$. If B is compact, then $Th(\xi)$ is the one-point compactification of E. Consider the complex bordism spectrum MU, which is given by

$$\begin{cases} MU_{2n} = MU(n) = Th(\gamma_n^{\mathbb{C}} \to BU(n)), \\ MU_{2n+1} = \Sigma MU_{2n}. \end{cases}$$

where Σ satisfies

$$\Sigma MU_{2n+1} = \Sigma^2 MU(n) \xrightarrow{\text{id}} MU(n+1) = MU_{2n+2}.$$

Another way to see it is to define the spectra $MU(n) = \Sigma^{\infty-2n} BU(n)/BU(n-1)$, which form a direct system

$$MU(0) \to MU(1) \to MU(2) \to \cdots .$$

In this case, the (homotopy) colimit of this sequence is called the complex bordism spectrum and is denoted by MU.

Example 1.1. In the second approach, $MU(0) \simeq \mathbb{S}$, the sphere spectrum. $MU(1)$ is the desuspension $\Sigma^{\infty-2}\mathbb{CP}^\infty$ of $\mathbb{CP}^\infty = BU(1)$.

Denote the related (co)homology theory rel. X by $MU_*(X)$ (resp. $MU^*(X)$). If X is specified or is unimportant, we might abbreviate the (co)homology theory to MU_* (resp. MU^*).

In Example 1.1, the inclusion $\Sigma^{\infty-2}\mathbb{CP}^\infty \simeq MU(1) \to MU$ determines a class $t \in \widetilde{MU}^2(\mathbb{CP}^\infty)$. This t is actually a complex orientation of MU.

Proposition 1.2. *(1) $MU^*(\mathbb{CP}^\infty) \cong MU^*[[y]]$.*

(2) ([2, Lecture 6], universal property of MU) Let E be a commutative ring spectrum, and $t \in \widetilde{MU}^2(\mathbb{CP}^\infty)$ be the complex orientation described as above. Then there exists a 1-1 bijection:

$$\hom(MU, E) \leftrightarrow \{\text{ complex orientations of } E\},$$

where every $\phi : MU \to E$ corresponds to a complex orientation $\phi(t)$ of E.

The complex bordism spectrum MU is the universal complex-oriented cohomology theory. The proposition suggests that to understand the complex-oriented cohomology theory and related formal group laws, we ought to focus on the complex bordism spectrum MU.

Now we shift our attention to an algebraic viewpoint. Let m be the multiplication structure on $\mathbb{CP}^\infty$:

$$\begin{aligned} \mathbb{CP}^\infty \times \mathbb{CP}^\infty &\xrightarrow{m} \mathbb{CP}^\infty \\ ([x],[y]) &\longmapsto [xy], \end{aligned}$$

where $[x]$ denotes a line passing through x in the underlying vector space $\mathbb{C}[x]$ of $\mathbb{CP}^\infty$. Here we regard $\mathbb{CP}^\infty$ as the projectivization of the polynomial ring $\mathbb{C}[x]$ as a vector space over $\mathbb{C}$. m induces

$$MU^*(\mathbb{CP}^\infty) \xrightarrow{m^*} MU^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty).$$

By Proposition 1.2 (1), this amounts to

$$MU^*[[y]] \to MU^*[[y \otimes 1, 1 \otimes y]], \tag{1.1}$$

where $\dim y = 2$. One can write RHS to be $MU^*[[x_1, x_2]]$, where $x_1 = y \otimes 1$ and $x_2 = 1 \otimes y$. (1.1) defines a **formal group law** (**abbr. fgl**) as follows:

$$m^*(y) = \sum a_{i,j} x_1^i x_2^j = \mathcal{F}_{MU}(x_1, x_2),$$

where $\mathcal{F}_{MU}$ satisfies $\mathcal{F}_{MU}(x,y) = \mathcal{F}_{MU}(y,x)$ and $\mathcal{F}_{MU}(x, \mathcal{F}_{MU}(y,z)) = \mathcal{F}_{MU}(\mathcal{F}_{MU}(x,y),z)$. In general,

Definition 1.3. A fgl over a commutative unital ring R is a power series $F(x,y) \in R[[x,y]]$ satisfying the following three conditions:

(1) $F(x,0) = F(0,x) = x$,
(2) $F(x,y) = F(y,x)$,
(3) $F(F(x,y),z) = F(x, F(y,z))$.

Example 1.4. (1) Additive fgl: $F(x,y) = x + y$.

(2) Multiplicative fgl: $F(x,y) = (x+1)(y+1) - 1 = x + y + xy$.

Any fgl $F(x, y) \in R[[x, y]]$ can be written as a formal sum $\sum c_{i,j}x^iy^j$ for some $c_{i,j} \in R$, where by definition $c_{i,j}$ satisfies

(1) $c_{i,0} = c_{0,i} = 1$ when $i = 1$, and $c_{i,0} = c_{0,i} = 0$ when $i \neq 1$,
(2) $c_{i,j} = c_{j,i}$,
(3) some complication relation determined by (3) in the definition.

So giving a fgl over R is equivalent to giving a collection of $c_{i,j} \in R$ satisfying the previous relations.

Definition 1.5. The commutative ring

$$L \coloneqq \mathbb{Z}[c_{i,j}]/Q$$

is called the **Lazard ring**, where $c_{i,j} \in R$ and Q is the ideal of $\mathbb{Z}[c_{i,j}]$ generated by those whose coefficients satisfies the relations mentioned above.

Theorem 1.6 (Lazard, part I)**.** *The Lazard ring L is isomorphic to a polynomial ring $\mathbb{Z}[t_1, t_2, \cdots]$, where each t_i has degree $2i$.*

Theorem 1.7 (Lazard, part II)**.** *There is a universal fgl defined over L of the form*

$$G(x, y) = \sum_{i,j} a_{i,j}x^iy^j,$$

where $a_{i,j} \in L$, such that for any other fgl F over R there is a unique ring homomorphism θ from L to R with

$$F(x, y) = \sum_{i,j} \theta(a_{i,j})x^iy^j.$$

By Lazard's theorem, the fgl over MU^*, i.e. $\mathcal{F}_{MU}(x_1, x_2) = m^*(y)$, induces a unique ring homomorphism θ from L to MU^*:

Theorem 1.8 (Quillen)**.** *There is a natural isomorphism*

$$\theta : L \to MU^*.$$

That is, the fgl associated with the complex cobordism is the universal one.

Let R be a ring. Denote $FGL(R)$ = all fgls over R. Let $F, G \in FGL(R)$. A morphism between F and G is given by

$$f(F(x, y)) = G(f(x), f(y)),$$

where $f \in \mathfrak{m} = \{\alpha \in R[[t]] : \alpha(0) = 0\}$.

Definition 1.9. f is an **isomorphism** if $f'(0)$ is invertible in R. If $f'(0) = 1$, then f is called a **strict isomorphism**. A strict isomorphism from F to the additive fgl is a **logarithm** for F, denoted by $\log_F(x)$.

Explicitly, $\log_F(F(x, y)) = \log_F(x) + \log_F(y)$.

Example 1.10. Let F be a multiplicative fgl, i.e. $F(x, y) = x + y + xy$. So $1 + F = (1 + x)(1 + y)$, yielding $\log(1 + F) = \log(1 + x) + \log(1 + y)$. Then

$$\log_F(x) = \sum_{i>0} \frac{(-1)^{i-1}x^i}{i}.$$

Example 1.11. Let $F(x,y) = (x+y)/(1+xy)$. One can check it is indeed a fgl over $\mathbb{R}$. If $x = \tanh u$ and $y = \tanh v$, then $F(x,y) = \tanh(u+v)$. So

$$\tanh^{-1}(F(x,y)) = \tanh^{-1} x + \tanh^{-1} y.$$

In other word, $\log_F(x) = \tanh^{-1}(x) = \sum_{i\geq 0} x^{2i+1}/(2i+1)$.

Theorem 1.12. *Let F be a fgl and $f(x) \in R \otimes \mathbb{Q}[[x]]$ be given by*

$$f(x) = \int_0^x \left(\frac{\partial F}{\partial y}\Big|_{(t,0)} \right)^{-1} dt,$$

then f is a logarithm for F, i.e. $F(x,y) = f^{-1}(f(x)+f(y))$. F is isomorphic to the additive fgl over $R \otimes \mathbb{Q}$.

Proof. Let $w = f(F(x,y)) - (f(x)+f(y))$. It suffices to show $w = 0$. By definition, $F(x, F(y,z)) = F(F(x,y),z)$. Differentiating w.r.t. z, we get

$$\frac{\partial F(x,F(y,z))}{\partial y} \cdot \frac{\partial F(y,z)}{\partial y} = \frac{\partial F(F(x,y),z)}{\partial y}.$$

Setting $z = 0$, we get

$$\frac{\partial F}{\partial y}(x, F(y,0)) \cdot \frac{\partial F}{\partial y}(y,0) = \frac{\partial F}{\partial y}(x,y) \cdot \frac{\partial F}{\partial y}(y,0) = \frac{\partial F}{\partial y}(F(x,y),0).$$

On the other hand,

$$\frac{\partial w}{\partial y} = f'(F(x,y)) \cdot \frac{\partial F}{\partial y} - f'(y).$$

So by assumption,

$$\frac{\partial w}{\partial y} = \left(\frac{\partial F}{\partial y}\Big|_{(F(x,y),0)} \right)^{-1} \cdot \frac{\partial F}{\partial y} - \left(\frac{\partial F}{\partial y}\Big|_{(y,0)} \right)^{-1}. \tag{1.2}$$

This equals 0 by (1.2). Similarly, $\frac{\partial w}{\partial x} = 0$, implying w is constant. Since $f(0) = 0$, $w \equiv 0$. □

Proposition 1.13. *For any fgl F over a torsion-free ring R, we have*

$$\log_F(F(x,y)) = \log_F(x) + \log_F(y).$$

Proof. Let $w(x,y) = \log_F(F(x,y)) - \log_F(x) - \log_F(y)$. It suffices to show that

$$\frac{\partial w}{\partial y} = 0,$$

since w is symmetric in x and y, and $\log_F$ has zero constant term. Repeating what we did in the proof of theorem 1.12, the differentiating w.r.t. z and setting $z = 0$ in the associativity condition yields

$$\frac{\partial F}{\partial y}(x,y) \cdot \frac{\partial F}{\partial y}(y,0) = \frac{\partial F}{\partial y}(F(x,y),0).$$

By fundamental theorem of calculus, we obtain

$$\begin{aligned}\frac{\partial w}{\partial y} &= \frac{\partial}{\partial y}\left[\int_0^{F(x,y)} \left(\frac{\partial F}{\partial y}(t,0) \right)^{-1} dt - \int_0^x \left(\frac{\partial F}{\partial y}(t,0) \right)^{-1} dt - \int_0^y \left(\frac{\partial F}{\partial y}(t,0) \right)^{-1} dt \right]. \\ &= \frac{\frac{\partial F}{\partial y}(x,y)}{\frac{\partial F}{\partial y}(F(x,y),0)} - \frac{1}{\frac{\partial F}{\partial y}(y,0)} = 0.\end{aligned}$$

□

Definition 1.14. A fgl over a torsion-free $\mathbb{Z}_{(p)}$-algebra is ***p*-typical** if its logarithm has the form $\sum_{i\geq 0} \ell_i x^{p^i}$ with $\ell_0 = 1$.

Just a quick reminder, $\mathbb{Z}_{(p)}$ is the subring of the rationals consisting of fractions with denominator prime to p, which is flat as $\mathbb{Z}$-module. If we replace $R \to R \otimes \mathbb{Q}$ by localizing at a prime p, i.e. $R \to R \otimes \mathbb{Z}_{(p)}$, then we have the following result:

Theorem 1.15 (Cartier)**.** *Every fgl F over a torsion-free $\mathbb{Z}_{(p)}$-algebra is canonically strictly isomorphic to a p-typical one.*

To prove the theorem, it suffices to prove that the universal fgl is isomorphic to a p-typical one over $L \otimes \mathbb{Z}_{(p)}$. We omit the details. In fact, if $\log_F(x) = \sum_{i\geq 0} a_i x^i$, then the logarithm of its corresponding p-typical fgl is $\log_{F^{typ}}(x) = \sum_{i\geq 0} a_{p^i-1} x^{p^i}$. For example,

$$\log_{\mathcal{F}_{MU}} = \sum_{i\geq 0} \frac{[\mathbb{CP}^i]}{i+1} x^{i+1} \in MU^* \otimes \mathbb{Q}[[x]],$$

and

$$\log_{\mathcal{F}_{MU}^{typ}} = \sum_{i\geq 0} \frac{[\mathbb{CP}^{p^i-1}]}{p^i} x^{p^i} \in MU^* = \log_{\mathcal{F}_{BP}}(x),$$

where $[\mathbb{CP}^n]$ is the cobordism class represented by $\mathbb{CP}^n$, and BP is the Brown-Peterson spectrum.

Notation 1.16. Let F be a fgl over R. We denote

$$x +_F y \coloneqq F(x, y).$$

It can be iterated in the sense: $x +_F y +_F z = F(F(x, y), z)$. For non-negative integers n, we write

$$[n]_F(x) \coloneqq F(x, [n-1]_F(x)) = \underbrace{x +_F x +_F \cdots +_F x}_{n}$$

with $[0]_F(x) = 0$. We use the notion $\sum^F$ to indicate the formal sum of the chosen elements. From the notion,

$$[n]_F(x) = \sum b_i x^i$$

for some coefficient $b_i \in R$.

Proposition 1.17. *We adapt the notations above. Let r_1, r_2 be two non-negative integers.*

(1) $[r_1 + r_2]_F(x) = F([r_1]_F(x), [r_2]_F(x))$,
(2) $[r_1 r_2]_F(x) = [r_1]_F([r_2]_F(x))$.

A natural question to ask is the universal p-typical fgl. Here is the theorem:

Theorem 1.18. *Let $V = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$ with $|v_i| = 2(p^i - 1)$. Then there is a universal p-typical fgl F defined over V; that is, for any p-typical fgl G defined over a commutative $\mathbb{Z}_{(p)}$-algebra R, there exists a unique homomorphism $\theta : V \to R$ such that $G(\theta(x), \theta(y)) = \theta(F(x, y))$. Moreover, V is isomorphic to a direct summand $L \otimes \mathbb{Z}_{(p)}$.*

An important consequence needed to keep in mind is that $BP_* = \pi_*(BP) \cong V$.

The v_n's in the theorem can be defined in terms of the log coefficients. There are two choices. One is given by Hazewinkel,

$$p\ell_n = \sum_{0 \le i < n} \ell_i v_{n-i}^{p^i}. \tag{1.3}$$

Another easier choice is given by Araki, which is nearly identical,

$$p\ell_n = \sum_{0 \le i \le n} \ell_i v_{n-i}^{p^i}, \tag{1.4}$$

where $v_0 = p$. $\ell_0 = 1$.

Theorem 1.19 (Hazewinkel, Araki)**.** *The sets of elements defined by (1.3) and (1.4) are obtained in and generate V as a ring, and they are congruent* $\mod (p)$.

Return to the p-series $[p]_F(x)$.

Definition 1.20. Let F be a fgl over a commutative $\mathbb{F}_p$-algebra R. F is said to have **height** n if $[p]_F(x)$ has leading term ax^{p^n}, with a invertible. If $[p]_F(x) = 0$, then we say F has height ∞.

The height of fgls is invariant under isomorphism. Indeed, if $f : F \to G$ is an isomorphism between fgls F and G, then $f([p]_F(x)) = [p]_G(f(x))$ since $f(0) = 0$, and the leading term of f is ux for u a unit in R.

Example 1.21. Consider the additive and multiplicative fgls under $\mod (p)$ reduction.

(1) If F is additive, then $[p]_F(x) = 0$. So F has height ∞.
(2) If F is multiplicative, then $[p]_F(x)$ is the leading term of $(x+1)^p - 1 \mod (p)$, which equals x^p. So F has height 1.

Theorem 1.22 (Classification of fgls)**.** *Two fgls over the algebraic closure of $\mathbb{F}_p$ are isomorphic iff they have the same height.*

See Appendix: Morava K-theory and Brown-Peterson theory.

We now discuss something about the (co)bordism with singularities. Such things establish a big source of interesting (co)homology theories, and enable us to construct cohomology theories with prescribed properties. The main ideas are due to Baas [6] and Sullivan [7].

Let $\psi : B \to B\mathcal{V}$ be a multiplicative structure map, where $\mathcal{V} = \mathcal{PL}$ or $\mathcal{O}$. Here $\mathcal{PL}$ (resp. $\mathcal{O}$) is the colimit of the geometric realization of the simplicial piecewise linear group PL_n (resp. orthogonal group O_n). By multiplicative structure maps we mean that ψ comes with a multiplication map $\mu_B : B \times B \to B$ and a homotopy $H : \psi\mu_B \xrightarrow{\simeq} \mu \circ (\psi \times \psi)$.

Definition 1.23. Let S be the class of all compact (B, ψ)-manifolds (smooth for $\mathcal{V}$ and PL (piecewise linear) for $\mathcal{V} = \mathcal{PL}$). S is closed under multiplication. Also, we require S satisfies

$$M \times N \cong (-1)^{\dim M \dim N} N \times M$$

for all $M, N \in S$. We call this class **a** S**-manifold**.

Let $P \in S$ with $\dim P = d$ be a closed S-manifold. Let M be a S-manifold with $\partial M = P \times Z$, where Z is a closed S-manifold. It forms a polyhedron called **Sullivan construction**

$$K \coloneqq Z \times C(P) \cup_\psi M,$$

where $C(P)$ is the cone over P and $\psi : Z \times P \to \partial M$ is a S-isomorphism (e.g. an oriented diffeomorphism when S is the class of oriented smooth manifolds). Every closed S-manifold N has this form if we put $Z = \varnothing$ and $N = M$. On the other hand, K is a manifold if we delete the set $Z \times \{*\}$ from it.

Definition 1.24. A **k-dimensional S-manifold with Sullivan-Baas P-singularity** is a quintuple $(V, \partial_0 V, \partial_1 V, \delta V, \psi)$, where

(1) V is a S-manifold with $\dim V = k$, with $\partial V = \partial_0 V \cup \partial_1 V$, where $\partial_i V$, $i = 1, 2$ is a S-manifold.
(2) $\partial\partial_0 V = \partial_0 V \cap \partial_1 V = \partial\partial_1 V$.
(3) δV is a certain S-manifold and $\psi : \delta V \times P \to \partial_1 V$ is a S-isomorphism.

For simplicity, We simply call such quintuple, or V, a **S^P-manifold**. The **boundary** of the quintuple $(V, \partial_0 V, \partial_1 V, \delta V, \psi)$ is also a S^P-manifold defined by

$$\partial^P(V, \partial_0 V, \partial_1 V, \delta V, \psi) \coloneqq (\partial_0 V, \varnothing, \partial_0 V \cap \partial_1 V, \partial\delta V, \psi\mid_{\partial\delta V}),$$

with $\partial^P \partial^P = 0$. A **closed S^P-manifold** is a S^P-manifold M such that $\partial^P M = 0$. That is, there is a fixed S-isomorphism $\psi : \delta M \times P \to \partial M$, where δM is a closed S-manifold.

From the definition, every S-manifold V is a S^P-manifold by setting $\delta V = \varnothing$.

From a more concrete view, one might start with a closed manifold P. Let V be a $(k+1)$-dimensional manifold with corners modeled on $[0,\infty)^2 \times \mathbb{R}^{k-1}$. The boundary of V decomposes into the union of two manifolds $\partial_0 V$ and $\partial_1 V$ with an obvious property

$$\partial\partial_0 V = \partial_0 V \cap \partial_1 V = \partial\partial_1 V.$$

Suppose further that there is a factorization, $\partial_1 V \underset{=}{\xrightarrow{\psi}} \delta V \times P$, where δV is a manifold with boundary. Then $(V, \partial_0 V, \partial_1 V, \delta V, \psi)$ is a S^P-manifold. The submanifold $\partial_0 V$ plays the role of the boundary of V in the ordinary sense.

Definition 1.25. A **k-dimensional singular S^P-manifold** in a pair (X, A) is a map $f : (V, \partial^P V) \to (X, A)$ of a k-dimensional S^P-manifold V such that the diagram is commutative:

$$\begin{tikzcd} V \arrow[r, "f"] & X \\ \delta V \times P \arrow[u, "\eta"] \arrow[r, "p_1"] & \delta V \arrow[u, "f_0"'] \end{tikzcd}$$

where η is the embedding $\delta V \times P \xrightarrow{\psi} \partial_1 V \subset \partial V \subset V$.

In other words, $f\mid_{\partial_1 V}$ factors through p_1, i.e. $f\psi(b, p) = f_0(b)$ for every $b \in \delta V$ and $p \in P$. The diagram formalizes the gluing of the cone in the Sullivan construction $K = Z \times C(P) \cup_\psi M$.

Definition 1.26. A singular closed S^P-manifold $f : M \to X$ **bounds**, if there exists a singular S^P-manifold $g : V \to X$ with $\partial^P V = M$ and $g\mid_M = f$. In this case, we write $\partial^P(V,g)=(M,f)$. In other words, we have a commutative diagram

$$\begin{tikzcd} \partial^P V = M \arrow[r, "f"] \arrow[d, "g|_M"'] & X \\ V \arrow[ur, "g"'] & \end{tikzcd}$$

Definition 1.27. Two closed singular S^P-manifold (M,f) and (N,g) are called **bordant** if $(M,f)\sqcup(-N,-g)=\partial^P(V,h)$ for some (V,h). This defines a S^P**-bordism relation** on the class of closed singular S^P-manifolds. The S^P-bordism class of a closed singular manifold $f:M\to X$ is denoted by $[M,f]$.

For every n, there is an n-dimensional S^P-bordism group of X under the disjoint union of S^P-manifolds, whose elements are S^P-bordism classes of n-dimensional S^P-manifolds. We denote this group by $L_n^P(X)$, and the n-dimensional S-bordism group by L_n. For example, $MU^{\mathbb{CP}^1}$ is the complex bordism theory with $\mathbb{CP}^1$-singularity.

The following homomorphisms are natural w.r.t. X:

(1) $P: L_i(X)\to L_{i+d}(X)$, sending $\{f:M\to X\}$ to $\{M\times P\xrightarrow{p_1}M\xrightarrow{f}X\}$.
(2) $r=r^X: L_i(X)\to L_i^P(X)$. Here we regard a S-manifold M as a S^P-manifold by letting $\delta M=\partial_1 M=\varnothing$.
(3) $\delta=\delta^X: L_i^P(X)\to L_{i-d-1}(X)$, sending $[M,f]$ to $[\delta M,f_0]$, where $f_0:\delta M\to X$ is identical to the one in Definition 1.25.

Theorem 1.28 (Bockstein-Sullivan-Baas exact sequence)**.** *For every space X, there is an associated l.e.s.*

$$\cdots\to L_n(X)\xrightarrow{P}L_{n+d}(X)\xrightarrow{r}L_{n+d}^P(X)\xrightarrow{\delta}L_{n-1}(X)\to\cdots.$$

Remark 1.29. The Bockstein homomorphism is

$$\beta^P: L_i^P(X)\to L_{i-d-1}^P(X)$$

as $\beta^P=r\delta$. This is a generalization of the classical Bockstein homomorphism β.

Summarize the idea, a singular S^P-manifold $f:(M,\partial^P M)\to(X,A)$ in (X,A) bounds if there exist a singular S^P-manifold $g:(V,\partial^P V)\to(X,A)$ such that $\partial_0 V=\partial_0' V\cup\partial_0'' V$ with $\partial_0' V=M$, $g\mid_{\partial_0' V}=f$ and $g(\partial_0'' V)\subset A$. The corresponding bordism classes form the bordism group $L_*^P(X,A)$ under the disjoint union. Let $\partial_n: L_n^P(X,A)\to L_{n-1}^P(A)$ be given by $\partial_n[M,f]=[\partial^P M, f\mid_{\partial^P M}]$. Then $\{L_*^P(X,A),\partial_*\}$ constitutes an additive homology theory.

Proposition 1.30. *If P and Q are bordant S-manifolds, then the homology theories L_*^P and L_*^Q are isomorphic.*

The proposition implies that the homology theory L_*^P is determined by the bordism class of P. So we write $L^{[P]}$ instead of L^P.

Proposition 1.30 and Theorem 1.28 provide a computational tool for the homology of S^P-manifolds. For example, we can consider the spectrum MU^{x_i}, where x_i is a generator in the homotopy group of MU (Recall $\pi_* MU=\mathbb{Z}[x_1,x_2,\cdots]$). By Bockstein-Sullivan-Baas exact sequence (Theorem 1.28), $\pi_*(MU^{x_i})=\pi_*(MU)/(x_i)$.

The preceding story can generalize to include not only a single closed S-manifold, but an iteration of it. Namely, if P' is a closed S-manifold and $V=(V,\partial_0 V,\partial_1 V,\delta V,\psi)$

is a S^P-manifold, $P \ncong P'$, then it is an easy exercise to check

$$P' \times V = (P' \times V, P' \times \partial_0 V, P' \times \partial_1 V, P' \times \delta V, \mathrm{id}_{P'} \times \psi)$$

is again a S^P-manifold. Proceed with the same recipe, we get

Definition 1.31. Let $\Sigma = \{P_i\}_{i \in N}$ be a finite set of closed S-manifolds with $\dim P_i = d_i$. A **closed k-dimensional S-manifold with Sullivan-Baas Σ-singularity** (abbr. a **closed S^Σ-manifold**) is a triple $(M, \delta_I M, \psi_{I,i})$, where I runs over all subsets of N and $i \in I$, such that the following hold:

(1) M is a S-manifold with dimension k.
(2) $\delta_I M$ is a S-manifold, and $\psi_{I,i} : \delta_I M \times P_i \to \partial(\delta_{I-\{i\}} M)$ is a S-embedding. Furthermore, $\dim \delta_I M + d_i + 1 = \dim \delta_{I-\{i\}} M$, and

$$\partial(\delta_J M) = \bigcup_{i \notin J} \psi_{J \cup \{i\}, i}(\delta_{J \cup \{i\}} M \times P_i).$$

Also $\delta_\varnothing M = M$, and so $\partial M = \cup_i \psi_{\{i\},i}(\delta_{\{i\}} M \times P_i)$.
(3) For every I and every $i, j \in I$, $i \neq j$, the diagram is commutative:

$$\begin{tikzcd}
\delta_I M \times P_i \times P_j \arrow[r, "\psi_{I,i} \times \mathrm{id}"] \arrow[d, "(\psi_{I,j} \times \mathrm{id})(\mathrm{id} \times T)"'] & \partial(\delta_{I-\{i\}} M) \times P_j \arrow[r, "\subset"] & \delta_{I-\{i\}} M \times P_j \arrow[d, "\psi_{I-\{i\},j}"] \\
\partial(\delta_{I-\{j\}} M) \times P_i \arrow[r, "\subset"] & \delta_{I-\{j\}} M \times P_i \arrow[r, "\psi_{I-\{j\},i}"] & \delta_{I-\{i,j\}} M
\end{tikzcd}$$

where T interchanges P_i and P_j. This rephrases the obvious compatibility condition as expected.

Let $i_0 \in N$, $P = P_{i_0}$ and $\tilde{\Sigma} = \Sigma - \{P\}$. Given a S^Σ-manifold $M = (M, \delta_I M, \psi^M_{I,i})$, there is a $S^{\tilde{\Sigma}}$-manifold $N = \delta_P M := (N, \delta_J N, \psi^N_{J,j})$ by setting $\delta_J N = \delta_{J \cup \{i_0\}} M$, with

$$\psi^N_{J,j} : \delta_J N \times P_j = \delta_{J \cup \{i_0\}} M \times P_j \xrightarrow{\psi^M_{J \cup \{i_0\}, j}} \partial(\delta_{J \cup \{i_0\} - \{j\}} M) = \delta(\delta_{J - \{j\}} N).$$

In other word, $N = \delta_{\{i_0\}} M$. Furthermore, $\dim M = \dim N + \dim P + 1$. In particular, given a family $\Sigma = \{P_i\}$ and a closed S-manifold P, let $\Sigma' = \Sigma \cup P$. We can assign a S^σ-manifold $\delta_P M$ to a $\delta^{\Sigma'}$-manifold M.

Definition 1.32. Let $\Sigma = \{P_i\}_{i \in N}$ and $\Sigma' = \Sigma \cup \{\mathrm{pt}\}$. A **$S^\Sigma$-manifold with boundary** is defined to be a closed $S^{\Sigma'}$-manifold. Let V be a S^Σ-manifold with boundary. We set $\partial^\Sigma V := \delta_{\mathrm{pt}} V$, and call this is the **S^Σ-boundary** of V.

Proposition 1.33. *Let $\Sigma = \{P_i\}_{i \in N}$.*

(1) If P is a closed S-manifold and M is a closed S^Σ-manifold, then $M \times P$ is a closed S^Σ-manifold, where $\delta_I(M \times P) = \delta_I(M) \times P$ and $\psi^{M \times P}_{I,i} = \psi^M_{I,i} \times \mathrm{id}_P$.
(2) Given a subset $M \subset N$. Let $\Sigma' = \{P_i : i \in M\}$. Then every closed $S^{\Sigma'}$-manifold M can regarded as a closed S^Σ-manifold if we put $\delta_I M = \varnothing$ for every I which is not a subset of T.

Definition 1.34. A **singular S^Σ-manifold** in pair (X, A) is a map $f : (V, \partial^\Sigma V) \to (X, A)$ where V is S^Σ-manifold with boundary and f is such that for every I there exists a commutative diagram

$$\begin{tikzcd}
V \arrow[r, "f"] & X \\
\delta_I V \times \prod_{i\in I} P_i \arrow[u] \arrow[r, "p_1"] & \delta_I V \arrow[u, "f_I"']
\end{tikzcd}$$

where the left map is the inclusion and the bottom map is the projection onto the first factor.

We can repeat what we do in $\Sigma = \{P\}$ now. The bordism classes can be defined in an identical way as before. Thus, we can define a bordism theory with Σ-singularity $L_*^\Sigma(X,A)$. If $\Sigma' = \Sigma \cup \{P\}$ with $\dim P = d$, then we have the Bockstein-Sullivan-Baas exact sequence as follows

$$\cdots \to L_n^\Sigma(X,A) \xrightarrow{P} L_{n+d}^\Sigma(X,A) \xrightarrow{r} L_{n+d}^{\Sigma'}(X,A) \xrightarrow{\delta} L_{n-1}^\Sigma(X,A) \to \cdots,$$

where P is described in (1) of Proposition 1.33, r is described in (2) of Proposition 1.33, and $\delta[V,f] = [\delta_P(V), f\mid_{\delta_P(V)}]$. It can be proved that L_*^Σ is a homology theory, so it can be represented by a spectrum L^Σ, which is unique up to equivalence.

If $\Sigma = \{P_i\}_{i\in N}$, N is chosen to be a countable set, then we can take a finite piece $\Sigma_n = \{P_1, \cdots, P_n\}$ and define $L_*^\Sigma = \lim L_*^{\Sigma_n}$. Since lim preserves exactness, L_*^Σ is a homology theory. It is also a bordism theory based on manifolds of the class $S^\Sigma := \cup_n S^{\Sigma_n}$.

With all the backgrounds above in hand, we are ready to talk about main property of (co)bordism with singularities, which we have already seen before.

Definition 1.35. Let R be a commutative ring, and M be an R-module. A sequence $\{x_1, x_2, \cdots\} \subset R$ (not necessarily finite) is called **M-proper** if multiplication by $x_1 : M \to M$ is monic and multiplication by $x_1 : M/(x_1, \cdots, x_{i-1})M \to M/(x_1, \cdots, x_{i-1})M$ is monic, for every i. If $M = R$, then we call such sequence **proper**.

Proposition 1.36. *Let $\Sigma = \{x_1, x_2 \cdots\}$ be a proper sequence in $\pi_*(L)$. Then there is a $\pi_*(L)$-module isomorphism*

$$\pi_*(L^\Sigma) \cong \pi_*(L)/(x_1, x_2, \cdots).$$

Example 1.37. If we take $\Sigma = \{x_1, x_2 \cdots\}$, where x_i is the generator of $\pi_* MU$, then by Proposition 1.36, $\pi_* MU^\Sigma = \mathbb{Z}$. This implies $MU^\Sigma = H\mathbb{Z}$. So classical homology can be interpreted as bordism with singularities.

Example 1.38. Let $m \in \mathbb{N}$, and $\mathfrak{m}$ be m discrete points (which is a manifold). Set $\Sigma' = \Sigma \cup \mathfrak{m}$ with $\Sigma = \{x_1, x_2 \cdots\}$, where x_i is the generator of $\pi_* MU$. Then it is to see $MU^{\Sigma'} = H\mathbb{Z}/m$ by Proposition 1.36. In this case, the associated Bockstein-Sullivan-Baas exact sequence is just the Bockstein exact sequence.

2. Lecture 2: 10/04/2022

NO LECTURE TODAY DUE TO THE NATIONAL DAY!

3. Lecture 3 (Shuhan Zheng): 10/11/2022

The big goals of this note are the nilpotent theorem and the periodicity theorem. Start with the nilpotent theorem.

Definition 3.1. A map $f : \Sigma^d X \to X$ is called a **self-map** of X. We denote the composite maps

$$\cdots \xrightarrow{\Sigma^{3d} f} \Sigma^{3d} X \xrightarrow{\Sigma^{2d} f} \Sigma^{2d} X \xrightarrow{\Sigma^{d} f} \Sigma^{d} X \xrightarrow{f} X$$

by f, f^2, f^3, etc. A self-map f is called **(stably) nilpotent** if some suspension of f^t for some $t > 0$ is null-homotopic. Otherwise it is called **periodic**.

Let X be a finite CW complex. We write $\overline{E}_*$ to indicate the reduced version of the given homology theory E_*.

Theorem 3.2 (Nilpotence Theorem)**.** *Let $f : \Sigma^d X \to X$ be a self-map. f is nilpotent iff some iterate of $\overline{MU}_*(f)$ is trivial.*

There are two other equivalent forms of the nilpotence theorem. For reasons too complicated to explain, we will leave these forms for later.

The periodicity theorem of a self map needs a bit more efforts. Fix a prime p. Let X be a CW complex such that for any homology theory $E_*(X)$, $\overline{E}_*(X) \otimes \mathbb{Z}_{(p)} \cong \overline{E}_*(X_{(p)})$. $X_{(p)}$ is called the p**-localization** of X.

Definition 3.3. Suppose X is a simply connected CW complex such that $\overline{H}_*(X)$ consists entirely of torsion. If it is all p-torsion, then X is p-local, i.e. $X_{(p)} \simeq X$. In this case, we call X is a p**-torsion complex**.

When X is a finite p-torsion complex, it is convenient to replace MU_* in the nilpotence theorem by the Morava K-theories $K(n)_*$ for each prime p. See Appendix C for detailed construction. The facts we need is the following:

Proposition 3.4. *For $n \geq 0$, $K(n)_*$ satisfies*

(1) $K(0)_* = H_*(X; \mathbb{Q})$ and $\overline{K(0)}_*(X) = 0$ when $\overline{H}_*(X)$ is all torsion.
(2) $K(0)_*(\mathrm{pt}) = \mathbb{Q}$ and $K(n)_*(\mathrm{pt}) = \mathbb{Z}/(p)[v_n, v_n^{-1}]$ where $|v_n| = 2(p^n - 1)$. The ring is a graded field in the sense that every graded module over it is free. $K(n)_*(X)$ is a module over $K(n)_*(\mathrm{pt})$.
(3) Suppose X is a p-local finite CW complex. If $\overline{K(n)_*}(X) = 0$, then $\overline{K(n-1)_*}(X) = 0$. In this case,

$$\overline{K(n)_*}(X) = \overline{K(n)_*}(\mathrm{pt}) \otimes \overline{H}_*(X; \mathbb{Z}/(p)).$$

(4) Künneth formula: $K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*(\mathrm{pt})} K(n)_*(Y)$.
(5) $\overline{K(n)_*}$ is $2(p^n - 1)$-periodic.

Definition 3.5. A p-local finite complex X has type k if k is the smallest integer such that $\{n \in \mathbb{N}_+ : \overline{K(n)_*}(X) \neq 0\} = \{k, k+1, k+2, \cdots\}$. If this set is empty, then we say that X has type ∞.

The Morava K-theories are extremely useful in detecting periodic self-maps.

Theorem 3.6 (Periodicity Theorem)**.** *Let X and Y be p-local finite CW complexes of type $n < \infty$.*

(1) *There is a self-map* $f : \Sigma^{d+i}X \to \Sigma^i X$ *for some* $i \geq 0$ *such that*

$$\begin{cases} K(n)_*(f) : K(n)_*(X) \to K(n)_{*+d}(X) & \text{isomorphism,} \\ K(m)_*(f) = 0 & m \geq n+1. \end{cases}$$

This self-map is called a v_n***-map***. *When* $n = 0$*, then* $d = 0$*; When* $n > 0$*,* d *is a multiple of* $2p^n - 2$.

(2) *Let* $h : X \to Y$ *be a continuous map. Assume that both have already been suspended enough times to be the target of a* v_n*-map. Let* $g : \Sigma^e Y \to Y$ *be a self-map as in (1). Then there are positive integers* i *and* j *(independent of choice of* h*) with* $di = ej$ *such that the following diagram commutes up to homotopy:*

$$\begin{tikzcd} \Sigma^{di}X \arrow[r, "\Sigma^{di}h"] \arrow[d, "f^i"'] & \Sigma^{di}Y \arrow[d, "g^j"] \\ X \arrow[r, "h"] & Y \end{tikzcd}$$

If we ask $Y = X$ in the theorem, then we can see the self-map f is **asymptotically unique** as shown in the following diagram: $(di = ej)$

$$\begin{tikzcd} \Sigma^{di}X \arrow[rr, "\Sigma^{ej}\mathrm{id}"] \arrow[dr, "f^i"'] & & \Sigma^{ej}X \arrow[dl, "g^j"] \\ & X & \end{tikzcd}$$

Now f^i is homotopic to g^j.

Proposition 3.7. *If* X *(as in periodicity theorem) has type* n*, then the cofiber of the* v_n*-map* $f : \Sigma^{d+i}X \to X$ *has type* $n+1$.

Proof. Let C be the cofiber of f. Consider the l.e.s.

$$\cdots \to \overline{K(m)}_t(\Sigma^d X) \xrightarrow{f_*} \overline{K(m)}_t(X) \to \overline{K(m)}_t(C) \to \overline{K(m)}_{t-1}(\Sigma^d X) \to \cdots.$$

If $m < n$, then $\overline{K(m)}_*(X) = 0$, yielding $\overline{K(m)}_*(W) = 0$. If $m = n$, f_* is an isomorphism, so $\overline{K(m)}_*(W) = 0$. If $m > n$, $f_* = 0$ and $\overline{K(m)}_*(X) \neq 0$. By Künneth formula,

$$\overline{K(m)}_*(W) \cong \overline{K(m)}_*(X) \oplus \overline{K(m)}_*(\Sigma^{d+1}X).$$

So W has type $n+1$. □

The motivation of nilpotence and periodicity theorem comes from the following example. Recall that the cofiber of $f : X \to Y$ is the space $C_f \coloneqq X \times [0,1] \cup Y / ((X \times \{0\}) \sim \text{pt},\ (x,1) \sim f(x)\,(x \in X))$. An extremely useful proposition says

Proposition 3.8. *Let* X*,* Y *be path connected CW complexes,* $f : X \to Y$ *be a morphism, and* E_* *be a homology theory.*

(1) *For any space* Z*, there is a l.e.s. induced by the cofiber of* f*:*

$$[X, Z] \xleftarrow{f^*} [Y, Z] \xleftarrow{i^*} [C_f, Z] \xleftarrow{j^*} [\Sigma X, Z] \xleftarrow{\Sigma f^*} [\Sigma Y, Z] \leftarrow \cdots.$$

(2) *There is a l.e.s.*

$$\cdots \xrightarrow{j_*} \overline{E}_m(X) \xrightarrow{f_*} \overline{E}_m(Y) \xrightarrow{i_*} \overline{E}_m(C_f) \xrightarrow{j_*} \overline{E}_{m-1}(X) \xrightarrow{f_*} \cdots.$$

(3) *Suppose that, furthermore, X and Y are $(k-1)$-connected, and W is a finite CW complex which is a double suspension with top cell in dimension less than $2k-1$. Then there is a l.e.s. of abelian groups:*

$$[W,X] \xrightarrow{f_*} [W,Y] \xrightarrow{i_*} [W,C_f] \xrightarrow{j_*} [W,\Sigma X] \xrightarrow{\Sigma f_*} [W,\Sigma Y] \to \cdots,$$

which will terminate at the point where the connectivity of the target exceeds the dimension of W.

Let $V(0)_k$ be the cofiber of the map $f : S^k \xrightarrow{\cdot p} S^k$ for some prime p. This is of type 1. For k large enough, Adams and Toda showed that there is a periodic map $\alpha : \Sigma^q V(0)_k \to V(0)_k$, where $q = 8$ when $p = 2$ and $2p-2$ when p is odd. Now the induced map

$$K(1)_*(\alpha) = \begin{cases} \cdot v_1 & , p \text{ is odd}, \\ \cdot v_1^4 & , p = 2. \end{cases}$$

Define for $p \geq 5$,

$$\mathrm{Cof}_\alpha =: V(1)_k.$$

$V(1)_k$ has type 2. For sufficient large k, Smith and Toda showed that there is a periodic map $\beta : \Sigma^{2p^2-2} V(1)_k \to V(1)_k$. Similarly, its induced map is a multiplication by v_2 in $K(2)$-theory. Continue this pattern, for $p \geq 7$, denote $\mathrm{Cof}_\beta =: V(2)_k$. This has type 3. There is a periodic map $\gamma : \Sigma^{2p^3-2} V(2)_k \to V(2)_k$ which passes to a v_3-multiplication in $K(3)$-theory. However, there are obstructions to extend this pattern to $V(3)_k \coloneqq \mathrm{Cof}_\gamma$, contradicting to the periodicity theorem at first sight.

Each of the maps above led to an infinite family of elements in the stable homotopy groups of spheres. Consider a slice of the cofiber sequence

$$S^k \xrightarrow{\cdot p} S^k \xrightarrow{i_1} V(0)_k \xrightarrow{j_1} S^{k+1}.$$

One can form the new sequence

$$S^{k+qt} \xrightarrow{i_1} \Sigma^{qt} V(0)_k \xrightarrow{\alpha^t} V(0)_k \xrightarrow{j_1} S^{k+1}.$$

The resulting element $[j_1 \alpha^t i_1]$ in π^S_{qt-1} is denoted by α_t for p odd and by α_{4t} for $p = 2$.

If we use β instead of α, then for $p \geq 5$,

$$S^{k+2t(p^2-1)} \to \Sigma^{(2p^2-2)t} V(0)_k \xrightarrow{i_2} \Sigma^{(2p^2-2)t} V(1)_k \xrightarrow{\beta^t} V(1)_k \xrightarrow{j_2} \Sigma^{2p-1} V(0)_k \xrightarrow{j_1} S^{k+2p},$$

where $i_2 : V(0)_k \to V(1)_k$ and $j_2 : V(1)_k \to \Sigma^{q+1} V(0)_k$ denote the maps in the cofiber sequence associated with α. In short, we have

$$S^{k+2t(p^2-1)} \xrightarrow{i_2 i_1} \Sigma^{k+2t(p^2-1)} V(1)_k \xrightarrow{\beta^t} V(1)_k \xrightarrow{j_1 j_2} S^{k+2p}.$$

The resulting element $[j_1 j_2 \beta^t i_2 i_1]$ in $\pi^S_{2t(p^2-1)-2p}$ is denoted by β_t. Smith showed it is essential for all $t > 0$.

Similarly, we can repeat the same process for γ when $p \geq 7$:

$$S^{k+2t(p^3-1)} \xrightarrow{i_3 i_2 i_1} \Sigma^{k+2t(p^3-1)} V(2)_k \xrightarrow{\gamma^t} V(2)_k \xrightarrow{j_1 j_2 j_3} S^{k+p(p+2)+3}.$$

In general, every periodic map on a finite CW complex leads to a periodic family of elements in π^S_*. Let X be a finite CW complex of type $n < \infty$ with dimension of cells in the range $[k, k+e]$. Suppose we have a self-map $f : \Sigma^d X \to X$. Then $f^t : \Sigma^{dt} X \to X$ fits into

$$S^{k+dt} \xrightarrow{i_0} \Sigma^{dt} X \xrightarrow{f^t} X \xrightarrow{j_0} S^{k+e}, \tag{3.1}$$

where $i_0 : S^k \to X$ and $j_0 : X \to S^{k+e}$ are the inclusions. This gives us a nontrivial element in π^S_{dt-e}, which is independent of the nontriviality of f^t. If the composite (3.1) is trivial, then we can still get a nontrivial element in $\pi^S_{dt-\varepsilon}$ for $\varepsilon \in (-e, e)$.

Write $X^u_v := \mathrm{Cof}\,(X^{v-1} \to X^u)$ for $k \le v \le u \le k+e$. In particular, $X^{k+e}_k = X$ and X^u_u is a wedge of u-dimensional spheres, one for each u-dimensional cell in X. Consider the diagram

$$\begin{tikzcd}
\Sigma^{dt} X \arrow[r, "f_e"] \arrow[dr, dashed, "f_{e-1}"] \arrow[ddr, dashed, "f_{e-2}"'] & X^{k+e}_k \arrow[r, "j"] & X^{k+e}_{k+e} \\
 & X^{k+e-1}_k \arrow[u, "i"] \arrow[r, "j"] & X^{k+e-1}_{k+e-1} \\
 & X^{k+e-2}_k \arrow[u, "i"] \arrow[r, "j"] & X^{k+e-2}_{k+e-2} \\
 & \vdots \arrow[u, "i"] &
\end{tikzcd}$$

where $f_e = f^t$, and $i : X^s_r \to X^{s'}_r$ $(s' > s)$, $j : X^s_r \to X^s_{r'}$ $(s \ge r' > r)$ are the inclusions. The ellipsis must terminate at some dimension $e_1 \in [0, e]$, because if all those composite jf_μ are null-homotopic, then $f_e = f^t$ will be null-homotopic by Proposition 3.8 (3). Let $g_0 = jf_{e_1}$. Consider the diagram

$$\begin{tikzcd}
X^{k+e_1}_{k+e_1} & & & \\
\Sigma^{dt} X^{k+e}_k \arrow[u, "g_0"] \arrow[r, "j"] & \Sigma^{dt} X^{k+e}_{k+1} \arrow[r, "j"] \arrow[ul, dashed, "g_1"] & \Sigma^{dt} X^{k+e}_{k+2} \arrow[r, "j"'] \arrow[ull, dashed, "g_2"] & \cdots \arrow[ulll, dashed] \\
\Sigma^{dt} X^k_k \arrow[u, "i"] & \Sigma^{dt} X^{k+1}_{k+1} \arrow[u, "i"] & \Sigma^{dt} X^{k+2}_{k+2} \arrow[u, "i"] & \cdots
\end{tikzcd}$$

If $g_0 i$ is null-homotopic, then by Proposition 3.8 (1), there is a map $g_1 j = g_0$ (note that $\Sigma^{dt} X^k_k \xrightarrow{i} \Sigma^{dt} X^{k+e}_k \xrightarrow{j} \Sigma^{dt} X^{k+e}_{k+1}$ is a cofiber sequence). Similarly, if $g_1 i$ is null-homotopic, then there is a map g_2 with $g_2 f = g_1$. This process must terminate at some $e_2 \in [0, e]$ since g_0 is not null-homotopic.

Summarize the whole procedure, we actually construct the following diagram:

$$\begin{tikzcd}
\Sigma^{dt} X^{k+e_2}_{k+e_2} \arrow[r, "i"] \arrow[d, equal] & \Sigma^{dt} X^{k+e}_{k+e_2} \arrow[r, "g_{e_2}"] & X^{k+e_1}_{k+e_1} \arrow[d, equal] \\
\bigvee_u S^{dt+k+e_2}_u \arrow[rr, "\phi"] & & \bigvee_v S^{k+e_1}_v
\end{tikzcd}$$

where ϕ is not null-homotopic. This corresponds to an element in $\pi^S_{dt+e_2-e_1}$.

The examples above leads to a natrual question: if every element in π_*^S is part of a periodic family? This question is answered by looking at the E_2-term of ANSS (see D) using a device called chromatic spectral sequence. We now introduce the concept of the chromatic filtration to end today's lecture.

Let Y be a p-local complex and $y \in \pi_k^S(Y)$, corresponding to a map $g : S^k \to Y$. If $|y| = p^i$, then it factors through the cofiber of the map of degree p^i on S^k, i.e.

$$\begin{tikzcd}
S^k \arrow[r, "p^i"] \arrow[dr, dashed] & S^k \arrow[r] \arrow[d, "g"] & \mathrm{Cof}(p^i) \arrow[dl, dashed, "g_1"] \\
& Y &
\end{tikzcd}$$

where $g_1 : \mathrm{Cof}(p^i) \to Y$. Denote the cofiber $\mathrm{Cof}(p^i)$ by $W(1)$. $W(1)$ has type 1 by Proposition 3.7. Mimicking what we have done before, there is a periodic map $f_1 : \Sigma^{d_1} W(1) \to W(1)$ inducing a $K(1)_*$-equivalence. We deal with the cases whether g_1 is null-homotopic after composing with some iterate of f_1 or not. If that is not the case, then we can extend the previous diagram into

$$\begin{tikzcd}
S^k \arrow[r, "p^i"] \arrow[dr, dashed] & S^k \arrow[r] \arrow[d, "g"] & W(1) \arrow[dl, dashed, "g_1"] \\
& Y & \Sigma^{d_1} W(1) \arrow[u, "f_1"] \arrow[l, dashed] \\
& & \Sigma^{2d_1} W(1) \arrow[u, "f_1"] \arrow[ul, dashed] \\
& & \Sigma^{3d_1} W(1) \arrow[u, "f_1"] \arrow[uul, dashed] \\
& & \vdots \arrow[u, "f_1"]
\end{tikzcd}$$

The inverse system gives a direct system of groups

$$[W(1), Y]_*^S \xrightarrow{f_1^*} [\Sigma^{d_1} W(1), Y]_*^S \xrightarrow{f_1^*} [\Sigma^{2d_1} W(1), Y]_*^S \xrightarrow{f_1^*} \cdots$$

Taking the limit of the direct system, we get a group denoted $v_1^{-1}[W(1), Y]_*^S$, which is independent of the choice of f_1 by the periodicity theorem 3.6 (2).

If g_1 is null-homotopic after composing with some iterate of f_1, i.e.

$$\begin{tikzcd}
S^k \arrow[r, "p^i"] \arrow[dr, dashed] & S^k \arrow[r] \arrow[d, "g"] & W(1) \arrow[dl, dashed, "g_1"] \\
& Y & \Sigma^{d_1 t} W(1) \arrow[u, "f_1^t"] \arrow[l, dashed, "\lambda"]
\end{tikzcd}$$

where λ is null-homotopic, then we can take the cofiber of f_1^t (denoted $W(2)$, which has type 2 and admits a periodic map $f_2 : \Sigma^{d_2}W(2) \to W(2)$) and continue

$$\begin{tikzcd}
S^k \arrow[r, "p^i"] \arrow[dr, dashed] & S^k \arrow[r] \arrow[d, "g"] & W(1) \arrow[r] \arrow[dl, dashed, "g_1"] & W(2) \arrow[r] & \cdots \\
 & Y & \Sigma^{d_1 t_1} W(1) \arrow[u, "f_1^{t_1}"] & \Sigma^{d_2 t_2} W(2) \arrow[u, "f_2^{t_2}"] & \cdots
\end{tikzcd}$$

It leads to the group $v_2^{-1}[W(2), Y]_*^S$. One might ask if such process can proceed indefinitely. This motivates the following definition:

Definition 3.9. If $y \in \pi_*^S(Y)$ extends to a complex $W(n)$ of type n as above, then y is called v_{n-1}**-torsion**. If in addition, y cannot be extended to a complex of type $n+1$, then it is called v_n**-periodic**. The chromatic filtration of $pi_*^S(X)$ is the decreasing family of subgroups consisting of the v_n-torsion elements for various $n \geq 0$.

4. Lecture 4 (Video Break): 10/18/2022

Today's lecture based on Denis Nardin's *Intro to Stable Homotopy Theory* (P15 & 16).

The intuition of the lecture is: $\mathbb{Z}[1/p]$-modules $\subset$ $\mathbb{Z}$-modules, (derived) p-complete $\mathbb{Z}$-module $\subset$ $\mathbb{Z}$-modules.

Let $E, X \in \mathsf{Sp}$. Recall that the E-homology of X is $E_*X = \pi_*(E \otimes X)$. If A is a group, then we can define the A-coefficient homology $H_*(Y; A) = \pi_*(HA \otimes \Sigma^\infty(Y_+))$.

Definition 4.1. $f : X \to Y \in \mathsf{Sp}$ is an ***E*-equivalence** if it is an equivalence in E-homology. That is, $E \otimes f : E \otimes X \xrightarrow{\cong} E \otimes Y$. It is ***E*-acyclic** if $X \to 0$ is E-equivalence, i.e. $E \otimes X \xrightarrow{\cong} 0$.

Remark 4.2. The functor $E \otimes -$ is exact. $f : X \to Y \in \mathsf{Sp}$ is E-equivalence iff the cofiber or the fiber of f is E-acyclic.

Now we always assume X and E are two appropriate spectra.

Definition 4.3. X is called ***E*-local** if for any E-equivalence $f : Y \to Z$, the morphism between mapping spectra $\hom(f, X) : \hom(Z, X) \to \hom(Y.X)$ is an equivalence.

We denote the full subcategory of E-local spectra by Sp_E.

Example 4.4. Let $p : \mathbb{S} \to \mathbb{S}$ be a multiplication by p on the sphere spectrum $\mathbb{S}$. Let $\mathbb{S}/p$ be the cofiber of p. It is obvious that

$$H_*(\mathbb{S}/p; \mathbb{Z}) = \begin{cases} \mathbb{Z}/p & , * = 0, \\ 0 & , \text{else}. \end{cases}$$

Now "X is $\mathbb{S}/p$-acyclic" is equivalent to "$p : \pi_*X \to \pi_*X$ is an equivalence." We will see that being $\mathbb{S}/p$-local is a "completion" condition on π_*X, i.e. $X \xrightarrow{\cong} \lim_n X/p^n$.

Example 4.5. Let

$$\mathbb{S}[p^{-1}] = \operatorname{colim}\left(\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \cdots\right).$$

So $\pi_*\mathbb{S}[p^{-1}] = (\pi_*\mathbb{S})[p^{-1}]$. X is $\mathbb{S}[p^{-1}]$-acyclic iff π_*X is locally p-power torsion. In this case,

$$\pi_*X = \bigcup_n \pi_*X[p^n],$$

where $[p^n]$ denotes the p^n-torsion in π_*X.

X is $\mathbb{S}[p^{-1}]$-local iff $p : X \to X$ is an equivalence.

Theorem 4.6. *The inclusion $\mathsf{Sp}_E \to \mathsf{Sp}$ has a left adjoint $L_E : \mathsf{Sp} \to \mathsf{Sp}_E$ called E-localization or localization of E. That is, for any $X \in \mathsf{Sp}$, there exists $L_EX \in \mathsf{Sp}_E$ and a map $X \to L_EX$ such that for any $Y \in \mathsf{Sp}_E$, there is an isomorphism*

$$\hom_{\mathsf{Sp}_E}(L_EX, Y) \xrightarrow{\cong} \hom_{\mathsf{Sp}}(X, Y).$$

To introduce the Bousfield localization, we need the uncountable regular cardinal κ. In other word, κ is an uncountable cardinality such that every unbounded subset $C \subset \kappa$ has cardinality κ. One crude explanation is that κ cannot be broken down into a small number of smaller parts.

Definition 4.7. A simplicial set I is **κ-small** if it has at most κ non-degenerate simplices.

Definition 4.8. A simplicial set S is **κ-filtered** if for every κ-small simplicial set I and every map $f : I \to S$, there exists an extension $\overline{f} : I^{\triangleright} \to S$. Here $I^{\triangleright}$ is the right cone, i.e. a simplicial set sending $[n]$ to $\{(f, \sigma) : f : \Delta^n \to \Delta^1, \sigma : f^{-1} \circ - \to I\}$.

Example 4.9. If $\kappa = \aleph_0$, then κ-filtered is just filtered.

Proposition 4.10. *If $\mathcal{C}$ is an ∞-category with all κ-small colimits, then $\mathcal{C}$ is κ-filtered.*

We denote the full subcategory of Sp generated bu desuspensions $\Sigma^n \mathbb{S}$ of the sphere under κ-small colimits by Sp^{κ}.

Proposition 4.11. *Sp^{κ} has only one set of equivalence classes of elements.*

Proof. Write $\mathsf{Sp}^{\kappa} = \bigcup_{\alpha<\kappa} \mathcal{C}_\alpha$ such that each $\mathcal{C}_\alpha$ is an essentially small category. Let $\mathcal{C}_0$ be the full subcategory spanned by $\{\Sigma^n \mathbb{S}\}_{n\in\mathbb{Z}}$. Suppose we have defined $\mathcal{C}_\alpha$ for ordinal number $\alpha < \kappa$. We let $\mathcal{C}_{\alpha+1}$ be the full subcategory of Sp spanned by all colimits of functors $F : I \to \mathcal{C}_\alpha$, where I is κ-small. It is clear that $\mathcal{C}_{\alpha+1}$ is also essentially small. For ordinal $\lambda < \kappa$, we define

$$\mathcal{C}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{C}_\alpha.$$

Our goal is to show that $\mathsf{Sp}^{\kappa} = \mathcal{C}_\lambda$. "$\supset$" is clear since if $\mathcal{C}_\alpha \subset \mathsf{Sp}^{\kappa}$, then $\mathcal{C}_{\alpha+1} \subset \mathsf{Sp}^{\kappa}$. For "$\subset$" part, note that $\mathcal{C}_\kappa \supset \mathcal{C}_0 = \{\Sigma^n \mathbb{S}\}_{n\in\mathbb{Z}}$. It suffices to show $\mathcal{C}_\kappa$ is closed under κ-small colimits of $F : I \to \mathcal{C}_\kappa$. We claim that F factors through $\mathcal{C}_\alpha$ for some α. Indeed, for every $i \in I$, there exists $\alpha_i < \kappa$, $F_i \in \mathcal{C}_{\alpha_i}$. Moreover $|I| < \kappa$, so from the definition of regular cardinal, $\alpha = \sup_{i\in I} \alpha_i < \kappa$, yielding the claim. Therefore, $\operatorname{colim}_I F \in \mathcal{C}_{\alpha+1} \subset \mathcal{C}_\kappa$. □

Theorem 4.12. *If I is κ-filtered, then* $\operatorname{colim} : \operatorname{Fun}(I, \mathsf{Spaces}) \to \mathsf{Spaces}$ *commutes with κ-small limits.*

Lemma 4.13. *If I is κ-filtered, then* $\hom_{\mathsf{Sp}}(X, -) : \mathsf{Sp} \to \mathsf{Space}$ *commutes with κ-filtered colimits.*

Proof. Suffice to show for $X = \Sigma^n \mathbb{S}$, which suffice to show that the subcategory of X such that $\hom_{\mathsf{Sp}}(X, -)$ commutes with κ-filtered colimits (say this is property P) is closed under κ-small colimits. If $F : I \to \mathsf{Sp}$ is a functor with I κ-small and F_i has the property P for all i, then

$$\hom_{\mathsf{Sp}}(\operatorname{colim}_I F, -) = \lim_I \hom_{\mathsf{Sp}}(F_i, -),$$

and the lemma follows from theorem 4.12. □

In fact, lemma 4.13 is an "iff". If $\hom_{\mathsf{Sp}}(X, -)$ commutes with κ-filtered colimits, then X is κ-small.

Proposition 4.14. *Let κ be uncountable. Then $|\pi_* X| < \kappa$ iff X is κ-small. In particular, for any $X \in \mathsf{Sp}$, there is an κ such that $X \in \mathsf{Sp}^{\kappa}$.*

Before we prove the theorem, we need the following theorem:

Theorem 4.15 (Serre). *For $i > n$, the homotopy group $\pi_i S^n$ is finite unless $i = 2n-1$ and n is even. In particular, $\pi_i \mathbb{S} = \pi_{2i+2} S^{i+2}$ is a finite group for every positive i.*

Proof of Proposition 4.14. First to prove the converse. By Serre's theorem, $\Sigma^n \mathbb{S} \in A$ for all $n \in \mathbb{Z}$, where $A = \{X \in \mathsf{Sp} : |\pi_* X| < \kappa\}$. It suffices to prove A is closed under κ-small colimits, or equivalently, under cofibers and κ-small coproducts. κ-small coproducts part is clear because $\pi_* \bigoplus_{i\in I} X_i \cong \bigoplus_{i \in I} \pi_* X_i$. So if $|I| < \kappa$, $|\pi_* X_i| < \kappa$, then $\left|\bigoplus_{i\in I} \pi_* X_i\right| < \kappa$ (note that $\bigoplus_{i\in I} \pi_* X_i$ is the union of the subgroups of those elements supported on a given finite subset). The cofibers part goes as follows: suppose $X, Y \in A$, $f : X \to Y$, $Z = \mathrm{Cof}(f)$. From the l.e.s.

$$\pi_* X \to \pi_* Y \to \pi_* Z \to \pi_{*-1} X \to \cdots,$$

there is an associated s.e.s

$$0 \to \operatorname{coker} \pi_* f \to \pi_* Z \to \ker \pi_{*-1} f \to 0.$$

Both $|\operatorname{coker} \pi_* f| < \kappa$ and $|\ker \pi_{*-1} f| < \kappa$ since $\ker \pi_{*-1} f$ is a subgroup of $\pi_{*-1} X$ and $\operatorname{coker} \pi_* f$ is quotient of $\pi_* Y$. It is straightforward that $|\pi_* Z| < \kappa$.

On the other hand, suppose we are given a spectrum X with $|\pi_* X| < \kappa$. Start with

$$X_0 = 0 \to X_1 \to X_2 \to \cdots \to X,$$

such that $X_i \in \mathsf{Sp}$ for all i and $\operatorname{colim} X_i = X$. Let $f_i : X_i \to X$ and $F_i = \mathrm{Fib}(f_i)$. We have the diagram

$$\begin{tikzcd}
F_i \arrow[r] \arrow[d, "h_i"] & X_i \arrow[r, "f_i"] \arrow[d] & X \arrow[d, equal] \\
F_{i+1} \arrow[r] & X_{i+1} \arrow[r, "f_{i+1}"] & X
\end{tikzcd}$$

Our next goal is to construct X_i such that $h_i = 0$ when passing to π_*, yielding $\operatorname{colim} F_i = \mathrm{Fib}(\operatorname{colim} X_i \to X) = 0$. Inductively, suppose we have already defined X_i. X_{i+1} is given through

$$\begin{tikzcd}
\bigoplus_{\substack{\alpha \in \pi_n F_i \\ n = |\alpha|, \forall n \in \mathbb{Z}}} \Sigma^{|\alpha|}\mathbb{S} \arrow[d, "\phi_1"] \arrow[dr] & & \\
F_i \arrow[r] \arrow[d, "\phi_2"] & X_i \arrow[r] \arrow[d] & X \\
F_{i+1} \arrow[r] & X_{i+1} \arrow[ur] &
\end{tikzcd}$$

where both $\bigoplus \Sigma^{|\alpha|}\mathbb{S} \to X_i \to X_{i+1}$ and $F_{i+1} \to X_{i+1} \to X$ are cofiber sequences, and the left lower square is a pushout. Here $\bigoplus \Sigma^{|\alpha|}\mathbb{S}$ is κ-small because it is a κ-small coproduct of spheres. ϕ_1 is surjective on π_*. ϕ_2 is 0 on π_* by the l.e.s. X_i is κ-small by hypothesis, so X_{i+1} is also κ-small. The composite $\bigoplus \Sigma^{|\alpha|}\mathbb{S} \to X_i \to X$ is null-homotopic because it factors through $F_i \to X_i$, which is null-homotopic after composing $X_i \to X$ since X is the fiber. Hence, we conclude our proof. ∎

Proposition 4.16. *Let $X \in \mathsf{Sp}$. Then the map*

$$\operatorname{colim}_{Y \in \mathsf{Sp}^\kappa / X} Y \to X$$

is an equivalence. In particular, every spectrum is a κ-filtered colimits of some κ-small spectrum.

Proof. Note that Sp^κ/X has all κ-small colimits, so it is κ-filtered. Recall that π_* commutes with filtered colimits, so it is enough to prove

$$\operatorname{colim}_{Y \in \mathsf{Sp}^\kappa/X} \pi_* Y \xrightarrow{\cong} \pi_* X.$$

It is surjective because if we have $\alpha \in \pi_n X$, then it corresponds to $[\alpha : \Sigma^n \mathcal{S} \to X] \in Y \in \mathsf{Sp}^\kappa/X$ from the fundamental class $1 \in \pi_n \Sigma^n \mathbb{S}$. Suppose now we have $f : Y \to X$, $Y \in \mathsf{Sp}^\kappa$, and $\alpha \in \pi_n Y$ going to 0 in $\pi_n X$. To prove the injectivity, we need to show this goes to 0 to some κ-small spectrum receiving a map from Y. This is true by looking at the diagram

$$\begin{tikzcd} \Sigma^n \mathbb{S} \arrow[d, "\alpha"] \arrow[dr, "0"] & \\ Y \arrow[r, "f"] \arrow[d] & X \\ \operatorname{Cof}(\alpha) \arrow[ur, "f'"'] & \end{tikzcd}$$

Now $\operatorname{Cof}(\alpha)$ is still κ-small. The diagram implies $[Y, \alpha] \mapsto 0$ in $\operatorname{Cof}(\alpha) \xrightarrow{f'} X$, and so it is 0 in the colimit. ☐

Proposition 4.17. *Let $E \in \mathsf{Sp}$ and κ be an uncountable regular cardinal. Then any E-acyclic spectrum is a κ-filtered colimit of some κ-small E-acyclic spectrum.*

In fancier language, the proposition said that $\ker(E \otimes - : \mathsf{Sp} \to \mathsf{Sp})$ is κ-accessible.

Corollary 4.18. *There exists a E-acyclic spectrum A such that X is E-local iff* $\hom(A, X) = 0$.

Proof. Let $\{A_i\}_{i \in I}$ be the set of κ-small E-acyclic spectrum, and $A = \bigoplus_{i \in I} A_i$. Now $\hom(A, X) = 0$ is the same as $\hom(A_i, X) = 0$ for all i. If A' is an E-acyclic spectrum, then $\hom(A', X) \cong \lim_i \hom(A_i, X)$ by Proposition 4.17. From the assumption $\hom(A, X) = 0$, we are done. ☐

Back to the localization $L_E X$. We abbreviate it to X_E. One crucial fact we will frequently use is the following:

Theorem 4.19. *Let $F : \mathbb{C} \to \mathcal{D}$ be a functor of ∞-category. For any $d \in \mathcal{D}$,* $\hom(d, F(-))$ *is corepresentable, then F has a left adjoint sending d to the corresponding element.*

Proof of Theorem 4.6. Given $X \in \mathsf{Sp}$, we want to construct $X \to L_E X$. Fix A as in the Corollary 4.18. Note $A \in \mathsf{Sp}^\kappa$ for some regular cardinal κ. Let $X_0 = X$, $X_1 = \operatorname{Cof}(\bigoplus_{\substack{f : \Sigma^A \to X \\ n \in \mathbb{Z}}} \Sigma^n A \to X)$. Inductively, we define

$$X_{\alpha+1} = \operatorname{Cof}\left(\bigoplus_{\substack{f : \Sigma^A \to X_\alpha \\ n \in \mathbb{Z}}} \Sigma^n A \to X \right),$$

and

$$X_\lambda = \operatorname{colim}_{\alpha<\lambda} X_\alpha,$$

where α is an ordinal and λ is a limit ordinal. We claim that $X \to X_\alpha$ is an E-equivalence for all $\alpha \leq \kappa$, and X_κ is E-local. In this case, $X_\kappa = L_E X$. Indeed, the first part of the claim follows from the fact that $X_\alpha \to X_{\alpha+1}$ is an E-equivalence by construction, and E-equivalences are stable under colimit. So $X \to X_\lambda = \operatorname{colim}_{\alpha<\lambda}(X \to X_\alpha)$ is an E-equivalence. To prove the second part of the claim, note that

$$X_\kappa = \operatorname{colim}_{\alpha<\kappa} X_\alpha,$$

and $\{\alpha \mid \alpha < \kappa\}$ is κ-filtered poset. Now

$$\hom(A, X_\kappa) \cong \operatorname{colim}_{\alpha<\kappa} \hom(A, X_\alpha).$$

Let $f \in \pi_n \hom(A, X_\kappa)$. Then f factor through X_α for some $\alpha < \kappa$. But its image in $X_{\alpha+1}$ is 0, so its image in $\pi_n \hom(A, X_\kappa)$ is 0. This indicates that X_κ is E-local. □

Remark 4.20. This proof is secretly basically the proof of adjoint functor theorem in our setting.

Fix X to be an E-acyclic spectrum.

Proposition 4.21. *For any $f \in \pi_n X$, there is a κ-small acyclic spectrum W with a map $g : W \to X$ such that f lifts to $\pi_* W$.*

We make the comment that if F is a finite spectrum, then $|E_*F| = |\pi_*(E \otimes F)| < \kappa$. Such spectra are closed under finite direct sums and cofibers.

Proof. To construct W as the colimit of

$$W_0 \to W_1 \to \cdots \to X$$

where every step is to add a few cells, and $W_0 = \Sigma^n \mathbb{S}$. The composite of maps is denoted $f = g_0$. Our construction will be made such that $E_*W_i \to E_*W_{i+1}$ is the zero map. It implies $E_*W = \operatorname{colim}_i E_*W_i = 0$. We assume $|E_*W_i| < \kappa$.

Inductively, suppose we have already constructed $g_i : W_i \to X$. Let $\overline{W_i} = \operatorname{Fib}(W_i \to X) = E_*\overline{W_i} \xrightarrow{\cong} E_*W_i$ (by the fact that X is E-acyclic). Note $\overline{W_i} = \operatorname{colim}_{j\in J} F_j$, where F_j is some finite spectrum, and J is filtered. So $E_*\overline{W_i} = \operatorname{colim}_{j\in J} E_*F_j$. For any $x \in E_*\overline{W_i}$, there is a finite spectrum F_x with a map $F_x \to \overline{W_i}$ such that x is in the image of $E_*F_x \to E_*\overline{W_i}$. Consider

$$\bigoplus_{x\in E_*\overline{W_i}} F_x \to \overline{W_i} \to W_i \to X. \tag{4.1}$$

$\bigoplus_{x\in E_*W_i} F_x$ is κ-small because $|E_*W_i| < \kappa$ by induction and all F_x are finite. Let $W_{i+1} = \operatorname{Cof}(\bigoplus_{x\in E_*W_i} F_x \to W_i)$. It is clearly κ-small. The composite (4.1) is canonically null-homotopic because one can insert a null-homotopic map $\overline{W_i} \to X$. So it factors through $W_{i+1} \to X$. By definition, $E_*(\bigoplus_{x\in E_*\overline{W_i}} F_x) \to E_*\overline{W_i}$ is surjective. We obtain

$$\begin{tikzcd}
E_*\left(\bigoplus_{x\in E_*\overline{W_i}} F_x\right) \arrow[r, two heads] & E_*\overline{W_i} \arrow[r, "\cong"] & E_*W_i \arrow[d, "\phi"] \\
& & E_*W_{i+1} \arrow[d] \\
& & E_{*-1}\left(\bigoplus_{x\in E_*\overline{W_i}} F_x\right)
\end{tikzcd}$$

where $\phi = 0$ by the l.e.s. in E_*, and $|E_*W_{i+1}| < \kappa$. □

Now we are ready to prove Proposition 4.17.

Proof of Proposition 4.17. Let J be the ∞-category of κ-small E-acyclic spectra with a map to X. In fact,

$$J = \mathsf{Sp}^\kappa \times_{\mathsf{Sp}} \mathsf{Sp}/X.$$

Take the colimit $\operatorname{colim}_{W\in J} W \to X$. We claim that (1) J has all κ-small colimits, and hence J is κ-filtered; (2) this map is an equivalence. Indeed, note that κ-small E-acyclic spectra are closed under κ-small colimits, so (1) holds. To prove (2), it suffices to show

$$\operatorname{colim}_{W\in J}\pi_*W \to \pi_*X$$

is an equivalence as before. We know it is surjective by Proposition 4.21. Take $[g: W\to X]\in J$, $f\in\pi_nW$ such that $gf=0$. Look at the diagram

$$\begin{tikzcd}
& \Sigma^n\mathbb{S} \arrow[dl] \arrow[d, "f"] & \\
F \arrow[r] & W \arrow[r, "g"] & X
\end{tikzcd}$$

where $F = \mathrm{Fib}(g)$. F is E-acyclic since W and X are. By Proposition 4.21, we can find κ-small E-acyclic $\overline{W}$ with a map $\overline{g}: \overline{W}\to F$ such that $\overline{f}$ lifts to $\tilde{f}$. Look at the new diagram

$$\begin{tikzcd}
& \Sigma^n\mathbb{S} \arrow[dl, "\tilde{f}"'] \arrow[d, "f"] & \\
\overline{W} \arrow[r, "h"] & W \arrow[r, "g"] \arrow[d, "h'"] & X \\
& W' \arrow[ur, "g'"'] &
\end{tikzcd}$$

Here gh has a canonical null-homotopy since it factors through f. So it is fine to take the cofiber $W' = \mathrm{Cof}(h)$ and factor g by g'. It is clear that W' is κ-small since W and $\overline{W}$ are. Now

$$g'h'f = g'h'h\tilde{f} = 0$$

since $h'h = 0$. □

Now we finish the basic set-up of the Bousfield localization. Let's go through some examples. First was introduced in Example 4.5.

Let $p\in\mathbb{Z}$ be a prime. Remember

$$\mathbb{S}[p^{-1}] = \operatorname{colim}\left(\mathbb{S}\xrightarrow{p}\mathbb{S}\xrightarrow{p}\mathbb{S}\xrightarrow{p}\cdots\right).$$

This is the $\mathbb{Z}[1/p]$-Moore spectrum since

$$H_*\mathbb{S}[p^{-1}] = \begin{cases} \mathbb{Z}[1/p] & , * = 0, \\ 0 & , \text{else}. \end{cases}$$

Our goal is to study $L_{\mathbb{S}[p^{-1}]}$. Notice that

$$\pi_*(\mathbb{S}[p^{-1}] \otimes X) = \operatorname{colim}\left(\pi_* X \xrightarrow{p} \pi_* X \xrightarrow{p} \cdots\right) = \pi_* X[1/p].$$

Lemma 4.22. *A spectrum X is $\mathbb{S}[p^{-1}]$-acyclic iff p acts locally nilpotently on $\pi_* X$, i.e. every element of $\pi_* X$ is p-power torsion.*

For example, $\mathbb{S}/p$ is $\mathbb{S}[p^{-1}]$-acyclic since

$$\mathbb{S}/p \otimes \mathbb{S}[p^{-1}] = \mathbb{S}[1/p]/p = 0.$$

Lemma 4.23. *A spectrum X is $\mathbb{S}[p^{-1}]$-local iff p acts invertibly on X (or equivalently on $\pi_* X$).*

Proof. Suppose X is $\mathbb{S}[p^{-1}]$-local. To prove $\mathbb{S} \otimes X = 0$, or equivalently $X \xrightarrow{p} X$ is an equivalence, or equivalently $\hom(\mathbb{S}, X) \xrightarrow{\hom(p,X)} \hom(\mathbb{S}, X)$ is an equivalence, we observe that the fiber of this map is $\hom(\mathbb{S}/p, X) = 0$, and $\hom(\mathbb{S}, X)$ is $\mathbb{S}[p^{-1}]$-acyclic. So X is $\mathbb{S}[p^{-1}]$-local.

Conversely, suppose p acts invertibly on X. Then the map

$$\hom(\mathbb{S}[p^{-1}], X) \to X,$$

which is the precomposition of $\mathbb{S} \to \mathbb{S}[p^{-1}]$, is an equivalence. Note

$$\hom(\mathbb{S}[p^{-1}], X) = \lim\left(\hom(\mathbb{S}, X) \xleftarrow{p} \hom(\mathbb{S}, X) \xleftarrow{p} \cdots\right) \xrightarrow{\cong} X.$$

Let W be $\mathbb{S}[p^{-1}]$-acyclic, then

$$\hom(W, X) \cong \hom(W, \hom(\mathbb{S}[p^{-1}], X)) \cong \hom(W \otimes \mathbb{S}[p^{-1}], X) = 0.$$

□

Theorem 4.24. *The map $X \to \mathbb{S}[p^{-1}] \otimes X$ exhibits $X[p^{-1}]$ as the $\mathbb{S}[p^{-1}]$-localization of X.*

Proof. Note that $\mathbb{S}[p^{-1}] \otimes X$ is $\mathbb{S}[p^{-1}]$-local by Lemma 4.23 (passing to π_*). So it is enough to show this map is $\mathbb{S}[p^{-1}]$-equivalence. Regard this map as a colimit of $\mathbb{S}[p^{-1}]$-equivalence. That is,

$$\begin{tikzcd} X = \operatorname{colim}(X \arrow[r, equal] \arrow[d] & X \arrow[r, equal] \arrow[d, "p"] & X \arrow[r, equal] \arrow[d, "p^2"] & X \arrow[r, equal] \arrow[d, "p^3"] & \cdots) \\ X[p^{-1}] = \operatorname{colim}(X \arrow[r, "p"] & X \arrow[r, "p"] & X \arrow[r, "p"] & X \arrow[r, "p"] & \cdots) \end{tikzcd}$$

We need to show $(X \to X[p^{-1}]) = \operatorname{colim}(X \xrightarrow{p^n} X)$ is an $\mathbb{S}[p^{-1}]$-equivalence. This is obvious, since their cofiber is $X \otimes \mathbb{S}/p$, which amounts to show $\mathbb{S}[p^{-1}] \otimes \mathbb{S}/p^n \cong \mathbb{S}[p^{-1}]/p = 0$. □

One should be warned that every element of $\pi_*\mathbb{S}/p$ is not p-torsion in general (though it is p-power torsion). For example, $\pi_1\mathbb{S}/2 = \mathbb{Z}/4$.

Remark 4.25. There is always a map $X \to (L_E\mathbb{S}) \otimes X$, which is always an E-equivalence. When the target is E-local for all X, $(L_E\mathbb{S}) \otimes X \cong L_E X$. In this case, the localization is called **smashing**.

We generalize the example to multiple primes. Let $P = \{p_1, p_2, \cdots\}$ be a set of primes, and

$$\mathbb{S}[P^{-1}] = \operatorname{colim}(\mathbb{S} \xrightarrow{p_1} \mathbb{S} \xrightarrow{p_1p_2} \mathbb{S} \xrightarrow{p_1p_2p_3} \cdots).$$

For any $X \in \mathbb{S}\mathsf{l}$, $\pi_*(\mathbb{S}[P^{-1}] \otimes X) = \pi_* X[P^{-1}]$.

Proposition 4.26. *A spectrum X is $\mathbb{S}[P^{-1}]$-local if all primes in P act invertibly on X.*

Proposition 4.27. *The map $X \to \mathbb{S}[P^{-1}] \otimes X =: X[P^{-1}]$ exhibits RHS as the $\mathbb{S}[P^{-1}]$-localization of X*

The proofs of these two propositions are identical to the previous proof when P contains only a single prime.

If P is the set of all primes, then $X[P^{-1}] =: X_{\mathbb{Q}}$ is called the **rationalization** of X. Now

$$\pi_*\mathbb{S}_{\mathbb{Q}} = \pi_*\mathbb{S} \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & , * = 0, \\ 0 & , \text{else}. \end{cases}$$

So $\mathbb{S}_{\mathbb{Q}} = H\mathbb{Q}$.

Proposition 4.28. *Let $X \in \mathsf{Sp}_{\mathbb{Q}}$. There is an equivalence*

$$X \cong \bigoplus_{n \in \mathbb{Z}} \Sigma^n H\pi_n X.$$

Proof. It suffices to construct maps $\Sigma^n H\pi_n X \to X$ which are isomorphic on π_n. Since $\pi_n X$ is a $\mathbb{Q}$-vector space, so it has a basis $\{e_i\}_{i \in I_n}$. We have a map

$$\bigoplus_{i \in I_n} \Sigma^n \mathbb{S} \xrightarrow{\{e_i\}} X,$$

where X is rational. By universal property, it factors through

$$\begin{tikzcd} \bigoplus_{i \in I_n} \Sigma^n \mathbb{S} \arrow[rr, "\{e_i\}"] \arrow[dr] & & X \\ & \bigoplus_{i \in I_n} \Sigma^n H\mathbb{Q} \arrow[ur] & \end{tikzcd}$$

Since the rationalization is a left adjoint, it commutes with the colimits. Therefore, $\bigoplus_{i \in I_n} \Sigma^n H\mathbb{Q} = \Sigma^n H\pi_n X$. We conclude our proof. ☐

Corollary 4.29. $KU_{\mathbb{Q}} \cong \bigoplus_{n \in \mathbb{Z}} \Sigma^{2n} H\mathbb{Q}$.

Remark 4.30. The map

$$KU \to \bigoplus_{n \in \mathbb{Z}} \Sigma^{2n} H\mathbb{Q} = KU_{\mathbb{Q}}$$

is the **Chern character**.

Corollary 4.31. *Let X be a finite space. Then*

$$KU^0(X)_{\mathbb{Q}} \cong \bigoplus_{n \geq 0} H^{2n}(X; \mathbb{Q}).$$

Proof. Note that $\hom(\Sigma^\infty X_+, -)$ commutes with filtered colimits in out setting. So

$$\begin{aligned} KU^0(X)_{\mathbb{Q}} &\cong \pi_0 \hom(\Sigma^\infty X_+, KU)_{\mathbb{Q}} \cong \pi_0 \hom(\Sigma^\infty X_+, KU_{\mathbb{Q}}) \\ &\cong \pi_0 \hom\left(\Sigma^\infty X_+, \bigoplus_{n\in\mathbb{Z}} \Sigma^{2n} H\mathbb{Q}\right) \\ &\cong \bigoplus_{n\in\mathbb{Z}} \pi_{-2n} \hom(\Sigma^\infty X_+, H\mathbb{Q}). \end{aligned}$$

□

In general, we have the following theorem:

Theorem 4.32 (Schwede-Shipley)**.** *The ∞-category $\mathsf{Sp}_{\mathbb{Q}}$ of rational spectra is equivalent to the derived ∞-category $\mathcal{D}(\mathbb{Q})$ of the rational numbers, i.e. $\mathcal{D}(\mathbb{Q}) =$* $\mathsf{Ch}(\mathbb{Q})[\text{quasi-isomorphisms}^{-1}]$

5. Lecture 5 (Video Break): 10/25/2022

Today's lecture based on the video (P8) of Taylor Lawson's talk *Homotopy Theory: Tools and Applications* (2017) at University of Illinois, Urbana-Champaign and his paper Secondary power operations and the Brown–Peterson spectrum at the prime 2 (2018).

The second part was the talk by Haynes Miller *Things I learned from Doug: the origins of chromatic homotopy theory* (2017) at Reed College, Portland, Oregon. I won't write them down because this is basically the mathematical history. However, I encourage you to watch for yourself as it will greatly motivate you to the development of modern chromatic homotopy theory.

We know there are a couple of spectra that admit E_∞-ring structures, such that MU and $H\mathcal{F}_p$. In particular, the E_∞-algebra structure is central to Quillen's relation between stable homotopy theories and fgls. After p-localization, MU decomposes into summands equivalent to the Brown-Peterson spectrum BP. One would ask that does BP also admit an E_∞-algebra structure? (See notes on operads in 2022 HMS seminar at Oxford held by Peize, Dekun, Shuwei and myself)

Unfortunately, this assertion is false when $p = 2$. We will briefly see how it goes in this lecture.

Let R be an E_∞-ring and H is the mod 2 homology theory. H_*R has the structure called the (Araki-Kudo-)Dyer-Lashof (abbr. Dyer-Lashof) operations $Q^s : H_nR \to H_{n+s}R$. They are natural transformations on the homotopy category of R-modules. They satisfies the following relations:

Proposition 5.1. *(1) The additive relations: $Q^s(x+y) = Q^s(x) + Q^s(y)$,*

(2) The instability relations: $Q^s(x) = x^2$ when $|x| = s$, and $Q^s(x) = 0$ when $|x| > s$.

(3) Cartan formula: $Q^s(xy) = \sum_{p+q=s} Q^p(x)Q^q(y)$.

(4) Adem relations: if $r > 2s$, then $Q^rQ^s = \sum_i \binom{i-s-q}{2i-r} Q^{r+s-i}Q^i$.

With such operations in hand, one can start to prove some non-existence of E_∞-structure. Before we provide an example, we need to introduce some terminologies.

Definition 5.2. Two spectra E and F are Bousfield equivalent if for each spectrum X, $E \wedge X$ is contractible iff $F \wedge X$ is contractible. We denote the Bousfield equivalence class of E by $\langle E \rangle$.

We use the notion $\langle E \rangle \geq \langle F \rangle$ if for each spectrum X, the contractibility of $E \wedge X$ implies that of $F \wedge X$. We say $\langle E \rangle > \langle F \rangle$ if $\langle E \rangle \geq \langle F \rangle$ and $\langle E \rangle \neq \langle F \rangle$.

Two equation of importance:

- $\langle E \rangle \wedge \langle E \rangle = \langle E \wedge F \rangle$.
- $\langle E \rangle \vee \langle F \rangle = \langle E \vee F \rangle$.

Definition 5.3. A class $\langle E \rangle$ has a complement $\langle E \rangle^c$ if $\langle E \rangle \wedge \langle E \rangle^c = \langle \mathrm{pt} \rangle$ and $\langle E \rangle \vee \langle E \rangle^c = \langle \mathbb{S} \rangle$. One should note that not all classes have complements.

For any spectrum E, we have

- $\langle \mathbb{S} \rangle \geq \langle E \rangle \geq \langle \mathrm{pt} \rangle$.
- $\langle \mathbb{S} \rangle \wedge \langle E \rangle = \langle E \rangle$.
- $\langle \mathbb{S} \rangle \vee \langle E \rangle = \langle \mathbb{S} \rangle$.
- $\langle \mathrm{pt} \rangle \vee \langle E \rangle = \langle E \rangle$.

- $\langle \mathrm{pt}\rangle \wedge \langle E\rangle = \langle \mathrm{pt}\rangle$.

Let $X(n)$ be a collection of ring spectra with $1 \leq n \leq \infty$ that used to prove the nilpotence theorem by Deviratz-Hopkins-Smith (we will see this later). Explicitly, $X(1) = \mathbb{S}$ and $X(\infty) = MU$ such that

$$\langle X(n)\rangle \geq \langle X(n+1)\rangle\,,$$

for each n, with

$$\left\langle X(p^k-1)_{(p)}\right\rangle > \left\langle X(p^k)_{(p)}\right\rangle$$

for some prime p and each $k \geq 0$. These spectra are constructed in terms of vector bundles and Thom spectra, Let SU = infinite special linear group. Bott periodicity gives

$$\Omega SU \xrightarrow{\simeq} BU.$$

Composing this with the loops on the inclusion of $SU(n)$ into SU, we get

$$\Omega SU(n) \to BU.$$

The associated Thom spectrum is $X(n)$. By a similar calculation to MU, we see

$$H_*(X(n)) \cong \mathbb{Z}[b_1, b_2, \cdots],$$

where $|b_i| = 2i$. $X(n)$ has a map down to the bottom of the Postnikov tower

$$X(n) \to H\mathbb{Z} \to H\mathbb{Z}/2.$$

If $X(n)$ had an E_∞ structure, then we take the $\mod 2$ homology and obtain

$$H_*X(n) \to H_*H\mathbb{Z}/2 = \mathcal{A}_*,$$

where $\mathcal{A}_*$ is the dual Steenrod algebra. Now LHS is "small" in the sense that it is a polynomial algebra on finitely many generators, and RHS is "large" with Dyer-Lashof operations in it (computed by Steinberg). In fact, the Dyer-Lashof operations act on a so big scale that there hardly exists some subalgebra that closed under such operations. Therefore, it is not possible for $H_*X(n)$ to be big enough for its image to be closed under Q^s. So $X(n)$ doesn't admit an E_*-structure.

Remark 5.4. Actually $X(n)$ doesn't admit an E_3-structure. But it does for E_2 by the properties of Thom spectra.

If we try to deal with BP with the same procedure, we can map it to

$$BP \to H\mathbb{Z}_{(2)} \to H\mathbb{Z}/2,$$

and look at the effect on homology. We get an inclusion

$$H_*BP = \mathcal{F}_2[\xi_1^2, \xi_2^2, \cdots] \to \mathcal{F}_2[\xi_1, \xi_2, \cdots].$$

However, this provides no contradiction using Q^s. This is because we can show that $H_*BP = \mathrm{im}\,(H_*MU \to \mathcal{A}_*)$, which is closed under the Dyer-Lashof operations.

There is a way to tackle the problem. Namely, we try to find some relations like

$$Q^8(\xi_1^2) = \xi_1^4 Q^4(\xi_1^2),$$

which doesn't hold for the corresponding elements in degree 2 of H_2MU. Summarize the idea (due to Hu-Kriz-May), if we have an E_∞-structure on BP with an

E_∞-map $BP \to MU$, then passing to homology

$$\begin{tikzcd} & H_*MU \arrow[dr] & \\ H_*BP \arrow[ur] \arrow[rr] & & \mathcal{A}_* \end{tikzcd}$$

some relations on H_*BP will not hold on H_*MU, whence the contradiction is obtained.

To do this, we need the secondary operations by Adams. Let $\mathcal{C}$ be a category enriched in pointed spaces under smash product $\wedge$ and $\mathrm{Map}_{\mathcal{C}}(x, y)$ be the mapping space between any pair of objects of $\mathcal{C}$. We make the convention that the basepoint of this mapping space is null (or $*$), satisfying $f* = *f = f$ for any f.

Definition 5.5. Suppose that we are given the following data:

(1) a sequence (X_0, X_1, X_2, X_3) of objects in $\mathcal{C}$,
(2) maps $f_{ij} : X_i \to X_j$ for $i < j$, and
(3) paths $h_{ijk} : f_{jk}f_{ij} \to f_{ik}$ in $\mathrm{Map}_{\mathcal{C}}(X_i, X_k)$ for $i < j < k$.

Then the associated **secondary composite** is the element of $\pi_1(\mathrm{Map}_{\mathcal{C}}(X_0, X_3), f_{03})$ represented by the path composite

$$h_{023}^{-1} \cdot (f_{23}h_{012})^{-1} \cdot (h_{123}f_{01}) \cdot h_{013},$$

viewed as a loop based at f_{03}:

$$\begin{tikzcd} f_{03} \arrow[r, "h_{023}^{-1}"] & f_{23}f_{02} \arrow[d, "f_{23}h_{012}^{-1}"] \\ f_{13}f_{01} \arrow[u, "h_{013}"] & f_{23}f_{12}f_{01} \arrow[l, "h_{123}f_{01}"] \end{tikzcd}$$

Definition 5.6. Suppose we have maps $X_0 \xrightarrow{f} X_1 \xrightarrow{g} X_2$. A **tethering** of the composite $g \circ f$ is a homotopy class of path $g \circ f \to *$ in the space $\mathrm{Map}_{\mathcal{C}}(X_0, X_2)$. We write $g \overset{h}{\leftrightsquigarrow} f$ to indicate such a tethering, and $g \leftrightsquigarrow f$ to indicate that there is a chosen tethering which is either implicit or not important to name.

Definition 5.7. Suppose we have maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

and tethering $h \overset{v}{\leftrightsquigarrow} g \overset{u}{\leftrightsquigarrow} f$. Then we define the element

$$\left\langle h \overset{v}{\leftrightsquigarrow} g \overset{u}{\leftrightsquigarrow} f \right\rangle \in \pi_1(\mathrm{Map}_{\mathcal{C}}(X, W), *)$$

to be the path composite $(h \circ u)^{-1} \cdot vf$ obtained by gluing together the two null-homotopies $hgf \to *$. This is an example of the secondary composite obtained by choosing $f_{02} = f_{03} = f_{13} = *$ and the trivial null-homotopies h_{013} and h_{023} in Definition 5.5.

Definition 5.8. Suppose we have maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

and tethering $h \overset{v}{\leftrightsquigarrow} g$. If gf is null-homotopic, we write

$$\left\langle h \overset{v}{\leftrightsquigarrow} g, f \right\rangle \subset \pi_1(\mathrm{Map}_{\mathcal{C}}(X, W), *)$$

for the set of all elements $\left\langle h \overset{v}{\leftrightsquigarrow} g \overset{u}{\leftrightsquigarrow} f \right\rangle$ as u ranges over possible tetherings, and refer to $\left\langle h \overset{v}{\leftrightsquigarrow} g, - \right\rangle$ as the **secondary operation** determined by the tethering. The set of maps f such that gf is null-homotopic is referred to as the **domain of definition** of this secondary operation, and the possibly multivalued nature of this function as the **indeterminacy** of the secondary operation. Similarly, we can define the secondary operations $\langle -, g \leftrightsquigarrow f \rangle$

Definition 5.9. Suppose we have maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

such that the double composites hg and gf are null-homotopic. We define the subset

$$\langle h, g, f \rangle \subset \pi_1(\mathrm{Map}_{\mathcal{C}}(X, W), *),$$

or **bracket**, to be the set of all secondary composites $\langle h \leftrightsquigarrow g \leftrightsquigarrow f \rangle$.

Proposition 5.10 (Juggling formulas)**.** *The bracket satisfies*

(1) $a\langle b, c, d \rangle \subset \langle ab, c, d \rangle$.
(2) $\langle ab, c, d \rangle \subset \langle a, bc, d \rangle$.
(3) $\langle a, bc, d \rangle \supset \langle a, b, cd \rangle$.
(4) $\langle a, b, cd \rangle \supset \langle a, b, c \rangle$.
(5) $a\langle b, c, d \rangle = [\langle a, b, c \rangle d]^{-1}$.

Example 5.11. Let $\mathcal{C} = \mathsf{Top}_*$. Consider the composite

$$S^n \xrightarrow{\alpha} S^m \xrightarrow{\beta} S^p \xrightarrow{y} X.$$

where $y \in \pi_p X$, $\beta \in \pi_m S^p$, $\alpha \in \pi_n S^m$ such that $\beta\alpha$ and $y\beta$ are null-homotopic. From this we get a bracket

$$\langle y, \beta, \alpha \rangle \subset \pi_1(\mathrm{Map}(S^n, X)) = \mathrm{Map}(S^{n+1}, X) = \pi_{n+1} X.$$

This recovers the Toda bracket.

Example 5.12. We can repeat the process in the sense of Eckmann-Hilton duality. Let A, B, C be some appropriate groups. Consider

$$X \xrightarrow{y} K(A, n) \xrightarrow{\theta} K(B, m) \xrightarrow{\psi} K(C, p),$$

where $y \in H^n(X; A)$, and $\theta : H^n(-; A) \to H^m(-; B)$ and $\psi : H^m(-; B) \to H^p(-; C)$ are cohomological operations satisfying $\theta(y) = 0$ and $\psi \circ \theta = 0$ (a relation between cohomological operations). From this we get a bracket

$$\langle \psi, \theta, y \rangle \in \pi_1(\mathrm{Map}(X, K(C, p))) = \pi_0 \mathrm{Map}(X, K(C, p-1)) = H^{p-1}(X; C),$$

which is a secondary operation (in the sense of secondary cohomological operations) applied to y. If we allow y to vary, then we get a secondary operation $\langle \psi, \theta, - \rangle$ with indeterminacy.

Example 5.13. Let $\mathcal{C}$ be the category of differential graded A-modules, where A is a dga. Then we can also define a Massey product which is a secondary operation. See [4] for details.

Let $f : Z \to X$ be a map. Then we can apply the juggling formula to get

$$f(\langle a, b, c\rangle) \subset \langle a, b, f(c)\rangle.$$

So functions between spaces preserve secondary operations.

Let $\mathcal{C}$ be the category of E_∞-algebras over the Eilenberg-MacLane spectrum $H = H\mathbb{Z}/2$. We denote the free algebra on S^n by $\mathbb{P}_H(S^n)$. Roughly speaking, $\mathbb{P}_H(S^n)$ is defined through the property that any map $\mathbb{P}_H(S^n) \to A$, where A is an E_∞ H-algebra, is precisely the same as $S^n \to A$ in Sp. This means the homotopy class of maps

$$[\mathbb{P}_H(S^n), A]_{E_\infty\text{-algebras}} = \pi_n A,$$

where the identification is canonical. By Yoneda lemma, if we can get the homotopy groups of one of these free algebras, then we obtain all the natural transformations we can have on homotopy. That is,

$$[\mathbb{P}_H(S^n), \mathbb{P}_H(S^m)] \xrightarrow{\text{よ}} (\eta : \pi_m(-) \Rightarrow \pi_n(-))$$

in E_∞ H-algebras. Here "よ" denotes the Yoneda (hiragana: よねだ) functor.

Recall that our goal is to show that $H_*BP \hookrightarrow H_*H = \mathcal{A}_*$ is not closed under secondary homotopy operations. Breaking down into pieces, we need to

(1) find some relations between homotopy operations;
(2) find some element in H_*BP such that the secondary operations can be applied to this elements and the operations give something which is not in H_*BP.

One of the canonical examples of relations that happens between the Dyer-Lashof operations Q^s is the Adem relations

$$Q^{20}Q^8 + Q^{18}Q^{10} + Q^{17}Q^{11} = 0.$$

If $x \in H_*BP$ with $Q^8x = Q^{10}x = Q^{11}x = 0$, then we can get a secondary operation. Luckily, we have the following proposition.

Proposition 5.14. *Let $x \in \pi_2 R$, where R is an E_∞ H-algebra (actually this condition can be weaken to E_{12} instead of E_∞). Then we have*

$$\begin{aligned}0 = {} & Q^{20}(Q^8x + x^2Q^4x) + Q^{18}(Q^{10}x + (Q^4x)^2) + Q^{17}Q^{11}x + \\ & x^4Q^{12}(Q^8x + x^2Q^4x) + (Q^7x)^2(Q^4x)^2 + (Q^5x)^2Q^9Q^5x + \\ & (Q^6x + x^4)^2Q^8Q^4x + (Q^9Q^7x)(Q^4x)^2 + Q^{10}(Q^6x + x^4)(Q^4x)^2 + \\ & (Q^3x)^2(Q^{11}Q^7x + Q^{10}Q^8x + x^4Q^6Q^4x).\end{aligned}$$

So there is a secondary operation Φ defined on elements x in $\pi_2 A$ (or $H_2R = \pi_2(H \wedge R)$), where A is an E_∞ H-algebra (resp. R is an E_∞-ring) which satisfy

(1) $Q^3x = Q^5x = Q^7x = Q^{11}x = 0$, and
(2) $Q^8x + x^2Q^4x = 0$, and
(3) $Q^{10}x + (Q^4x)^2 = 0$.

We denote these 3 conditions by (#). This secondary operation takes value (up to indeterminacy) in $\pi_{31}(A)$ (or $H_{31}R$). It is also defined on $\xi_1^2 \in \pi_2(H \wedge H) = \pi_*(\mathcal{A}_*)$ since ξ_1^2 satisfies all 3 conditions in (#) by Cartan formulas and complicated calculations. Note that the value of Φ on ξ_1^2 is shown to be exactly ξ_5 mod decomposables, which is not in H_*BP. Whence a contradiction is arrived under the assumption that BP has an E_∞-structure. In general, the contradiction arises for anything with the same homology as BP through degree 31. For example, the truncated Brown-Peterson spectrum $BP\langle n\rangle$ for $n \geq 4$.

The last problem is to calculate the secondary operations. The idea is to express a relationship between secondary operations by juggling after pushing to the secondary operations related to H_*MU, since most of relations in (#) hold in H_*MU ((2) of (#) doesn't hold). It will reduce the question to a much smaller and answerable problem. At last, we ends up the spectrum $H \wedge_{MU} H$, which is an H-algebra with Dyer-Lashof operations. $\pi_*(H \wedge_{MU} H) = \bigwedge_{\mathbb{F}_2}[\beta, \sigma x_1, \sigma x_2, \cdots]$, where β is the Bockstein (see [5, Proposition 2.7.6]). This is an exterior algebra on classes in odd degrees. There are maps relating other spectra to $H \wedge_{MU} H$. For example, there exists a map

$$\ell : H \wedge H \to H \wedge_{MU} H$$

hitting the generators in degrees 1, 3, 7, 13, etc. In this case, one can reduce the calculation of secondary operations to the question: does $Q^{10}(\sigma x_2)$ equal to σx_7 mod decomposables (note that σx_2 is of degree 5, which is not in the image of ℓ)? Luckily, we have a "yes" to this question. So knowing the information about how the Dyer-Lashof operations act on $H \wedge_{MU} H$ will not only give the analog of how we see the Nishida relations in the homology of MU-algebras, but also give us the information about ℓ.

6. Lecture 6 (Kun Chen): 11/01/2022

Definition 6.1. Let Γ be the group of power series over $\mathbb{Z}$ having the form

$$\gamma = x + b_1x^2 + b_2x^3 + \cdots ,$$

where the group operation is functional composition.

Let $G(x, y)$ be the universal fgl over L. By a theorem of Mischenko,

$$\log_G(x) = \sum_{i\geq 0} m_i x^{i+1},$$

where $m_n = \frac{[\mathbb{CP}^n]}{n+1} \in \pi_{2n}(MU) \otimes \mathbb{Q}$.

Let $\gamma \in \Gamma$. Now $\gamma^{-1}(G(\gamma(x)), \gamma(y))$ is another fgl over L, inducing by an endomorphism ϕ of L. Since γ is invertible in $\mathbb{Z}[[x]]$, ϕ is an automorphism. So there is a natural Γ-action on L.

Write $\mathcal{C}\Gamma$ and FH to denote the category of finitely presented graded L-modules with compatible Γ-action and category of finite CW complexes and homotopy classes of maps between them, respectively. We will continue this topic in the thick subcategory theorem (TST). See my lectures on TST.

The following paragraph concerns mainly the Morava stabilizer groups. Recall that the Quillen's theorem 1.8 indicated that $L \cong \pi_*(MU)$ as rings. The proof of this theorem is rather complicated and becomes our main focus for the following lectures.

Consider the map $\psi : L \to \mathbb{Z}[b_1, b_2, \cdots]$. It classifies the formal group law $b(b^{-1}(x) + b^{-1}(y))$ as stated previously, where $b = x + \sum_{i\geq 1} b_i x^{i+1}$. Denote the additive fgl by G_a. One might wonder if this ψ is the desired isomorphism in the Quillen's theorem. Unfortunately, this is not the case because under ψ, $\mathbb{Z}[b_1, b_2, \cdots]$ can be interpreted as classifying fgls that strict isomorphic to G_a, which is clearly not the class of all fgls. So the rest of proof follows by studying how ψ fails to be an isomorphism. In order to do that, we need the following notion.

Definition 6.2. Let R be a non-negatively graded ring and I be its ideal of positive degree elements. The **module of indecomposables** is the graded module I/I^2.

We can now split L into $\mathbb{Z} \otimes I$, where $\mathbb{Z}$ is the part of degree 0 elements, and I is the part of degree > 0 elements. ψ induces an augmentation map

$$\psi : I/I^2 \to J/J^2$$

on the corresponding modules of indecomposables. In fact, $(J/J^2)_{2n} \cong \mathbb{Z}\{b_n\}$, the free $\mathbb{Z}$-module generated by b_n. The image of ψ modulo decomposables can be identified as follows.

Lemma 6.3. *For $n \geq 1$, the images of $\psi_{2n} : (I/I^2)_{2n} \to (J/J^2)_{2n}$ are*

$$\psi_{2n}(x_n) = \begin{cases} pb_n & , n+1 = p^k \\ b_n & , \text{ else.} \end{cases}$$

where $k \geq 1$, $x_n \in L_{2n}$ with $|x_n| = 2n$. In particular, $(I/I^2)_{2n} \cong \mathbb{Z}\{x_n\}$.

Theorem 1.8 immediately follows from Lemma 6.3. Assume that the lemma is proved. Choose a lift $t_n \in I_{2n}$ of generator, which specifies a map $\phi : \mathbb{Z}[t_1, t_2, \cdots] \to L$ with $|t_n| = 2n$. This map is an isomorphism on modules of indecomposables, and is surjective in degree zero since L is connected. We use the induction to show ϕ is surjective. When $n > 0$, ϕ is surjective onto I^2_{2n} by inductive hypothesis since any element inside is a linear combination of products of lower degree elements. Hence, it is also a surjection onto $(I/I^2)_{2n}$, yielding the desired result.

To show that ϕ is an isomorphism, we consider the composite

$$\psi \circ \phi : \mathbb{Z}[t_1, t_2, \cdots] \to \mathbb{Z}[b_1, b_2, \cdots],$$

and claim this is injective. If this is the case, then ϕ is injective, finishing the proof that ϕ is an isomorphism. To see $\psi \circ \phi$ is injective, one can transform the question into the rational case by tensoring $\mathbb{Q}$:

$$\mathbb{Q} \otimes (\psi \circ \phi) : \mathbb{Q}[t_1, t_2, \cdots] \to \mathbb{Q}[b_1, b_2, \cdots],$$

since both sides of the composite are torsion-free. Note that $\psi : L \to \mathbb{Z}[b_1, b_2, \cdots]$ is an isomorphism after tensoring with $\mathbb{Q}$ because any fgl is uniquely strictly isomorphic to G_a in characteristic zero case by Theorem 1.12. Lemma 6.3 tells us that $\mathbb{Q} \otimes (\psi \circ \phi)$ is an isomorphism on modules of indecomposables. Reproducing the proof of surjectivity of ϕ, we conclude that $\mathbb{Q} \otimes (\psi \circ \phi)$ is surjective. As both sides of $\mathbb{Q} \otimes (\psi \circ \phi)$ are of same finite dimension over $\mathbb{Q}$ in each degree, $\mathbb{Q} \otimes (\psi \circ \phi)$ is then an isomorphism. The result follows.

It suffices to prove Lemma 6.3. We do it by interpreting both sides of $\psi_{2n} : (I/I^2)_{2n} \to (J/J^2)_{2n}$ using formal groups, and then calculating the desired result by deformation theory. This requires some background algebraic geometry. See Appendix E. A rigorous treatment can be found in the course note Oxford C2.6 Intro to Schemes (typed) or Oxford C2.6 Intro to Schemes (handwritten). We will keep the notations in the Appendix.

7. Lecture 7 (Kun Chen): 11/08/2022

The goal of today's lecture is to prove the Landweber exact functor theorem.

The motivation for this theorem is question: given a graded ring R and a fgl classified by $L \cong MU_* \to R$, is the functor $X \mapsto MU_*(X) \otimes_L R$ from Sp to the category of graded abelian groups a homology theory? That is, such a functor must satisfy the Eilenberg-Steenrod axioms.

One example to support the result is to take $X = KU_*$. In this case, $KU_*(X) = MU_*(X) \otimes_{MU_*(X)} KU_*$ is clearly a homology theory. In general, the answer is yes. We state the result as follows:

Theorem 7.1 (Landweber exact functor theorem)**.** *Consider a fgl over a graded ring R, corresponding to a ring map $L \to R$. If the corresponding map* $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$ *is a flat morphism of algebraic stacks, then the functor $X \mapsto MU_*(X) \otimes_L R$ is a homology theory.*

Before we proceed, we need to introduce the concept of stacks. Let $\mathcal{C}$ be a category with finite fiber products.

Definition 7.2. A **Grothendieck topology** on $\mathcal{C}$ is a covering $\mathrm{Cov}(\mathcal{C}) = \{U_i \to U\}$ with fixed targets satisfying

(1) if $V \to U$ is an isomorphism, then $\{V \to U\}$ is in $\mathrm{Cov}(\mathcal{C})$;
(2) if $\{U_i \to U\}$ forms a cover of U, and for each i, $\{V_{ij} \to U_i\}$ forms a cover of U_i, then the set of morphisms $\{V_{ij} \to U\}$ obtained by taking all compositions covers U;
(3) if $\{U_i \to U\}$ is a covering of U, and $V \to U$ is some morphism, then $\{U_i \times_U V \to V\}$ forms a covering of V.

A category equipped with a Grothendieck topology is called a **site**. A site for which all representable functors are sheaves is called **subcanonical**.

Example 7.3. Let $\pi : X \to Y$ be a morphism of schemes. It is called **fpqc** if it is faithfully flat, and either of the two equivalent properties is satisfied:

(1) every quasi-compact open subset of Y is the image of a quasi-compact open subset of X;
(2) there exists an affine open covering $\{\mathrm{Spec}(R_i)\}$ of Y such that each $\mathrm{Spec}(R_i)$ is the image of a quasi-compact open subset of X.

If X is quasi-compact and Y is affine, then $fpqc$ conditions degenerate to faithful flatness. The **fpqc topology** on the category of schemes Sch/S over a fixed base scheme S is the one which the coverings $\{U_i \to U\}$ are collections of morphisms such that the induced morphism $\sqcup_i U_i \to U$ is fpqc. The resulting site, denoted $(\mathsf{Sch}/S)_{fpqc}$ is subcanonical.

Now to define the stacks. Essentially, a stack is almost the same as a sheaf, except it takes values in categories instead of in sets. There are two ways to formalize this: via fibered categories and via pseudo-functors.

Definition 7.4. Let $\mathcal{C}$ be a category. A **(contravariant) pseudo-functor** F on $\mathcal{C}$ consists of the following data:

(1) for every object $U \in \mathcal{C}$, a category $F(U)$;
(2) for every morphism $f : U \to V$, a functor $f^* : F(V) \to F(U)$;
(3) for every object U, a natural isomorphism of functors $\varepsilon_U : \mathrm{id}^* \to \mathrm{id}_{F(U)}$;

(4) for any two composable morphisms $f : U \to V$ and $g : V \to W$, a natural isomorphism $a_{f,g} : f^* \circ g^* \xrightarrow{\cong} (g \circ f)^*$.

Let $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} X$ be composable arrows and $\eta \in F(X)$. The following conditions are required to be satisfied in our definition: the commutative diagram

$$\begin{tikzcd}
f^*g^*h^*\eta \arrow[r, "a_{f,g}(h^*\eta)"] \arrow[d, "f^*a_{g,h}(\eta)"'] & (g \circ f)^*h^*\eta \arrow[d, "a_{g\circ f,h}(\eta)"] \\
f^*(h \circ g)^*\eta \arrow[r, "a_{f,h\circ g}(\eta)"'] & (h \circ g \circ f)^*\eta
\end{tikzcd}$$

, as well as the equalities $a_{\mathrm{id}_U,f}(\xi) = \varepsilon_U(f^*\xi)$ and $a_{f,\mathrm{id}_V}(\xi) = f^*\varepsilon_V(\xi)$ for all $\xi \in F(V)$.

Definition 7.5. Let $\mathcal{C}$ be a category. A **category over** $\mathcal{C}$, denoted $\mathcal{F}$, is a category with a functor $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$. A morphism $\phi : f \to g$ in $\mathcal{F}$ is a Cartesian morphism if for any diagram

$$\begin{tikzcd}
h \arrow[rr] \arrow[dr, dashed, "\exists!"] \arrow[dd] & & g \arrow[dd] \\
& f \arrow[ur, "\phi"] \arrow[dd] & \\
p_{\mathcal{F}}h \arrow[rr] \arrow[dr] & & d \\
& c \arrow[ur] &
\end{tikzcd}$$

the dashed arrow exists to make everything commute.

Definition 7.6. A **fibered category** $\mathcal{F}$ over $\mathcal{C}$ is a category such that for every $c \to d$ in $\mathcal{C}$ and an object g such that $g \mapsto d$, there is a Cartesian morphism $f \to g$ making the diagram commute:

$$\begin{tikzcd}
f \arrow[r] \arrow[d] & g \arrow[d] \\
c \arrow[r] & d
\end{tikzcd}$$

The **fiber** $\mathcal{F}(c)$ for $c \in \mathcal{C}$ is the subcategory of $\mathcal{C}$, whose objects are objects of $\mathcal{F}$ sent to c under $p_{\mathcal{F}}$, and whose arrows are those arrows of $\mathcal{F}$ sent to id_c under $p_{\mathcal{F}}$.

A **morphism of fibered categories** $F : \mathcal{F} \to \mathcal{G}$ over $\mathcal{C}$ is a functor from $\mathcal{F}$ to $\mathcal{G}$ which commutes with the projections $p_{\mathcal{F}}$ and $p_{\mathcal{G}}$, sending Cartesian morphisms to Cartesian morphisms.

Definition 7.7. A **cleavage** of a fibered category $\mathcal{F} \to \mathcal{G}$ is a class $\mathcal{A}$ of Cartesian morphism in $\mathcal{F}$ such that for each arrow $c \to d$ in $\mathcal{C}$ and each object $\eta \in \mathcal{F}(d)$, there is a unique Cartesian morphism in $\mathcal{A}$ which provides a lift. The existence of a cleavage on every fibered category is ensured by the axiom of choice.

Definition 7.8. Let $\mathcal{C}$ be a site, and $U \in \mathcal{C}$ with a covering $\{\sigma_i : U_i \to U\}$. the **descent data** at U consist of the following:

(1) for all i, a choice of object ξ_i in the fiber $\mathcal{F}(U_i)$;

(2) for all i, j, a choice of isomorphism ϕ_{ij} from the pullback $p_1^*\xi_j$ to the pullback $p_2^*\xi_i$, satisfying the cocycle conditions: $p_{13}^*\phi_{ik} = p_{12}^*\phi_{ij} \circ p_{23}^*\phi_{jk}$ for all i, j, k.

A **morphism of descent data** $(\xi_i, \phi_{ij}) \to (\eta_i, \psi_{ij})$ is a collection of arrows $a_i : \xi_i \to \eta_i$ in $\mathcal{F}(U_i)$ such that for each i, j, the following diagram commutes:

$$\begin{tikzcd} p_2^*\xi_j \arrow[r, "p_2^*a_j"] \arrow[d, "\phi_{ij}"'] & p_2^*\eta_j \arrow[d, "\psi_{ij}"] \\ p_1^*\xi_i \arrow[r, "p_1^*a_i"'] & p_1^*\eta_i \end{tikzcd}$$

The collection of descent data at $U \in \mathcal{C}$ becomes a category, denoted by $\mathcal{F}(U)$.

For each object $\xi \in \mathcal{F}(U)$, there is a natrual choice of descent data for ξ, simply by letting the objects ξ in $\mathcal{F}(U_i)$ be $\sigma_i^*\xi_i$. If $\xi \to \eta$ is a morphism in $\mathcal{F}(U)$, then there is an associated morphism on the canonical choices of descent data. Let $\mathcal{U}$ be the covering $\{\sigma_i : U_i \to U\}$ of U. The preceding facts yield a functor $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$.

Definition 7.9. $\mathcal{F}$ is a **stack** over $\mathcal{C}$ if the functor $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ is an equivalence of categories. If $\mathcal{C}$ is further subcanonical, then a representable functor h_X, which is a sheaf, can be viewed as a stack, denoted $\mathcal{C}/X$. This is called a **representable stack**.

Proposition 7.10 (Stackification)**.** *Let $\mathcal{F}$ be a fibered category over a site $\mathcal{C}$. There is always a stack $\mathcal{F}'$ over $\mathcal{C}$ together with a morphism of fibered categories $\mathcal{F} \to \mathcal{F}'$, such that the following holds:*

- *For every element $c \in \mathcal{F}'(U)$, there is a covering $\{h_i : U_i \to U\}$ such that h_i^*c is in the essential image of functor $\mathcal{F} \to \mathcal{F}'$.*
- *Let $c, c' \in \mathcal{F}(U)$. For every morphism $h : V \to U$, $\hom_{\mathcal{F}}(h^*c, h^*c')$ is a presheaf over $\mathcal{C}/U$. The morphism $\mathcal{F} \to \mathcal{F}'$ yields natural maps $\hom_{\mathcal{F}}(h^*c, h^*c') \to \hom_{\mathcal{F}'}(h^*c, h^*c')$, which is precisely the sheafification of the presheaf.*

8. Lecture 8 (Kun Chen): 11/15/2022

9. Lecture 9 (Congzheng Liu): 11/22/2022

Today's theme is about the computation of algebraic K-theory groups of fields. Typed notes can be found here (by Congzheng Liu, Sichuan University).

10. Lecture 10 (Kun Chen): 11/29/2022

Appendix A. Brown-Peterson theory and complex orientations

Fix an arbitrary prime p. The Brown-Peterson spectrum BP is originally, the smallest ingredient of the infinite wedge decomposition of p-localized MU.

Proposition A.1. *BP satisfies*

(1) $\pi_*(BP) \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$, *where* $\dim v_n = 2(p^n - 1)$.
(2) $H_*(BP) \cong \mathbb{Z}_{(p)}[t_1, t_2, \cdots]$, *where* $\dim t_n = 2(p^n - 1)$.
(3) *v_i can be chosen in such a way that the Hurewicz map $h_* : \pi_* \to H_*$ is given by*
$$h(v_i) = pt_i + \text{ decomposables }.$$

Let L be the Lazard ring. Recall that in Lecture 1, there is a universal p-typical fgl over $BP^* \cong V$ induced by a homomorphism
$$\tau : L \otimes \mathbb{Z}_{(p)} \to V,$$
which is topologically determined by $BP^*(\mathbb{CP}^\infty)$. Cartier's theorem asserts that an arbitrary fgl over a torsion-free $\mathbb{Z}_{(p)}$-algebra is strictly isomorphic to a p-typical one. That is to say that, there is an idempotent endomorphism
$$\epsilon_p : L \otimes \mathbb{Z}_{(p)} \to L \otimes \mathbb{Z}_{(p)}$$
which factors through τ defined as above. We will describe this idempotent map in details in Appendix B. Before we finish this section, we need to build some preliminary background.

Let E be an associative commutative ring spectrum. Let $\xi = \{E \to M\}$ be an n-dimensional vector bundle. Recall that, classically, an orientation on ξ is an element called Thom class in $\tilde{H}^n(Th(\xi))$ which is sent to the generator by the induced map $\tilde{H}^n(Th(\xi)) \to \tilde{H}^n(E_x, E_x - x) \cong \tilde{H}^n(Th(E_x \to \{x\})) \cong \mathbb{Z}$ for any $x \in M$.

Theorem A.2 (Thom isomorphism)**.** *If $\xi = \{E \to M\}$ is oriented with Thom class $c \in \tilde{H}^n(Th(\xi))$, then there is an isomorphism:*
$$H^*(M) \cong H^*(E) \xrightarrow{c\smile -} H^{*+n}(E, E-M) \cong \tilde{H}^{*+n}(Th(\xi)).$$

Generalizing the orientation to E-orientation, we get

Definition A.3. An **E-orientation** on $\xi = \{V \to M\}$ is an element in $\tilde{E}^n(Th(\xi))$ which is sent to the generator (as $\pi_0(E)$-modules) by the induced map
$$\tilde{E}^n(Th(\xi)) \to \tilde{E}^n(Th(V_x \to \{x\})) \cong \tilde{E}^n(S^n) \cong \pi_0 E$$
for any $x \in M$.

If $E = H\mathbb{Z}$, then the E-orientation is the classical orientation.

Definition A.4. A **complex orientation** on E is a class $x_V \in \tilde{E}^{2n}(Th(\xi))$ for any positive integer n together with a vector bundle of rank n: $\xi = \{V \to M\}$, such that

(1) For any $x \in M$, x_V is mapped into a generator by
$$\tilde{E}^n(Th(\xi)) \to \tilde{E}^n(Th(V_x \to \{x\})) \cong \tilde{E}^{2n}(S^{2n}) \cong \pi_0 E.$$
(2) For any map $f : N \to M$, $f^*(x_V) = x_{f^*V}$.
(3) For different complex vector bundles $\xi_1 : V_1 \to M$ and $\xi_2 : V_2 \to M$, $x_{V_1 \oplus V_2} = x_{V_1} \cdot x_{V_2}$.

Alternatively, a complex orientation E can also be defined as a class $x_E \in \tilde{E}^2(\mathbb{CP}^\infty)$ whose restriction to $\pi_0(E) \cong \tilde{E}^2(S^2) \cong \tilde{E}^2(\mathbb{CP}^1)$ is 1. For example, when $E = H\mathbb{Z}$, $\tilde{H}^2(\mathbb{CP}^\infty) \cong \tilde{H}^2(\mathbb{CP}^1) \cong \mathbb{Z}$. We will use this definition as our setting for later.

Example A.5. Let $E = H\mathbb{Z}$. The usual generator of $H^2(\mathbb{CP}^\infty)$ is a complex orientation $x_{H\mathbb{Z}}$.

Example A.6. Let $E = MU$. By definition, $\mathbb{CP}^\infty = BU(1) \xrightarrow{\simeq} MU(1) \to MU$ gives a complex orientation $x_{MU} \in MU^2(\mathbb{CP}^\infty)$.

Lemma A.7. *Let E be a complex oriented ring spectrum. The following holds:*

(1) $E^*(\mathbb{CP}^\infty) \cong E^*(\mathrm{pt})[[x_E]]$.
(2) $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E^*(\mathrm{pt})[[x_E \otimes 1, 1 \otimes x_E]]$.
(3) Let $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ be the H-space structure map stated in Lecture 1, and let $F_E(x, y) \in E^*(\mathrm{pt})[[x, y]]$ be defined by $F_E(x_E \otimes 1, 1 \otimes x_E) = m^*(x_E)$. Then F_E is a fgl over $E^*(\mathrm{pt})$.

We have discussed this lemma in Lecture 1 when $E = MU$. Lemma **??** generalizes the discussion to every complex oriented ring spectrum E. As an easy observation, $E_*(\mathbb{CP}^\infty)$ is a free π_*E-module on elements β_i^E dual to x_E^i. Let b_i^E be the image of β_{i+1}^E of the stable map $\mathbb{CP}^\infty \to \Sigma^2 MU$. By Atiyah-Hirzebruch spectral sequence, we have the following result:

Lemma A.8. *If E is a complex oriented ring spectrum, then*

$$E_*(MU) = \pi_*(E)[b_1^E, b_2^E, \cdots].$$

Appendix B. Hopf algebroids

Hopf algebroids are the generalization of Hopf algebras. Before we explain this sentence, we first introduce the definition of Hopf algebroids.

Definition B.1. A **Hopf algebroid** over a commutative ring R is a groupoid object in the category of (graded or bigraded) commutative R-algebras. That is, a pair (A, Γ) of commutative R-algebra with structure maps such that for any other commutative R-algebra B, the sets $\hom(A, B)$ and $\hom(\Gamma, B)$ are the objects and morphisms of a groupoid. The structure maps are

$$\begin{array}{cc} \eta_L : A \to \Gamma & \text{left unit (source),} \\ \eta_R : A \to \Gamma & \text{right unit (target),} \\ \Delta : \Gamma \to \Gamma \otimes_A \Gamma & \text{coproduct (composition),} \\ \varepsilon : \Gamma \to A & \text{counit (identity),} \\ c : \Gamma \to \Gamma & \text{conjugation (inverse).} \end{array}$$

Here Γ is a left A-module via η_L and a right A-module via η_R, $\Gamma \otimes_A \Gamma$ is the usual tensor product of bimodules, and Δ and ε are A-bimodule maps. The defining properties of a groupoid correspond to the following relations among the structure maps:

(1) $\varepsilon\eta_L = \varepsilon\eta_R = \mathrm{id}_A$, the source and the target of an identity morphism are the object on which it is defined.
(2) $(\Gamma\otimes\varepsilon)\Delta = (\varepsilon\otimes\Gamma)\Delta = \mathrm{id}_\Gamma$, composition with the identity leaves a morphism unchanged.
(3) $(\Gamma \otimes \Delta)\Delta = (\Delta \otimes \Gamma)\Delta$, composition of morphisms is associative.
(4) $c\eta_R = \eta_L$ and $c\eta_L = \eta_R$, inverting a morphism interchanges source and target.
(5) $cc = \mathrm{id}_\Gamma$, the inverse of the inverse is the original morphism.
(6) Maps exist such that the following commute

$$\begin{tikzcd} \Gamma & \Gamma \otimes_R \Gamma \arrow[l, "c\cdot\Gamma"'] \arrow[r, "\Gamma\cdot c"] \arrow[d] & \Gamma \\ & \Gamma \otimes_A \Gamma \arrow[ul, dashed] \arrow[ur, dashed] & \\ A \arrow[uu, "\eta_R"] & \Gamma \arrow[l, "\varepsilon"'] \arrow[r, "\varepsilon"] \arrow[u, "\Delta"] & A \arrow[uu, "\eta_L"'] \end{tikzcd}$$

where $c \cdot \Gamma(\gamma_1 \otimes \gamma_2) = c(\gamma_1)\gamma_2$ and $\Gamma \cdot c(\gamma_1 \otimes \gamma_2) = \gamma_1 c(\gamma_2)$, composition of a morphism with its inverse on either side gives an identity morphism.

If $\eta_R = \eta_L$, then Γ is a commutative Hopf algebra over A, or a **cogroup object** in the category of commutative A-algebras. Generally, if $D \subset A$ is the subalgebra on which $\eta_R = \eta_L$, then Γ is also a Hopf algebroid over D.

Definition B.2. Let (A, Γ) be a Hopf algebroid over R. A **left Γ-comodule** M is a left A-module M together with a left A-linear map $\phi : M \to \Gamma \otimes_A M$, such that

$$\begin{array}{cc} (\varepsilon \otimes M)\phi = M & \text{counitary,} \\ (\Delta \otimes M)\phi = (\Gamma \otimes \phi)\phi & \text{coassociative.} \end{array}$$

An element $m \in M$ is **primitive** if $\phi(m) = 1 \otimes m$. Similarly we can define a right Γ-comodule.

Definition B.3. Let $(A, \Gamma$ be a Hopf algebroid over R. A **comodule algebra** M is a comodule which is also a commutative associative A-algebra such that the structure map ϕ is an algebra map. If M and N are left Γ-comodules, their **comodule tensor product** is $M \otimes_A N$ with structure map being the composite

$$M \otimes N \xrightarrow{\phi_M \otimes \phi_N} \Gamma \otimes M \otimes \Gamma \otimes N \to \Gamma \otimes \Gamma \otimes M \otimes N \to \Gamma \otimes M \otimes N,$$

where the second map interchanges the second and third factors and the third map is the multiplication on Γ. All tensor products are over A using only the left A-module structure on A. A **differential comodule** C^* is a cochain complex in which each C^n is a comodule and the coboundary operator is a comodule map.

Theorem B.4. *Let E be a flat homotopy commutative ring spectrum, i.e. a ring spectrum such that the multiplication map $m : E \wedge E \to E$ is commutative up to homotopy, and $E \wedge E$ is equivalent to a wedge of suspensions of E. Some good examples of such spectra are $E = H\mathbb{Z}/(p)$ (in which case E_*E is the dual Steenrod algebra), MU, and BP. Then $(\pi_*(E), E_*E)$ over $\mathbb{Z}$ is a Hopf algebroid. If E is p-local, then it is a Hopf algebroid over $\mathbb{Z}_{(p)}$.*

The motivation example of a Hopf algebroid is when $E = H\mathbb{Z}/(p)$. In this case, $E^*(X) = H^*(X; \mathbb{Z}/(p))$, which is a graded $\mathbb{Z}/(p)$-vector space whose structure is enriched by Steenrod operations (see my notes [**?**] for details). These form the Steenrod algebra A over which $H^*(X; \mathbb{Z}/(p))$ has a natural module structure, i.e. a map

$$A \otimes E^*(X) \to E^*(X)$$

with certain properties. Taking $\mathbb{Z}/(p)$-linear dual we get a homomorphism

$$A_* \otimes E_*(X) \xleftarrow{\phi} E_*(X)$$

with similar properties. ϕ gives a comodule structure on $E_*(X)$. Note that the Steenrod algebra A can be identified with the mod p cohomology of the mod p Eilenberg-MacLane spectrum E, i.e. E^*E. It follows that its dual A_* can be identified with E_*E, and the comodule structure map can be rewritten as

$$\pi_*(E \wedge E \wedge X) \xleftarrow{\phi} \pi_*(E \wedge X),$$

which is induced by

$$E \wedge E \wedge X \xleftarrow{E \wedge \eta \wedge X} E \wedge S^0 \wedge X = E \wedge X,$$

where $\eta : S^0 \to E$ is the unit map for E. The structure map ϕ is valid for any ring spectrum E, but the isomorphism

$$\pi_*(E \wedge E \wedge X) \cong E_*(E) \otimes_{E_*} E_*(X)$$

only holds when E is flat. This isomorphism is induced by

$$(E \wedge E) \wedge (E \wedge X) \xrightarrow{1 \wedge m \wedge 1} E \wedge E \wedge X.$$

To prove it, we only need to focus on $X = S^n$, and then use the induction on the number of cells, and pass to colimits for an arbitrary X.

When $E = H\mathbb{Z}/(p)$, the dual Steenrod algebra $A_* = E_*E$ is a commutative Hopf algebra. In other words, it has the following structure maps:

$$\begin{array}{lcc} & \Delta : A_* \to A_* \otimes A_* & \text{coproduct,} \\ & \mu : A_* \otimes A_* \to A_* & \text{product,} \\ \text{(A.2.1)} & \eta : \mathbb{Z}/(p) \to A_* & \text{unit,} \\ & c : A_* \to A_* & \text{conjugation,} \\ & \epsilon : A_* \to \mathbb{Z}/(p) & \text{argumentation,} \end{array}$$

satisfying certain conditions. A_* is a cogroup object in the category of unitary, graded commutative $\mathbb{Z}/(p)$-algebras. For any other such algebra C, there is a natural group structure on the set $\hom(A_*, C)$ of graded algebra homomorphisms. In particular, Δ on A_* induces

$$\hom(A_*, C) \times \hom(A_*, C) = \hom(A_* \otimes A_*, C) \xrightarrow{\Delta^*} \hom(A_*, C),$$

which is the group operation on $\hom(A_*, C)$. c induces the map that sends each element to its inverse, and ϵ gives the identity element. η and μ, on the other hand, define A_* itself as a unitary, graded commutative algebra. Now the defining properties of a group can be translated into the properties of the structure maps that define a graded commutative Hopf algebra such as A_*.

Theorem B.5 (Milnor)**.** *For $p = 2$,*

$$A_* = \mathbb{Z}/(2)[\xi_1, \xi_2, \cdots]$$

with $|\xi_n| = 2^n - 1$, and the coproduct is given by

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i,$$

where $\xi_0 = 1$. For $p > 2$,

$$A_* = \mathbb{Z}/(p)[\xi_1, \xi_2, \cdots] \otimes E(\tau_0, \tau_1, \cdots)$$

with $|\xi_n| = 2(p^n - 1)$ and $|\tau| = 2p^n - 1$. The coproduct is given by

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i,$$

$$\Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \tau_i.$$

If we identify $\mathbb{Z}/(p)$ with $\pi_*(E)$, A_* with $\pi_*(E \wedge E)$ and $A_* \otimes A_*$ with $\pi_*(E \wedge E \wedge E)$, then some of the maps in (A.2.1) can be seen as the induced maps of

$$\begin{array}{lcr} & E \wedge E = E \wedge S^0 \wedge E \xrightarrow{E\wedge\eta\wedge E} E \wedge E \wedge E & \text{coproduct,} \\ \text{(A.2.2)} & E \wedge E \wedge E \xrightarrow{E\wedge m} E \wedge E & \text{product,} \\ & E \wedge E \xrightarrow{T} E \wedge E & \text{conjugation,} \\ & E \wedge E \xrightarrow{m} E & \text{argumentation,} \end{array}$$

where η and m are the unit and multiplication maps for the ring spectrum E, and T is the map which interchanges the two factors. For the unit map $\eta : \pi_*(E) \to \pi_*(E \wedge E)$, there are two candidates, namely

$$\eta_R : E = S^0 \wedge E \xrightarrow{\eta\wedge E} E \wedge E,$$

and

$$\eta_L : E = E \wedge S^0 \xrightarrow{E\wedge\eta} E \wedge E.$$

They determine two different E_*-module structure on E_*E. Since E is flat, the two choices are identical, E_*E is then a two-side E_*-module. Now we actually proved theorem B.4. As we mentioned before, η_L and η_R are not necessarily identical in general, for which E is not flat. We need to accommodate the distinct right and left units, and this is when groupoids get involved.

Consider the pair $(\pi_*(MU), MU_*MU)$. We will discuss the structure of MU_*MU. This pair is obviously a Hopf algebroid, by theorem B.4. So $(\pi_*(MU), MU_*MU)$ is a groupoid object in the category of (graded) commutative R-algebras, where R is a commutative algebra. To describe the structure of MU_*MU, we can define the groupoid-valued functor represented by the pair $(\pi_*(MU), MU_*MU)$, whose value on a (graded) commutative algebra R is the set of fgls over R and the groupoid of isomorphisms between them.

Recall that $\pi_*(MU) \cong L \cong \mathbb{Z}[x_1, x_2, \cdots]$, where $|x_i| = 2i$. This can be computed by Adams spectral sequence (see Appendix D). Milnor and Novikov also showed that the generators x_i can be chosen in such a way that the Hurewicz map $h : \pi_*(MU) \to H_*(MU)$ is given by

$$h(x_i) = \begin{cases} pb_i + \text{ decomposables} & \text{, if } i = p^k - 1 \text{ for some prime } p, \\ b_i + \text{ decomposables} & \text{, else.} \end{cases}$$

Let $m_n = \frac{[\mathbb{CP}^n]}{n+1} \in \pi_{2n}(MU) \otimes \mathbb{Q}$.

Theorem B.6 (Landweber-Novikov)**.** *There is a ring isomorphism*

$$MU_*MU \cong MU_*[b_1, b_2, \cdots],$$

with $|b_i| = 2i$*. The coproduct is given by*

$$\sum_{i\geq 0} \Delta(b_i) = \sum_{i\geq 0} b_i \otimes \left(\sum_{j\geq 0} b_j\right)^{i+1},$$

where $b_0 = 1$*. The left unit* η_L *is the standard inclusion*

$$MU_* \to MU_*MU,$$

while the right unit on $MU_*MU \otimes \mathbb{Q}$ *is given by*

$$\sum_{i\geq 0} \eta_R(m_i) = \sum_{i\geq 0} m_i \left(\sum_{j\geq 0} c(b_j)\right)^{i+1},$$

where $m_0 = 1$ *and* c *is the conjugation satisfying* $c(m_n) = \eta_R(m_n)$ *and*

$$\sum_{i\geq 0} c(b_i) \left(\sum_{j\geq 0} b_j\right)^{i+1} = 1.$$

Our next goal for the section is to understand the right unit map in the theorem.

Remark B.7. In the statement of the theorem, one might notice that the ground ring in the defining equation for Δ is $\mathbb{Z}$ instead of MU_*. This means that $B = \mathbb{Z}[b_1, b_2, \cdots]$ is a Hopf algebra over $\mathbb{Z}$ and the Hopf algebroid structure is determined by B along with the right unit map, which is essentially determined by a comodule structure $\phi : MU_* \to MU_*MU \otimes_{MU_*} MU_* \cong MU_*MU$.

The remark is closely related to a term *split*.

Definition B.8. Consider the category $\mathcal{C} = G \ltimes X$ whose objects are in X (a set), and $\hom_{\mathcal{C}}(x, y) = \{g \in G : gx = y\}$, where G is a group. A groupoid is **split** if it is equivalent to $G \ltimes X$ for some X and G.

Definition B.9. A Hopf algebroid (A, Γ) over a commutative ring R is **split** if $\Gamma = A \otimes B$, where B is a Hopf algebra over which A is a comodule algebra.

Equivalently, if (A, Γ) is a split Hopf algebroid over a commutative ring R, then for any commutative R-algebra C,

$$\hom(\Gamma, C) = \hom(A, C) \times \hom(B, C),$$

where $\hom(B, C)$ is a group acting on the set $\hom(A, C)$. Take $(A, \Gamma) = (\pi_*(MU) = MU_*, MU_*MU)$. By Lazard's and Quillen's theorem, we can identify the set $\hom(MU_*, R)$ with $FGL(R)$.

Definition B.10. Let Γ be the group of power series over $\mathbb{Z}$ having the form

$$\gamma = x + b_1 x^2 + b_2 x^3 + \cdots ,$$

where the group operation is functional composition.

The Hopf algebra B is a ring of $\mathbb{Z}$-valued algebraic function of Γ (See Lecture 6). This is seen by looking at an arbitrary element $\gamma = \sum_{i \geq 0} b_i x^{i+1} \in \Gamma$ with $b_0 = 1$. Each b_i can be regarded as a function assigning to γ the coefficient of x^{i+1} in the power series. Equipped B with the coproduct given in theorem B.6. Then for a commutative ring R, $\hom(B, R)$ is the group of power series with coefficient in R of the form $\sum_{i \geq 0} b_i x^{i+1}$, whose group operation is the functional composition. Denote this group Γ_R.

Let F be a fgl over R, and $f(x) = \sum_{i \geq 0} b_i x^{i+1}$ with $b_0 = 1$, $b_i \in R$. It is obvious that

$$F^f(x, y) = f(F(f^{-1}(x), f^{-1}(y))) \tag{A.2.3}$$

is another fgl over R. This gives an Γ_R-action on $FGL(R)$. We want to compute the logarithm of F^f in terms of that of F. Rewrite (A.2.3) as

$$f^{-1}(F^f(x, y)) = F(f^{-1}(x), f^{-1}(y)).$$

Taking the logarithm of both sides yields

$$\log_F(f^{-1}(F^f(x, y))) = \log_F(f^{-1}(x)) + \log_F(f^{-1}(y)),$$

which is equivalent to

$$\log_{F^f}(F^f(x, y)) = \log_{F^f}(x) + \log_{F^f}(y).$$

Therefore, $\log_{F^f}(x) = \log_F(f^{-1}(x))$ by comparing the terms degree-wise.

Given a morphism $\theta : MU_*MU \to R$, we have

$$\begin{tikzcd} & B \arrow[d, "\psi"] & \\ MU_* \arrow[r, shift left, "\eta_L"] \arrow[r, shift right, "\eta_R"'] & MU_*MU \arrow[r, "\theta"] & R \end{tikzcd}$$

Here B is the Hopf algebra mentioned above (i.e. a ring of $\mathbb{Z}$-valued algebraic function of Γ). $\theta\psi \in \hom(B, R)$, $\theta\eta_L$ gives the fgl F, and $\theta\eta_R$ gives the fgl F^f. By Mischenko's and Lazard's theorem,

$$\log_F(x) = \sum_{i\geq 0} \theta(m_i) x^{i+1},$$

and

$$\begin{aligned}\log_{F^f}(x) &= \sum_{i\geq 0} \theta(m_i)(f^{-1}(x))^{i+1} \\ &= \sum_{i\geq 0} \theta(m_i) \left(\sum_{j\geq 0} \theta(c(b_j)) x^{j+1} \right)^{i+1}\end{aligned}$$

where

$$f^{-1}(x) = \sum_{j\geq 0} \theta(c(b_j)) x^{j+1}.$$

Now it follows that the right unit map in MU_*MU is as stated in theorem B.6.

Appendix C. Morava K-theory

Appendix D. Adams spectral sequences

Appendix E. Background in algebraic geometry

The central point of interest is the functor-of-points approach by Grothendieck. It said that complicated structures can be described by specifying how they interact with the simple ones. Concretely, if X is the geometric object (e.g. a scheme) needed to be analyzed, then the functor on affine schemes $F(X)(\mathrm{Spec}(A)) = \hom_{\mathcal{G}}(\mathrm{Spec}(A), X)$, where $\mathcal{G}$ is a category of geometric objects containing all affine schemes, would retain all information about X. So the key to our focus is the study of such functors and how to retrieve info from them. For the rest of this section, the familiarity of sheaves is assumed.

Definition E.1. A homomorphism of rings $A \to B$ is **flat** if the functor $M \mapsto B \otimes_A M$ from A-modules to B-modules is exact. In this case, B is said to be a **flat A-algebra**. A flat homomorphism $A \to B$ is **faithfully flat** if it satisfies one of the equivalent conditions:

(1) if an A-module $M \neq 0$, then $B \otimes_A M \neq 0$;
(2) if a sequence of A-modules $M' \to M \to M''$ is not exact, then neither is $B \otimes_A M' \to B \otimes_A M \to B \otimes_A M''$;
(3) the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

Definition E.2. Let $\psi : A \to B$ be a ring map, and M be a B-module. An **A-derivation** into M is a map $D : B \to M$ which is additive, annihilates elements of $\psi(A)$, and satisfies the Leibniz rule: $D(ab) = aD(b) + bD(a)$.

The set of all A-derivations forms an B-module. Indeed, given two A-derivations D, D', $D + D' : B \to M$ sending a to $D(a) + D'(b)$ is still an A-derivation. The readers are encouraged to check the scalar multiplication by elements in A is also an A-derivation. Denote this B-module by $\mathrm{Der}_A(B, M)$. Consider the map of free B-modules

$$\bigoplus_{(a,b)\in B^2} B[(a,b)] \oplus \bigoplus_{(f,g)\in B^2} B[(f,g)] \oplus \bigoplus_{r\in A} B[(r)] \to \bigoplus_{a\in B} B[a],$$

which sends $[(a,b)]$ to $[a+b]-[a]-[b]$, $[(f,g)]$ to $[fg]-f[g]-g[f]$, and $[r]$ to $[\psi(r)]$. Denote the cokernel of this map by $\Omega_{B/A}$. There is a map $d : B \to \Omega_{B/A}$ sending a to $da = [a]$. From the definition,

$$\begin{aligned} d(a+b) &= da + db, \\ d(fg) &= fdg + gdf, \\ d\psi(a) &= 0. \end{aligned}$$

So this is an A-derivation.

Definition E.3. $(\Omega_{B/A}, d)$ is called the **module of Kähler differentials**.

Proposition E.4. *The map* $\hom_B(\Omega_{B/A}, M) \to \mathrm{Der}_A(B, M)$*,* $\alpha \mapsto \alpha \circ d$*, is an isomorphism of functors. This is the universal property of the module of differentials of B over A.*

Proposition E.5. *If $\psi : A \to B$ is surjective, then* $\Omega_{B/A} = 0$.

Definition E.6. Let $\psi : A \to B$ be a ring map. It is of **finite presentation** if there exists integers $n, m \in \mathbb{N}$ and polynomials $f_1, f_2, \cdots, f_m \in A[x_1, \cdots, x_n]$ and an isomorphism of A-algebras $A[x_1, \cdots, x_n]/(f_1, \cdots, f_m) \cong B$.

Definition E.7. Let $\psi : A \to B$ be a ring map. It is **G-unramified** if ψ is of finite presentation and $\Omega_{B/A} = 0$.

Definition E.8. Let $\psi : A \to B$ be a ring map. It is **étale** if ψ is flat and G-unramified.

Now we are able to define the critical object that will be used throughout the section.

Definition E.9. An **étale sheaf** is a functor $X : \mathsf{CommRing} \to \mathsf{Set}$ satisfying that for any finite set of étale maps $A \to B_i$ of commutative rings such that $A \to \prod B_i$ is faithfully flat, the diagram

$$X(A) \to \prod_i X(B_i) \rightrightarrows \prod_{i,j} X(B_i \otimes_A B_j)$$

is an equalizer.

Definition E.10. If B is a ring, then the corresponding **affine scheme** is the sheaf $\mathrm{Spec}(B)$, with $\mathrm{Spec}(B)(A) \coloneqq \hom_{\mathsf{CommRing}}(B, A)$.

Note that the definition here is different from the classical construction of $\mathrm{Spec}(B)$ as locally ringed space. Our new language is somewhat tautologous, and more convenient to describe some concrete objects. For example, the affine line $\mathbb{A}^1$ is the sheaf defined by $\mathbb{A}^1(R) = R$, which is an affine scheme isomorphic to $\mathrm{Spec}(\mathbb{Z}[x])$.

Let R be a k-algebra over some algebraic closed field k.

Definition E.11. Let X be an étale sheaf. An **R-valued point** of X is a point $x \in X(R)$. The **category of elements** $\mathrm{Elt}(X)$ is the category of pairs (R, x), where $R \in \mathsf{CommRing}$ and $x \in X(R)$, whose morphisms $f : (R, x) \to (S, y)$ are given by ring homomorphisms $f : R \to S$ such that $f^*x = y \in X(S)$, where $f^* = X(f) : X(R) \to X(S)$ is the induced map.

The forgetful functor $\mathrm{Elt}(X) \to \mathsf{CommRing}$ induces an étale Grothendieck topology on the opposite of the category of elements, where a family is covering iff its image is covering.

By Yoneda lemma, $\mathrm{Elt}(X)$ can be identified with the opposite of the full subcategory of the comma category $\mathrm{Fun}_{\text{ét}}(\mathsf{CommRing}, \mathsf{Set})/X$ consisting of the representables. Hence, any sheaf over X determines a covariant functor on $\mathrm{Elt}(X)$, and so the restriction

$$\mathrm{Fun}_{\text{ét}}(\mathsf{CommRing}, \mathsf{Set})/X \to \mathrm{Fun}_{\text{ét}}(\mathrm{Elt}(X), \mathsf{Set})$$

is an equivalence, where on the RHS the functors satisfies the étale conditions in Definition E.9 with respect to the induced topology described above.

Example E.12. Let S be a ring. An R-valued point of $\mathrm{Spec}(S)$ is the same as a ring homomorphism $S \to R$. It follows that $\mathrm{Elt}(\mathrm{Spec}(S))$ is equivalent to the category of S-algebras. The induced topology is the usual étale one that only depends on the underlying ring.

Definition E.13. Let $X : \mathsf{CommRing} \to \mathsf{Set}$ be an étale sheaf. A **quasi-coherent sheaf** M over X is an association of an R-module $M(x)$ for every R-valued point $x \in X(R)$ and of a map $f^* : M(x) \to M(f(x))$ of R-modules adjoint to an isomorphism $S \otimes_R M(x) \cong M(y)$ for every ring homomorphism $f : R \to S$. Note that these have to be compatible in the sense that if $f : R \to S$ and $g : S \to T$ are composable maps of rings, then $g^* f^* = (g \circ f)^*$ as maps $M(x) \to M(g(f(x)))$.

Let X be an étale sheaf. The associated category of quasi-coherent sheaves is denoted as $\mathrm{QCoh}(X)$. It is symmetric monoidal with tensor product given levelwise by tensor product of modules. From the definition, every object in $\mathrm{QCoh}(X)$ specifies an object $M(x)$ for every $x \in R$. Equivalently, we have a family of modules indexed by $\mathrm{Elt}(X)$.

Definition E.14. The **structure sheaf** of X is the quasi-coherent sheaf $\mathcal{O}_X$ defined by $\mathcal{O}_X(x) := R$ for any $x \in X(R)$, which is the unit of the monoidal structure.

By definition, in the language of higher category theory, $\mathrm{QCoh}(X)$ can be identified with the limit of the covariant functor $(R, x) \mapsto \mathrm{QCoh}(\mathrm{Spec}(R)) := \mathrm{Mod}(R)$. Since the category of affine schemes Aff is equivalent to the opposite of $\mathsf{CommRing}$, we deduce that the association $\mathrm{QCoh} : \mathsf{Shv}_{\text{ét}}^{op} \to \mathsf{Cat}$ is the right Kan extension of the functor $\mathrm{QCoh} : \mathsf{Aff}^{op} \to \mathsf{Cat}$ given by $\mathrm{QCoh}(\mathrm{Spec}(R)) = \mathrm{Mod}(R)$. Actually, we have

Proposition E.15. $\mathrm{QCoh} : \mathsf{Shv}_{\text{ét}}^{op} \to \mathsf{Cat}$ *takes colimits of étale sheaves to limits of categories.*

Remark E.16. The definition of quasi-coherent sheaves above is different from the classical one. However, if we take S to be the scheme in the classical sense, then $\mathrm{QCoh}(S)$ coincides with the classical definition of the quasi-coherent sheaf. To see this, note that S is the colimit of the poset of its affine opens in the category of étale sheaves. This indicates that locally, any map $\mathrm{Spec}(R) \to S$ factors through an affine open. Hence, a quasi-coherent sheaf on S is the same as a specification of an R-module for each affine open $\mathrm{Spec}(R) \subset S$, coinciding with the classical definition of quasi-coherent sheaves.

We would like to define a quasi-coherent sheaf which corresponds to the module of Kähler differentials. To do so, we need a rephrasing of the derivation.

Definition E.17. Let M be an R-module. The **square-zero extension** of M is a pair $(R \oplus M, \pi)$, where the multiplication in $R \oplus M$ is given by

$$(r_1, m_1) \cdot (r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1),$$

where $r_1, r_2 \in R$ and $m_1, m_2 \in M$, and $\pi : R \oplus M \to R$ is a canonical projection. In this setting, M becomes an ideal which squares to zero, hence gets its name.

Proposition E.18. *Let $s : R \to R \oplus M$ be a section of $\pi : R \oplus M \to R$. Then for any $r \in R$, $s(r) = (r, d(r))$ for a unique R-derivation $d : R \to M$.*

Rephrasing the universal property (Proposition E.4) of the module of Kähler differentials in the language of square-zero extensions leads to the following definition.

Definition E.19. Let X be an étale sheaf over $\mathrm{Spec}(k)$. The **sheaf of Kähler differentials**, provided existence, is a quasi-coherent sheaf $\Omega_{X/k}$ such that for every point $f : \mathrm{Spec}(R) \to X$ and every R-module M, there is a natural bijection between the set of R-linear maps $\hom_R(\Omega_{X/k}(R), M)$ and the set of dotted arrows making diagrams of the form commute:

$$\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(R \oplus M) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(k)
\end{tikzcd}$$

This definition shows that the Kähler differentials control the ways a given morphism from an affine scheme can be extended along the *first order thickening* (or *infinitesimal thickening*) $\mathrm{Spec}(R) \to \mathrm{Spec}(R \oplus M)$.

Appendix F. Deformation theory

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