Math 514: Complex Geometry

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November 9, 2021

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1 Preface

These notes were taken in University of Illinois, Urbana Champaign (UIUC)'s Math 514 (Complex Algebraic Geometry) class in Fall 2020, taught by Professor Pierre Albin. Please send questions, comments, complaints, and corrections to `jinghui.yang@stcatz.ox.ac.uk`.

This course is an introduction to the geometry of Kähler manifolds and the Hodge structure of cohomology.

Kähler manifolds are at the intersection of complex analytic geometry, Riemannian geometry, and symplectic geometry. Moreover, every smooth projective variety is a Kähler manifold. All of this structure is reflected in a rich theory of geometric and topological invariants. In this course we will develop techniques from sheaf theory and linear elliptic theory to study the cohomology of Kähler manifolds.

The course web page can be founded here: `https://faculty.math.illinois.edu/~palbin/Math514.Fall2020/home.html`.

2 Introduction

Let $f_i(x_1, \cdots, x_n)$, $i \in \{1, \cdots, k\}$ be polynomials with coefficients in $\mathbb{R}$ and $\mathbb{C}$. An **affine algebraic variety** is the common zero set of $X = X(f_1, \cdots, f_n) = \{x : f_i(x) = 0, \forall i\}$. We incorporate the field into the notation

$$X(\mathbb{C}) = \{x \in \mathbb{C}^n : f_i(x) = 0, \forall i\}$$
$$X(\mathbb{R}) = \{x \in \mathbb{R}^n : f_i(x) = 0, \forall i\} \text{ (when } f_i \in \mathbb{R}[x]) \ .$$

These can be thought of naturally as topological spaces with the topology they inherit from $\mathbb{C}^n$ or $\mathbb{R}^n$. (Alternatively, one can see the Zariski topology induced by declaring that zero sets of polynomials are closed.)

Remark 1. $X(\mathbb{C})$ is essentially never compact.

To remedy this, we shift our attention to projective space.

Definition 1. The **complex projective space** $\mathbb{C}P^n$ is defined as follow: for all $\lambda \neq 0$,

$$\begin{aligned}\mathbb{C}P^n &= \mathbb{C}^{n+1} \backslash \{0\}/\text{dilations} \\ &= \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} \backslash \{0\}\}/(z_0, \cdots, z_n) \sim (\lambda z_0, \cdots, \lambda z_n) \\ &= S^{n+1}/U(1).\end{aligned}$$

By definition, it is compact.

Given homogeneous polynomials $F_i(x_0, \cdots, x_n)$, we obtain a "projective variety" $X = X(F_1, \cdots, F_k) = \{x \in \mathbb{R}^k : F_i(x) = 0, \forall i\}$. As before, if the polynomial have real coefficients, the case becomes $X(\mathbb{R}) = \{x \in \mathbb{R}P^n : F_i(x) = 0, \forall i\}$. We can ask about the relation between the topology and geometry of $X(\mathbb{R})$ and $X(\mathbb{C})$ and the algebraic properties of X. For example, say X is the zero set of a single homogeneous polynomial F of degree d, can we recover d from $X(\mathbb{C})$ and $X(\mathbb{R})$? This is only a sensible question for irreducible F.

It turns out that $X_F(\mathbb{C})$ determines a homology class $[X_F(\mathbb{C})] \in H_{2n-2}(\mathbb{C}P^n; \mathbb{Z})$ and this group is cyclic with generator $[H]$ induced by $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ and $[X_F(\mathbb{C})] = d \cdot [H]$.

We can recover d from the intrinsic geometry at $X_F(\mathbb{C})$ using its "Chern class" in $H^*(X_F(\mathbb{C}))$. Over the real numbers, $H_{n-1}(\mathbb{R}P^b; \mathbb{Z}_2)$ is cyclic with generator $[H]$ and $[X_F(\mathbb{R})] = d \cdot [H]$, this gives us the recovery of $d \mod 2$. It's possible to show that $X_F(\mathbb{R})$ does not provide an upper bound for d.

From a different point of view, the Nash embedding theorem shows that any smooth, closed manifold over $\mathbb{R}$ is diffeomorphic to $X(\mathbb{R})$ for some real, smooth, projective variety. For complex manifolds, the analogue statement is false.

To be diffeomorphic to a complex projective variety, a manifold must be complex, Kähler, Hodge and then it will have an embedding into $\mathbb{C}P^N$ for some integer N and Chow's theorem guarantees that it's algebraic. In this course, we'll show that compact submanifolds of $\mathbb{C}P^n$ satisfy these properties is a complex projective variety.

3 Several Complex Variables

3.1 Basic Settings

First recall some concepts and theorems for one complex variable. Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a function.

Definition 2. f is called **holomorphic** if f satisfies the Cauchy-Riemann equation. That is, write $z = x + iy$ and $f(x, y) = u(x, y) + iv(x, y)$, where $x, y \in \mathbb{R}$ and u, v are $\mathbb{R}$-valued C^1-functions, and they satisfies

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Or, $df(z)$ is $\mathbb{C}$-linear.

Definition 3. f is called **analytic** if for any $z_o \in U$, there exists an $\epsilon > 0$ such that for every $z \in B_\epsilon(z_0)$, the ball with radius ϵ centered at z_0,

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n.$$

Proposition 1. f is analytic if and only if f is holomorphic. This is also equivalent to $f \in C^1$ satisfying the Cauchy integral formula: for every $z_0 \in U$, there exists a small $\epsilon > 0$ such that

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(z)}{z - z_0} dz.$$

Let's introduce the differential operators

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

We justify the notations by

$$\partial_z z = \partial_{\overline{z}} \overline{z} = 1, \quad \partial_z \overline{z} = \partial_{\overline{z}} z = 0.$$

In terms of the new notations, Cauchy-Riemann equation can be written by $\partial_{\overline{z}} f = 0$. Geometrically, $f : U \subset \mathbb{C} = \mathbb{R}^2 \to \mathbb{C} = \mathbb{R}^2$, then

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix}$$

induces $D_{z_0} f : T_{z_0}\mathbb{R}^2 \to T_{f(z_0)}\mathbb{R}^2$ with respect to the standard bases. This is the **(real) Jacobian** of f,

$$J_{\mathbb{R}}(f) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

Example 1.

$$\begin{aligned}(D_{(x_0,y_0)}f)(\partial_x) &= \partial_t|_{t=0} f(x_0+t,y_0)\\ &= \partial_t|_{t=0}\begin{bmatrix}u(x_0+t,y)\\ v(x_0+t,y)\end{bmatrix}\\ &= \begin{bmatrix}\partial_x u\\ \partial_x v\end{bmatrix}\end{aligned}$$

After we complexify $D^{\mathbb{C}}_{z_0}f : T_{z_0}\mathbb{R}^2\otimes\mathbb{C}\to T_{f(z_0)}\mathbb{R}^2\otimes\mathbb{C}$, we can write this matrix in the basis $\partial_z, \partial_{\overline{z}}$ for both domain and codomain:

$$\begin{bmatrix}\partial_z(u+iv) & \partial_{\overline{z}}(u+iv)\\ \partial_z(u-iv) & \partial_{\overline{z}}(u-iv)\end{bmatrix} = \begin{bmatrix}\partial_z f & \partial_{\overline{z}} f\\ \partial_z \overline{f} & \partial_{\overline{z}}\overline{f}.\end{bmatrix}$$

Here we have $\partial_{\overline{z}}\overline{f} = \overline{\partial_z f}$ and $\overline{\partial_z\overline{f}} = \partial_{\overline{z}} f$. The function f is holomorphic if and only if this matrix is diagonal. The **(complex) Jacobian** for f is

$$J_{\mathbb{C}}(f) = \begin{bmatrix}\partial_z f\end{bmatrix}.$$

Holomorphic functions of one variable satisfy the following important theorems:

Theorem 1 (**Maximum Principle**)**.** Suppose we have an open and connected set $U\subset\mathbb{C}$ and a holomorphic function $f : U\to\mathbb{C}$ that is non-constant. Then $|f|$ has no local maximum in U. If U is bounded and f can extend to a continuous function $\tilde{f} : \overline{U}\to\mathbb{C}$, then $\max|f|$ occurs on $\partial\overline{U}$.

Theorem 2 (**Identity Theorem**)**.** Suppose we have two holomorphic functions $f, g : U\to\mathbb{C}$, and $U\subset\mathbb{C}$ is connected. If $\{z\in U : f(z) = g(z)\}$ contains an open set, then it is all of U.

Theorem 3 (**Extension Theorem**)**.** Suppose we have a bounded holomorphic function $f : B_\epsilon(z_0)\backslash\{z_0\}\to\mathbb{C}$ defined on some ball of radius $\epsilon > 0$ centered at z_0, then it extends to a holomorphic function $\tilde{f} : B_\epsilon(z_0)\to\mathbb{C}$.

Theorem 4 (**Liouville's Theorem**)**.** If $f : \mathbb{C}\to\mathbb{C}$ is holomorphic and bounded, then f is constant.

Theorem 5 (**Riemann Mapping Theorem**)**.** If $U\subset\mathbb{C}$ is a simply connected proper open set, then U is biholomorphic to the unit ball $B_1(0)$.

Theorem 6 (**Residue Theorem**)**.** If $f : B_\epsilon(0)\backslash\{0\}\to\mathbb{C}$ is holomorphic and $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ is its Laurent series, then

$$a_{-1} = \frac{1}{2\pi i}\int_{\partial B_{\epsilon/2}(0)} f(z)dz.$$

We now begin the several complex variables part.

Definition 4. Let $U\subset\mathbb{C}^n$ be an open set, $f : U\to\mathbb{C}$ is continuous differentiable. Then f is **holomorphic** at $a = (a_1,\cdots .a_n)\in U$ if for all $j\in\{1,\cdots,n\}$, the function of one variable

$$z_j\mapsto f(a_1,\cdots,a_{j-1},z_j,a_{j+1},\cdots,a_n)$$

is holomorphic at a_j, i.e. $\partial_{\overline{z_j}} f = 0$. If we write

$$(df)_{\mathbb{C}} = \underbrace{\sum \frac{\partial f}{\partial z_j} dz_j}_{\partial f} + \underbrace{\sum \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j}}_{\overline{\partial} f},$$

then f is holomorphic if and only if $\overline{\partial} f = 0$.

Definition 5. For $a \in \mathbb{C}^n$, $R \in (\mathbb{R}^+)^n$, the **polydisc** around a with multiradius R is the set

$$D(a,R) = \{z \in \mathbb{C}^n : |z_j - a_j| < R_j, \forall j \in \{1, \cdots, n\}\}.$$

If $R = (1, \cdots, 1)$ and $a = 0$, we abbreviate $D(0,1)$ by $\mathbb{D}^n$ and refer to it as the unit disc in $\mathbb{C}^n$.

Repeatly applying the Cauchy formula in one variable, we obtain

Theorem 7. Let $f : D(\omega, \epsilon) \to \mathbb{C}$ be a holomorphic function and $z \in D(\omega, \epsilon)$, then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D(\omega,\epsilon)} \frac{f(\xi_1, \cdots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} d\xi_1 \cdots d\xi_n.$$

Using this theorem, we can show that for any $\omega \in U$, there exists $D(\omega, \epsilon) \subset U$ such that for all $z \in D(\omega, \epsilon)$,

$$f(z) = \sum_{|\alpha|=0}^{\infty} \frac{\partial_z^\alpha f}{\alpha!} (z - \omega)^\alpha,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{N})^n$ is a multi-index. To be explicit,

$$(z - \omega)^\alpha = \prod_{k=1}^{n} (z_k - \omega_k),$$
$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!,$$
$$\partial_z^\alpha f = \partial_{z_1}^{\alpha_1} \partial_{z_2}^{\alpha_2} \cdots \partial_{z_n}^{\alpha_n} f.$$

From the list above, the Maximum Principle, the Identity Theorem, and the Liouville's Theorem generalize easily. Riemann Extension Theorem holds but is harder to prove. However, Riemann Mapping Theorem fails in several variables. There are also phenomena that do NOT have analogues in one complex variable. One example we shall see later is the Hartogs Extension Theorem.

3.2 Equation $\overline{\partial} u = f$

In this section, we will continue to list the counterexamples that do NOT have analogues in one complex variable. These are examples of the Hartogs phenomenon.

Example 2. Consider $H = \{(z,\omega) \in \mathbb{C}^2 : |z| < 1, \frac{1}{2} < |z| < 1\} \cup \{(z,\omega) \in \mathbb{C}^2 : |z| < \frac{1}{2}, |\omega| < 1\}$. Let f be holomorphic on H. Claim: there exists a holomorphic function F defined on $\mathbb{D} = \{(z,\omega) \in \mathbb{C}^2 : |z| < 1, |\omega| < 1\}$ such that $F|_H = f$.

In fact, we have for $r \in (\frac{1}{2}, 1)$

$$F(z,\omega) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(z,\xi)}{\xi - \omega} d\xi,$$

so F is holomorphic. Indeed, $\partial_{\overline{z}}\left(\frac{f(z,\xi)}{\xi-\omega}\right) = \partial_{\overline{\omega}}\left(\frac{f(z,\xi)}{\xi-\omega}\right) = 0$. For any fixed z with $|z| < \frac{1}{2}$, $\omega \mapsto f(z,\omega)$ is holomorphic on all of $\{(z,\omega) \in \mathbb{C}^2 : |\omega| < 1\}$. So $F(z,\omega) = f(z,\omega)$ for any $|z| < \frac{1}{2}$, $|\omega| < r$ by the Cauchy integral formula, which implies $F = f$ on H.

Lemma 1. Let $f \in C^1(\overline{\Omega})$, then

$$\int_{\partial\Omega} f dz = \int_{\Omega} \frac{\partial f}{\partial \overline{z}} d\overline{z} dz = 2i \int_{\Omega} \partial_{\overline{z}} f dxdy.$$

Proposition 2. Let $u \in C^1(\overline{\Omega})$, then

$$u(\zeta) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{u(\zeta)}{z-\zeta} dz + \frac{1}{\pi} \int_{\Omega} \frac{\frac{\partial u}{\partial \overline{z}}(z)}{z-\zeta} dxdy.$$

Proof. Fix ζ, let $\epsilon < d(\zeta, \partial\overline{\Omega})$ and let $\Omega_\epsilon = \{z \in \Omega : |z-\zeta| > \epsilon\}$. Apply Lemma 1 to $f(z) = \frac{u(z)}{z-\zeta}$, we obtain

$$\int_{\partial\Omega_\epsilon} \frac{u(z)}{z-\zeta} dz = 2i \int_{\Omega_\epsilon} \frac{\partial_{\overline{z}} u}{z-\zeta} dxdy.$$

As $\epsilon \to 0$, LHS converges to

$$-\int_{\partial\Omega} \frac{u(\zeta)}{z-\zeta} dz + 2\pi i u(\zeta),$$

which concludes our result. □

Theorem 8. Let $\phi \in C_c^\infty(\mathbb{C})$ and $u(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\phi(z)}{z-\zeta} dxdy$, then u is an analytic function outside of $\operatorname{supp}\phi$ and u is smooth on $\mathbb{C}$. Moreover, $\partial_{\overline{z}} u = \phi$.

Proof. Interchanging derivatives and the integral we see that $u \in C^\infty(\mathbb{C})$. By a change of variables, we have

$$u(\zeta) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(\zeta - z)}{z} dxdy.$$

So

$$\partial_{\overline{\zeta}} u(\zeta) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial_{\overline{\zeta}} \phi(\zeta - z)}{z} dxdy = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial_{\overline{z}} \phi(z)}{z-\zeta} dxdy.$$

Applying Proposition 2 to any disc containing $\operatorname{supp}\phi$, we get $\partial_{\overline{\zeta}} u = \phi$. □

Remark 2. Even though ϕ has compact support, there is no solution of $\partial_{\overline{z}} u = \phi$ that can have compact support if $\int_{\mathbb{C}} \phi \neq 0$. Indeed, if $u(\zeta) = 0$ for any $|\zeta| > R$, then

$$0 = \int_{|z|=R} u(z) dz = \int_{|z|<R} \partial_{\overline{z}} u dxdy = 2i \int_{|z|<R} \phi dxdy.$$

Theorem 9. Suupose $f_j \in C_c^\infty(\mathbb{C}^n)$, $j \in \{1, \cdots, n\}$, $n > 1$, satisfy $\partial_{\overline{z_j}} f_k = \partial_{\overline{z_k}} f_j$ for every $j, k \in \{1, \cdots, n\}$, then there is a $u \in C_c^\infty(\mathbb{C}^n)$ such that $\partial_{\overline{z_j}} u = f_j$ for every $j \in \{1, \cdots, n\}$.

Proof. Define

$$\begin{aligned} u(z) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta, z_2, \cdots, z_n)}{\zeta - z_1} d\zeta d\overline{\zeta} \\ &= -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(z_1 - \zeta, z_2, \cdots, z_n)}{\zeta} d\zeta d\overline{\zeta}. \end{aligned}$$

We note that $u \in C^\infty(\mathbb{C})$. Since f_1 has compact support, u vanishes if $|z_2| + \cdots + |z_n| >> 0$. By Theorem 8, $\partial_{\overline{z_1}} u = f_1$. Also differentiating, by Proposition 2,

$$\partial_{\overline{z_j}} u = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial_{\overline{z_1}} f_j(\zeta, z_2, \cdots, z_n)}{\zeta - z_1} d\zeta d\overline{\zeta} = f_j.$$

Hence u solves the system of equations. Let $K = \bigcup_{j=1}^n \operatorname{supp} f_j$, then u is holomorphic on $\mathbb{C}^n \backslash K$. We know that u is zero if $|z_2| + \cdots + |z_n| >> 0$, so by the Identity Theorem u must vanish on the unbounded component of $\mathbb{C}^n \backslash K$, which implies $u \in C_c^\infty(\mathbb{C}^\infty)$. □

3.3 Hartogs Extension Theorem

We will introduce the famous Hartogs Extension Theorem.

Theorem 10 (**Hartogs Extension Theorem**). Let U be a domain in $\mathbb{C}^n$, $n > 1$; that is, U is a non-empty connected open set. Let K be a compact subset of U such that $U \backslash K$ is connected. Then every holomorphic function on $U \backslash K$ extends uniquely to a holomorphic function on U.

Proof. Given an analytic function f defined on $U \backslash K$. Choose $\theta \in C_c^\infty(U)$ such that $\theta|_K = 1$. Define $f_0 \in C^\infty(U)$ by setting

$$f_0(z) = \begin{cases} 0 & , z \in K \\ (1-\theta) f & , z \in U \backslash K. \end{cases}$$

We shall construct $v \in C^\infty(\mathbb{C}^n)$ such that $f_0 + v$ is the required holomorphic extension of f. In order for $f_0 + v$ to be holomorphic we need

$$\partial_{\overline{z_j}}(f_0 + v) = \partial_{\overline{z_j}}(1-\theta) f + \partial_{\overline{z_j}} v = -(\partial_{\overline{z_j}} \theta) f + \partial_{\overline{z_j}} v,$$

that is, we need $\partial_{\overline{z_j}} v = (\partial_{\overline{z_j}} f)$ for every $j \in \{1, \cdots, n\}$. By Theorem 9, we can find $v C_c^\infty(\mathbb{C}^n)$ solving this system of equations. Since v has compact support and is holomorphic outside the support of θ, it must vanish on the unbounded component of $\mathbb{C}^n \backslash \operatorname{supp} \theta$, Since $\operatorname{supp} \theta \subset U$, there is an open set in $U \backslash K$ where $v \equiv 0$ and so $f_0 + v = f_0 = f$. But $U \backslash K$ is connected and $f, f_0 + v$ are holomorphic, so they must coincide on all of $U \backslash K$. Thus $f_0 + v$ is the desired holomorphic extension of f. □

Corollary 1. Let $U \subset \mathbb{C}^n$ be a domain, $n > 1$, and f is holomorphic on U. The zero set $f^{-1}(0)$ of f is never a compact subset of U.

Proof. Assume $K = f^{-1}(0)$ is compact and let $g : U \backslash K \subset \mathbb{C}^n \to \mathbb{C}^n$ be $g(z) = \frac{1}{f(z)}$. Then g is holomorphic on $U \backslash K$. Proceeding as in the proof of the Hartogs Extension Theorem, we pick $\theta \in C_c^\infty(U)$ with $\theta|_K = 1$. Define $g_0 \in C^\infty(U)$ to be 0 on K and $(1-\theta) g$ otherwise. We can find $v \in C_c^\infty(\mathbb{C}^n)$ such that v is holomorphic on $\mathbb{C}^n \operatorname{supp} \theta$ and $g_0 + v$ is holomorphic on U. So v vanishes on the unbounded component of $U \backslash \operatorname{supp} \theta$. Thus there is an open set in $U \backslash K$ on which $g = g_0 = g_0 + v$ since g and $g_0 + v$ are holomorphic on $U \backslash K$, they coincide on the corresponding connected component of $U \backslash K$, say W. Finally, pick $(w_k) \subset W$, $w_k \to w_\infty \in K$. We have $|g(w_k)| = \frac{1}{|f(w_k)|} \to \infty$, but from $g(w_k) = (g_0 + v)(w_k) \to (g_0 + v)(w_\infty)$ we conclude $|g(w_k)|$ is bounded. Contradiction! □

Heuristically, we might expect that the zero set of a nontrivial holomorphic function $f : U \subset \mathbb{C}^n \to \mathbb{C}$ will have complex codimension 1. For example, if f is a polynomial of degree 1, then its zero set is an affine subspace of complex dimension $n-1$. If $D_p f : \mathbb{C}^n \to \mathbb{C}$ is nonzero at each $p \in f^{-1}(0)$, then f is "well-approximated" by its linear approximation and $f^{-1}(0)$ should be locally modeled by open subsets of $\mathbb{C}^{n-1}$. Indeed, the complex version of the implicit function theorem holds and shows that $f^{-1}(0)$ is a smooth submanifold of complex dimension $n-1$.

Things are more complicated if the derivative vanishes. In one complex variable, if nonzero function f satisfies $f(0) = 0$, then "0 is a root of a finite order" (say p) means that there is a holomorphic function g such that $g(0) \neq 0$ and $f(z) = z^p g(z)$. In several complex variables, after a change of coordinates, we can write the nontrivial function f to be $F(z_n) = f(0, z_n)$, where $0 \in \mathbb{C}^{n-1}$. So it has a zero of finite order, sat p, such that $F(z_n) = z_n^p g_0(z_n)$.

Using the continuity of f, we can apply Roche's Theorem to conclude that there is a polydisc $D(0, \epsilon) \subset \mathbb{C}^{n-1}$ such that for all $z' \in D(0, \epsilon)$, the function $z \mapsto f(z', z)$ has exactly p zeros in $D(0, \epsilon_n) \subset \mathbb{C}$. In particular, we see again that the zeros of a holomorphic function of several variables are not isolated.

3.4 Weierstrass Preparation Theorem

In this section, we will discussion what a holomorphic function looks like near a zero. In the one variable case, $f(z) = z^p g(z)$, where p is a positive integer and g is a holomorphic function with $g(0) \neq 0$ or $f \equiv 0$. Suppose we are given a holomorphic function $f(z_1, \cdots, z_{n-1}, \omega)$ near $0 \in \mathbb{C}^n$, and $f(0, \cdots, 0) = 0$, and ω-axis is not in $f^{-1}(0)$. Write $f_z(\omega) = f(z, \omega)$, where $z \in \mathbb{C}^{n-1}$, then $f_0(\omega)$ is not identically zero. We know that $f_0(\omega) = \omega^p g(\omega)$, where $g(0) \neq 0$, p is a positive integer. There exists a $r > 0$ such that $|f_0(\omega)| > \delta > 0$ whenever $|\omega| = r$. So there exists an $\epsilon > 0$ such that $|z| < \epsilon$, $|\omega| = r$, then $|f_z(\omega)| > \frac{\delta}{2} > 0$.

Writing $f_z(\omega) = \tilde{f}_z(\omega) \prod_{j=1}^p (\omega - a_j(z))$, we see that

$$\sum_j a_j(z)^q = \frac{1}{2\pi i} \int_{|\omega|=r} \omega^q \frac{f_z'(\omega)}{f_z(\omega)} d\omega.$$

This shows that the LHS is a holomorphic function of z for any q. Hence the elementary symmetric functions of the $a_j(z)$ are holomorphic functions of z. Denote them by $\sigma_j(z)$. Then $g_z(\omega) = \omega^p - \sigma_1(z)\omega^{p-1} + \cdots + (-1)^N \sigma_N(z) = \prod(\omega - a_j(z))$ is also holomorphic function of z. So, $g(z, \omega) = g_z(\omega)$ is holomorphic on $\{(z, \omega) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z| < \epsilon, |\omega| < r\}$. It also has the same zeros as $f(z, \omega)$ on this set.

Define

$$h(z, \omega) = \frac{f(z, \omega)}{g(z, \omega)}.$$

One can check it is well-defined and holomorphic off the zero set. Fix z, $h_z(\omega)$ has only removable singularities so it extends to a function on $D(0, \epsilon) \times D(o, r)$. The extended one is holomorphic in ω for each z and holomorphic off the zero set. Writing

$$h(z,\omega) = \frac{1}{2\pi i}\int_{|u|=r} \frac{h(z,u)}{u-\omega}du,$$

we see that h is holomorphic in z.

Definition 6. A **Weierstrass polynomial** in ω is a polynomial of the form

$$\omega^p + \alpha_1(z)\omega^{p-1} + \cdots + \alpha_p(z),$$

where $\alpha_j(z)$ is holomorphic for each j with $\alpha_j(0) = 0$.

Theorem 11 (**Weierstrass Preparation**)**.** Let f be a holomorphic function near the origin in $\mathbb{C}^n$ and $f(0) = 0$. f is also assumed not to be identically zero on the ω-axis. Then there is a neighborhood of 0 in which f can be written uniquely as $f = g \cdot h$, where g is a Weierstrass polynomial of degree p in ω and $h(0) \neq 0$.

Theorem 12 (**Riemann Extension**)**.** Suppose $f(z,\omega)$ is holomorphic in a ball $\mathbb{B} \subset \mathbb{C}^n$, and $g(z,\omega)$ is holomorphic on $\mathbb{B}\backslash f^{-1}(0)$ and bounded. Then g extends to a holomorphic function on $\mathbb{B}$.

Proof. WLOG, assume that ω-axis is not contained in $f^{-1}(0)$. As before, there are r, ϵ such that $|f(z,\omega)| > \delta > 0$ whenever $|z| < \epsilon$, $|\omega| = r$. The one variable version of Riemann extension then applies to each g_z and the extension $\tilde{g}_z$ satisfies

$$\tilde{g}_z(\omega) = \frac{1}{2\pi i}\int_{|\omega|=r} \frac{g_z(\xi)}{\xi - \omega}d\xi.$$

Hence $\tilde{g}(z,\omega) = g_z(\omega)$ is holomorphic in (z,ω) for all $|z| < \epsilon, |\omega| < r$. □

Now, we will discuss the failure of Riemann mapping theorem in several complex variables.

Example 3. Consider $H = \{z \in \mathbb{C}^n : \Im z_1 > 0\}$ and $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$. If $\psi : H \to \mathbb{B}^n$ is holomorphic, then for each z_1 with $\Im z_1 > 0$, the function $(z_2, \cdots, z_n) \mapsto \psi(z_1, z_2, \cdots, z_n)$ is holomorphic and bounded on C^{n-1}. Hence it is constant.

Theorem 13 (Poincaré)**.** For $n > 1$, the unit polydisc $\mathbb{D}^n$ and the unit ball $\mathbb{B}^n$ are not biholomorphic.

Proof. Assume $\psi : \mathbb{D}^n \to \mathbb{B}^n$ is a biholomorphic with $\psi(0) = 0$ and let $\Phi : \mathbb{B}^n \to \mathbb{D}^n$ be the inverse. We claim that we must have $D_0\psi(\mathbb{D}^n) \subset \mathbb{B}^n$ and $D_0\Phi(\mathbb{B}^n) \subset \mathbb{D}^n$. Given the claim we have $D_0\psi(\mathbb{D}^n) = \mathbb{B}^n$ and $D_0\psi(\partial\mathbb{D}^n) = \partial\mathbb{B}^n$. This is impossible since $D_0\psi$ is linear and $\partial\mathbb{D}^n$ contains linear pieces of positive dimension, which $\partial\mathbb{B}^n$ does not.

Now to prove the claim.

1. $D_0\psi(\mathbb{D}^n) \subset \mathbb{B}^n$:

   Write $\psi = (\psi_1, \cdots, \psi_n)$, where $r \in \mathbb{D}^n$ and $u = (u_1, \cdots, u_n) \in \mathbb{B}^n$. Applying Schwarz's lemma to the function

   $$t \mapsto \sum_{j=1}^{n} u_j \psi_j(tv),$$

   we see that $|\langle \overline{u}, (D_0\psi)(v)\rangle| \leq 1$. As this holds for all $u \in \mathbb{B}^n$, we must have $|D_0\psi(v)| \leq 1$.

2. $D_0\Phi(\mathbb{B}^n) \subset \mathbb{D}^n$:

   Write $\Phi = (\Phi_1, \cdots, \Phi_n)$ and $u = (u_1, \cdots, u_n) \in \mathbb{B}^n$. Applying Schwarz's lemma to the function

$$t \mapsto \Phi_j(tu_1, \cdots, tu_n),$$

   we see that $|\sum u_k \partial_{z_k} \Phi_j(0)| \leq 1$ for $1 \leq j \leq n$. Hence $D_0\Phi(u) \in \mathbb{D}^n$.

$\square$

4 Manifolds and Bundles

We will focus on the complex manifolds, complex structures and bundle theory in this chapter. To begin with, we assume the familiarization of point-set topology.

4.1 Manifolds

Let M be a metrizable topological space.

Definition 7. A **real (or complex) coordinate chart** is a homeomorphism $\varphi : U \to V$ between an open subset $U \subset M$ and an open subset $V \subset \mathbb{R}^n$ (or $\mathbb{C}^n$). A **smooth (or holomorphic) atlas** is a collection of charts $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ such that

1. $M = \bigcup_\alpha U_\alpha$ and,
2. the transition map $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ between open subsets of $\mathbb{R}^n$ (or $\mathbb{C}^n$) is smooth (or holomorphic) whenever $U_\alpha \cap U_\beta \neq \varnothing$.

Two atlases are **equivalent** if their union is again an atlas.

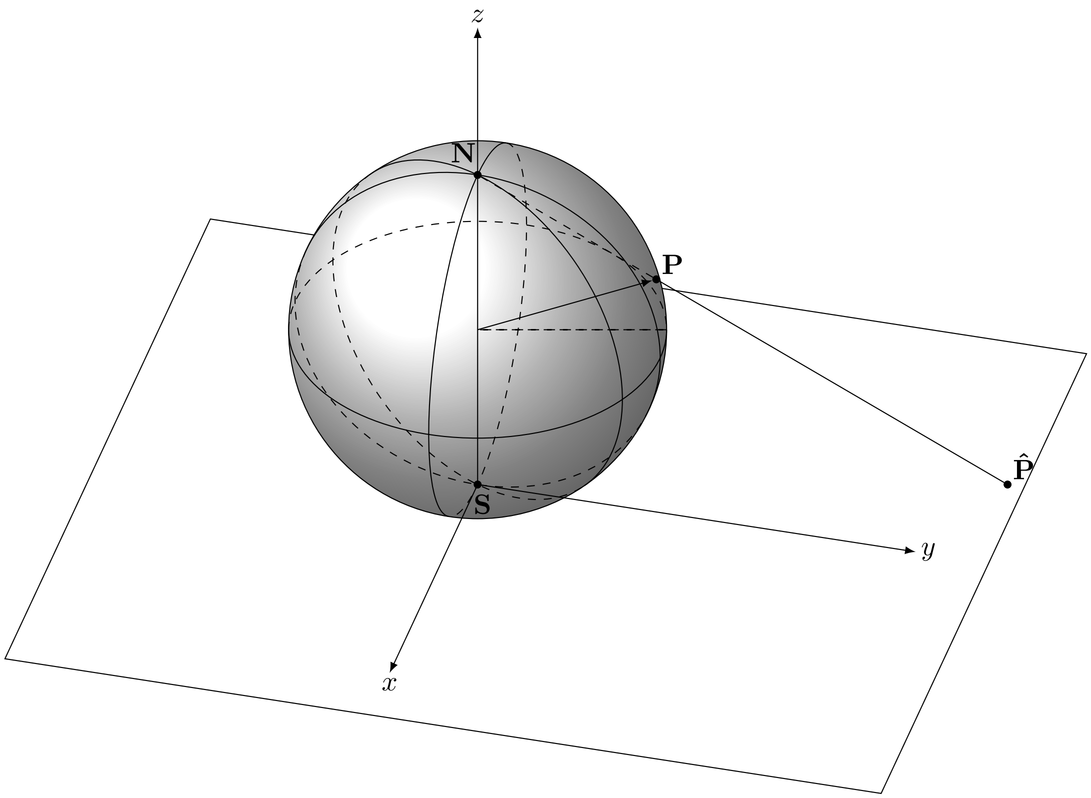
Definition 8. A **smooth (or complex) manifold** is a metrizable topological space M together with an equivalence class of smooth (or holomorphic) atlases.

We will assume once for all that all charts use the same dimensional Euclidean space and refer to n as the real dimension of M (or n as the complex dimension of M).

Example 4. Any open subset of $\mathbb{R}^n$ is an n-dimensional real manifold with a single chart. Similarly any open subset of $\mathbb{C}^n$ is an n-dimensional complex manifold with a single chart.

Example 5. The sphere S^2 is a 2-dimensional real manifold and a 1-dimensional complex manifold.

1. To see it is a 2-dimensional real manifold, first notice $S^2 \subset \mathbb{R}^3$. We can use the stereographic projection. Choose a pair of antipodal points on the ball, say N and S, to be the "north pole" and the "south pole" of the ball, respectively. Define the coordinate charts $\phi_N : S^2 \backslash \{N\} \to \mathbb{R}^2$ to be $(x, y, z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z})$, and $\phi_S : S^2 \backslash \{S\} \to \mathbb{R}^2$ to be $(x, y, z) \mapsto (\frac{x}{1+z}, \frac{y}{1+z})$.



To see this makes sense, one can compute the inverse maps ϕ_N^{-1} as follows: let $(u,v) = (\frac{x}{1-z}, \frac{y}{1-z})$, then $u^2+v^2 = (\frac{x}{1-z})^2 + (\frac{y}{1-z})^2 = \frac{1+z}{1-z}$. This implies $z = \frac{u^2+v^2-1}{u^2+v^2+1}$. In the same way we can obtain x, y in terms of u, v. So $\phi_N^{-1}(u,v) = (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1})$. Similarly we can get ϕ_S^{-1} (exercise). It is straight calculation that $\phi_S \circ \phi_N^{-1}(u,v) = (\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2})$, which is smooth. Hence we obtain a smooth atlas $\{\phi_N, \phi_S\}$ that ensures S^2 is a 2-dimensional smooth manifold.

2. To see it is a 1-dimensional complex manifold, we define

$$\phi_N : S^2 \backslash \{N\} \to \mathbb{C}, \quad (x,y,z) \mapsto \frac{x+iy}{1+z}$$
$$\phi_S : S^2 \backslash \{S\} \to \mathbb{C}, \quad (x,y,z) \mapsto \frac{x-iy}{1-z}.$$

Then $\phi_S \circ \phi_N^{-1}(\omega) = \frac{1}{\omega} : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$, which is holomorphic. Hence it yields that S^2 is a 1-dimensional complex manifold. This is usually called the **Riemann sphere**.

Exercise 1. If we define

$$\tilde{\phi}_S : S^2 \backslash \{S\} \to \mathbb{C}, \quad (x,y,z) \mapsto \frac{x+iy}{1-z},$$

in the above Example 5, can we still obtain an atlas $\{\phi_N, \tilde{\phi}_S\}$ that makes S^2 a 1-dimensional complex manifold? Why?

Definition 9. If M is a smooth (or complex) manifold, a function $f : M \to \mathbb{R}$ (or $\mathbb{C}$) is **smooth** (or **holomorphic**) at $\zeta \in M$ if there is a chart $\varphi : U \to V$, where U is a neighborhood of ζ in M, such that $f \circ \varphi^{-1} : V \to \mathbb{R}$ is smooth (or holomorphic).

Definition 10. If M and M' are smooth (or complex) manifolds, a function $f : M \to M'$ is smooth (or holomorphic) at $\zeta \in M$ if there are charts $\varphi : U \to V$ and $\varphi' : U' \to V'$, where U is a neighborhood of ζ in M and U' is a neighborhood of $f(\zeta)$ in M', such that $\varphi' \circ f \circ \varphi^{-1} : V \to V'$ is smooth (or holomorphic) at $\varphi(\zeta)$.

Example 6. Consider the complex projective space of dimension n, $\mathbb{C}P^n$. There is a natural surjective map $\pi : \mathbb{C}^{n+1}\{0\} \to \mathbb{C}P^n$, which sends z to $[z] \in \mathbb{C}P^n$, where $[z]$ is the equivalence class of $\{\lambda z : \lambda \in \mathbb{C}\}$. We equip $\mathbb{C}P^n$ through π with quotient topology. Then for any $z = (z_0, \cdots, z_n)$, we denote $\pi(z)$ by $(z_0 : z_1 : \cdots : z_n)$ and call them "homogeneous coordinate". Note $(z_0 : z_1 : \cdots : z_n) = (\lambda z_0 : \lambda z_1 : \cdots : \lambda z_n)$ for every $\lambda \in \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$.

For $0 \le j \le n$, we denote $U_j = \{(z_0 : z_1 : \cdots : z_n) \in \mathbb{C}P^n : z_j \neq 0\}$. These U_j form the standard open cover of $\mathbb{C}P^n$. Define coordinate charts $\varphi_j : U_j \to \mathbb{C}^n$ by sending $(z_0 : z_1 : \cdots : z_n)$ to $\frac{1}{z_j}(z_0, \cdots, \widehat{z_j}, \cdots, z_n)$. These are homeomorphisms with $\varphi_j^{-1} : \mathbb{C}^n \to U_j$, sending $(\omega_1, \cdots, \omega_n)$ to $(\omega_1 : \cdots : \omega_{j-1} : 1 : \omega_{j+1} : \cdots : \omega_n)$ with $\varphi_j(U_j \cap U_k) = \{(\omega_1, \cdots, \omega_n) \in \mathbb{C}^n : \omega_k \neq 0\}$. Also

$$\begin{aligned} \varphi_k \circ \varphi_j^{-1} : &\varphi_j(U_j \cap U_k) \to \varphi_k(U_j \cap U_k) \\ &(\omega_1, \cdots, \omega_n) \mapsto \frac{1}{\omega_k}(\omega_1, \cdots, \omega_{j-1}, 1, \omega_{j+1}, \cdots, \widehat{\omega_k}, \cdots, \omega_n). \end{aligned}$$

So $\varphi_k \circ \varphi_j^{-1}$ is holomorphic. The atlas $\{\varphi_j : U_j \to \mathbb{C}^n\}$ gives $\mathbb{C}P^n$ the structure of a complex manifold of complex dimension n. As smooth manifolds, $\mathbb{C}P^n \simeq S^{2n+1}/S^1$, hence $\mathbb{C}P^n$ is compact.

4.2 Bundle Theory

Definition 11. Let M be a smooth manifold, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A **rank** k $\mathbb{F}$**-vector bundle** over M consists of

- A smooth manifold E together with a surjective map $E \xrightarrow{\pi} M$;
- For each $\zeta \in M$, the fiber $E_\zeta = \pi^{-1}(\zeta)$ is an $\mathbb{F}$-vector space of dimension k;
- **(local trivialization)** For each $\zeta \in M$, there is a neighborhood U and a diffeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{F}^k$ such that the following diagram is commutative

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad h \quad} & U \times \mathbb{F}^k \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U & \end{array}$$

  and $h|_{\pi^{-1}(\zeta)} : E_\zeta \to \mathbb{F}^k$ is an isomorphism.

M is called the **base space**, E is called the **total space**, and $\mathbb{F}^k$ is called the **fiber** of the vector bundle. We often denote this by $\mathbb{F}^k \to E \to M$. If $k = 1$, we call an rank 1 $\mathbb{F}$-vector bundle an $\mathbb{F}$**-line bundle**.

Definition 12. The **trivial rank** k $\mathbb{F}$**-vector bundle** over M is $M \times \mathbb{F}^k \xrightarrow{pr_1} M$. Denote the total space of this bundle by $\underline{\mathbb{F}^k}$.

Example 7. Let $M = S^1$. We have two rank 1 real vector bundles. The trivial bundle $E_1 = S^1 \times \mathbb{R} \to S^1$ and the infinite **Möbius band** E_2 : Möbius $\to S^1$. Both of them are local trivial. However, the first bundle is globally trivial while the second one is not.

Another way of thinking about a vector bundle comes from looking at the transitions between different local trivializations.

If (U_α, h_α) and (U_β, h_β) are local trivializations such that $U_\alpha \cap U_\beta \neq \varnothing$, then the map $h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{F}^k \to (U_\alpha \cap U_\beta) \times \mathbb{F}^k$ has the form

$$(\zeta, \nu) \mapsto (\zeta, g_{\alpha\beta}(\zeta)\nu),$$

and so is equivalent to a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k(\mathbb{F})$. We call these maps the **transition maps**. The transition maps $(U_\alpha, g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k(\mathbb{F}))$ make up a **Čech cocycle**. That is,

- $g_{\alpha\alpha} = \text{id}$,
- $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \text{id}$ whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$.

This data determines a rank k $\mathbb{F}$-vector bundle over M, and at the same time is determined from a rank k $\mathbb{F}$-vector bundle over M.

Definition 13. Let M be a complex manifold. A $\mathbb{C}$-vector bundle over M is **holomorphic** if its transition maps are holomorphic maps $U_\alpha \cap U_\beta \to GL_k(\mathbb{C})$.

Definition 14. A **vector bundle morphism** between $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ is a smooth map $\Phi : E \to E'$ such that

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E' \\ \downarrow & & \downarrow \\ M & \xrightarrow{F} & M' \end{array}$$

for some smooth $F : M \to M'$, and the restrictions $\Phi|_{\pi^{-1}(\zeta)} : E_\zeta \to E'_{F(\zeta)}$ are linear. A **vector bundle isomorphism** is a bijective vector bundle.

Definition 15. A **section** of a vector bundle $E \xrightarrow{\pi} M$ is a map $M \xrightarrow{s} E$ such that $\pi(s(\zeta)) = \zeta$ for each $\zeta \in M$. We denote the set of sections by $C^\infty(M; E)$. Note that $C^\infty(M; E) \subset C^\infty(M, E)$.

Example 8 (Tangent bundle)**.** Let M be an $\mathbb{F}$-manifold of dimension n and $(U_\alpha, \phi_\alpha : U_\alpha \to V_\alpha)$ be an atlas. So $U_\alpha \subset M$, $V_\alpha \subset \mathbb{F}^n$ and $\{U_\alpha\}$ forms a cover of M. The restriction maps on overlaps

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are diffeomorphisms. If we take derivatives, we obtain

$$D(\phi_\alpha \circ \phi_\beta^{-1}) : \phi_\beta(U_\alpha \cap U_\beta) \times \mathbb{F}^n \to \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{F}^n.$$

Define

$$TM = \left(\bigsqcup U_\alpha \times \mathbb{F}^n \right) / \sim,$$

where $(u,v) \sim (u',v')$ if for any $u,u' \in U_\alpha \cap U_\beta$, we have $D(\phi_\alpha \circ \phi_\beta^{-1})(\phi_\beta(u),v) = (\phi_\alpha(u'),v')$. That is, we take tangent vectors in each coordinate chart and identify them if they correspond to each other under the chain rule.

Remark 3. We have a *Meta-theorem* here: any canonical construction in linear algebra gives rise to a geometric version for smooth (or holomorphic) vector bundle.

Example 9. Let E, F be vector bundles over a manifold M.

1. $E \oplus F \to M$ is the vector bundle where the fiber $(E \oplus F)_\zeta$ is canonically isomorphic to $E_\zeta \oplus F_\zeta$.
2. $E \otimes F \to M$ is the vector bundle where the fiber $(E \otimes F)_\zeta$ is canonically isomorphic to $E_\zeta \otimes F_\zeta$.
3. Taking alternating or symmetric parts of $\bigotimes_{i=1}^k E \to M$ produces $\bigwedge^k E \to M$ (known as k-th exterior power) or $S^k E \to M$ (known as k-th symmetric power).
4. $\mathrm{Hom}(E,F) \to M$ has fiber canonically isomorphic to $\mathrm{Hom}(E_\zeta, F_\zeta)$.
5. $E^* \to M$ is the vector bundle $\mathrm{Hom}(E, \underline{\mathbb{F}})$.

Example 10. If E has transition maps $g_{\alpha\beta}$, F has transition maps $h_{\alpha\beta}$, then $E \oplus F$ is the vector bundle with transition maps $\left(\begin{smallmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta} \end{smallmatrix}\right)$, and E^* is the vector bundle with transition maps $f_{\alpha\beta} = (g_{\beta\alpha}^{-1})^*$.

4.3 Tangent vectors

There are several ways to define a tangent vector. We hereby give two models of doing this.

4.3.1 Algebraic Approach

Another way of thinking about manifolds, real or complex, in line with modern algebraic geometry is through "geometric structures".

Definition 16. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, X be a topological space. The for any open $U \subset X$, let $C(U) = C^0(U)$ denote the continuous functions $U \to \mathbb{F}$. Then a **geometric structure** $\mathcal{A}$ on X is a collection of subrings $\mathcal{A}(U) \subset C(U)$ such that for any open $U \subset X$, we have

1. The constant functions are in $\mathcal{A}(U)$;
2. If $f \in \mathcal{A}(U)$, the for any open $V \subset U$, $f\mid_V \in \mathcal{A}(V)$;
3. If $\{U_i\}$ is a collection of open subsets of X, $U = \cup_i U_i$, and we are given $f_i \in \mathcal{A}(U_i)$ such that $f_i \mid_{U_j} = f_j \mid_{U_i}$ whenever $U_i \cap U_j \neq \varnothing$, then there exists a unique $f \in \mathcal{A}(U)$ such that for any i, $f\mid_{U_i} = f_i$.

We call the pair $(X, \mathcal{A})$ a **geometric space**, and functions in $\mathcal{A}(U)$ **distinguished**. 2 and 3 in the conditions above imply that being distinguished is an open property.

Example 11. Differentibility and analyticity.

Remark 4. In the language of sheaves, $\mathcal{A}$ is a subsheaf of the sheaf of continuous functions.

Example 12. 1. Let $U \subset \mathbb{R}^n$ be open and C^∞ be the geometric structure $V \mapsto C^\infty(V)$, then (U, C^∞) is a geometric space.

2. Let $U \subset \mathbb{C}^m$ be open and θ be the geometric structure $V \mapsto \theta(V)$, where $\theta(V)$ is the set of holomorphic functions $V \to \mathbb{C}$, then (U, θ) is a geometric space.

3. Let X be a real (or complex) manifold with atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ for $U_\alpha \subset X$ and $V_\alpha \subset \mathbb{F}^n$. We define a geometric structure C^∞_X (or θ_X, respectively) as follows: For $U \subset X$ open, define

$$C^\infty_X(U) = \{f \in C(U) : (f\mid_{U\cap U_\alpha}) \circ \varphi_\alpha^{-1} : \varphi_\alpha(U \cap U_\alpha) \to \mathbb{R} \text{ smooth for any } U \cap U_\alpha \neq \varnothing\},$$
$$\theta_X(U) = \{f \in C(U) : (f\mid_{U\cap U_\alpha}) \circ \varphi_\alpha^{-1} : \varphi_\alpha(U \cap U_\alpha) \to \mathbb{C} \text{ holomorphic for any } U \cap U_\alpha \neq \varnothing\}.$$

A equivalent definition can be stated as follows: A smooth (or complex) manifold of $\mathbb{F}$-dimension n is a geometric structure $(X, \mathcal{A})$ in which every point $x \in X$ has a neighborhood U such that $(U, \mathcal{A}\mid_U) \simeq (\Omega, C^\infty_\Omega)$, where $\Omega \subset \mathbb{R}^n$ open, or such that, $(U, \mathcal{A}\mid_U) \simeq (\Omega, \theta_U)$, where $\Omega \subset \mathbb{C}^n$ open, respectively.

Definition 17. A **morphism** of geometric spaces $f : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$ is a continuous map $f : X \to Y$ with the property that if $g \in \mathcal{A}_Y(U)$ then $g \circ f \in \mathcal{A}_X(f^{-1}(U))$. From now on, we will write this map $f^* : \mathcal{A}_Y(U) \to \mathcal{A}_X(f^{-1}(U))$. f is further an **isomorphism** if there is an inverse morphism.

Example 13. 1. $f : (U, C^\infty_U) \to (V, C^\infty_V)$ is the same as $f \in C^\infty(U, V)$.

2. $f : (U, \theta_U) \to (V, \theta_V)$ is the same as $f \in \theta(U, V)$.

3. Let $U \subset X$ be open, then $(U, \mathcal{A}\mid_U) \to (X, \mathcal{A})$ is a morphism.

Definition 18. Given a geometric space $(X, \mathcal{A})$. Let $\xi \in X$ be a point. We define the **localization** of $\mathcal{A}$ to ξ as follows:

1. There is a ring of equivalence classes of functions associated to the point ξ, denoted as $\mathcal{A}_\xi$;

2. Each $[f] = [f]_\xi \in \mathcal{A}_\xi$ is represented by $f \in \mathcal{A}(U)$, where $U \subset X$ is open and $\xi \in U$;

3. Two representatives $f_1 \in \mathcal{A}(U_1)$, $f_2 \in \mathcal{A}(U_2)$ are equivalent, if there is an open W with $\xi \in W$ such that $f_1\mid_W = f_2\mid_W$.

We call $\mathcal{A}_\xi$ the **stalk** of $\mathcal{A}$ at ξ, and $[f]_\xi$ the **germ** of f at ξ.

Definition 19. A **derivation** of the algebra $\mathcal{A}_\xi$ is an $\mathbb{F}$-linear map $D : \mathcal{A}_\xi \to \mathbb{F}$ that satisfies the Leibniz rule at ξ, i.e.

$$D([f][g]) = f(\xi)D([g]) + D([f])g(\xi),$$

for any $[f], [g] \in \mathcal{A}_\xi$. The **real tangent space** of X at ξ is the $\mathbb{R}$-vector space of derivations of $C^\infty_{X,\xi}$ and the **complex tangent space** of X at ξ is the $\mathbb{C}$-vector space of derivations of $\theta_{X,\xi}$.

Example 14. 1. For $U \subset \mathbb{R}^n$ open, $D_j : C^\infty_\xi \to \mathbb{F}$, $D_j([f]) = \frac{\partial f}{\partial x_j}(\xi)$ is a derivation.

2. For $U \subset \mathbb{C}^n$ open, $D_j : \theta_\xi \to \mathbb{F}$, $D_j([f]) = \frac{\partial f}{\partial z_j}(\xi)$ is a derivation.

Remark 5. For a manifold, the stalk of $\mathcal{A}$ at ξ is the same as the stalk of $\mathcal{A}_U$ at ξ, for a coordinate chart U containing ξ.

Let $J_\xi \subset \mathcal{A}_\xi$ be the ideal of germs that vanish at ξ. Naturally,

$$J_\xi^2 = \{[f] \in \mathcal{A}_\xi : [f] = [g][h] \text{ for some } [g], [h] \in J_\xi\}.$$

The map $\mathcal{A}_\xi \to \mathcal{A}_\xi / J_\xi$ given by $[f] \mapsto f(\xi)$ is the **evaluation** at ξ. The map $J_\xi \to J_\xi / J_\xi^2$ for a manifold is the total derivation, or **exterior derivation**. Given $f \in \mathcal{A}(U)$, where $U \subset \mathbb{F}^n$, we can write

$$f(\omega) = f(\xi) + (\partial_{\omega_1} f)(\xi)(\omega_1 - \xi_1) + (\partial_{\omega_2} f)(\xi)(\omega_2 - \xi_2) + \cdots + (\partial_{\omega_n} f)(\xi)(\omega_n - \xi_n) + O((\omega - \xi)^2).$$

If $[f] \in J_\xi$, then $f(\xi) = 0$, while the reminder is in J_ξ^2. So $[f] \in J_\xi / J_\xi^2$, and $[f]$ is the class of j=1

$$[f] = [(\partial_{\omega_1} f)(\xi)(\omega_1 - \xi_1) + \cdots + (\partial_{\omega_n} f)(\xi)(\omega_n - \xi_n)] = \sum_{j=1}^{n} (\partial_{\omega_j} f)(\xi) \, [\omega_j - \xi_j] \, .$$

Denote the last one by

$$\sum_{j=1}^{n} (\partial_{\omega_j} f)(\xi) d\omega_j,$$

then $J_\xi / J_\xi^2 \cong \mathbb{F}^n$ with each choice of coordinates inducing a basis of J_ξ / J_ξ^2.

Lemma 2.

$$\mathrm{Der}(\mathcal{A}_\xi) \cong \left(J_\xi / J_\xi^2\right)^*,$$

where the right hand side is the annihilator of J_ξ^2 in J_ξ^*, and the left hand side is the collection of all derivations of $\mathcal{A}_\xi$.

Proof. ($\Longrightarrow$): Define the map $\mathcal{G} : \mathrm{Der}(\mathcal{A}_\xi) \to (J_\xi)^*$ by restriction. By Leibniz's rule, elements in the image will vanish on J_ξ^2.

($\Longleftarrow$): Given $\phi \in J_\xi^*$ with $\phi \mid_{J_\xi^2} \equiv 0$. Define $D \in \mathrm{Der}(\mathcal{A}_\xi)$ by $D([f]) = \phi([f] - [f(\xi)])$. Then by

$$\begin{aligned}
D([f][g]) &= \phi([f \cdot g] - [(f \cdot g)(\xi)]) \\
&= \phi([f - f(\xi)][g - g(\xi)] + [f(\xi)(g - g(\xi))] + [g(\xi)(f - f(\xi))]) \\
&= f(\xi)\phi([g - g(\xi)]) + g(\xi)\phi([f - f(\xi)]) \\
&= f(\xi)D([g]) + g(\xi)D([f]),
\end{aligned}$$

we are done. □

4.3.2 Geometric Approach

In Lecture 4.3.1, we actually defined T_pM at $p \in M$ to be the derivation of $\mathcal{A}_p$, which is independent of the choice of coordinate charts. The **tangent bundle** is then the collection of all tangent spaces: $TM = \cup_p T_pM$, and the derivation D is well-defined on tangent bundles. An alternative way of thinking about the tangent vectors to M (smooth or complex) at $p \in M$ is as an equivalence class of paths through p. For that, we first need the notation of a path.

Definition 20. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A **path** through $p \in M$ is a pair (U, γ) with $U \subset \mathbb{F}$ being a neighborhood of the origin, and $\gamma : U \to M$ with $\gamma(0) = p$ being a smooth map if $\mathbb{F} = \mathbb{R}$ and a holomorphic map if $\mathbb{F} = \mathbb{C}$. Two paths $(W, \gamma_1), (V, \gamma_2)$ through p are **equivalent** if there is a coordinate chart $\varphi : W \to V$, $p \in W$ such that

$$\partial_t \mid_{t=0} (\varphi \circ \gamma_1) = \partial_t \mid_{t=0} (\varphi \circ \gamma_2), \quad \mathbb{F} = \mathbb{R}$$
$$\partial_z \mid_{z=0} (\varphi \circ \gamma_1) = \partial_z \mid_{z=0} (\varphi \circ \gamma_2), \quad \mathbb{F} = \mathbb{C}$$

Now define the equivalence class of paths through p to be the tangent vector at p.

Exercise 2. Check the definition of equivalence class of paths is well-defined, i.e. it is not determined by the choice of charts.

Given a smooth (or holomorphic) map $F : M \to N$ between smooth (or complex) manifolds. Define its derivation, a smooth (or holomorphic) map $DF : TM \to TN$, by

$$\begin{aligned} D_pF : T_pM &\to T_{f(p)}N \\ [\gamma] &\mapsto [F \circ \gamma]. \end{aligned}$$

Exercise 3. Prove that the tangent space $T_p(-)$ is functorial for every $p \in M$.

Definition 21. A **section** of the tangent bundle is a map $M \to TM$ that assigns to each point a tangent vector at that point. Sections of the tangent bundle are called **vector fields**.

Example 15. Let $U \subset M$ be an open set and $V \in C^\infty(U, TM \mid_U)$. One can think of V as assigning to each $p \in U$ a derivation of $C^\infty_{M,p}$ (or $\theta_{M,p}$). Put these together, we obtain the next important definition.

Definition 22. The **Lie derivative** of the vector field V is

$$\mathscr{L}_V : \mathcal{A}(U) \to \mathcal{A}(U),$$

such that $\mathscr{L}_V(f)(\xi) = V(\xi)[f]_\xi$. This define a derivation of $\mathcal{A}(U)$ with Leibniz's rule

$$\mathscr{L}_V(fg) = f\mathscr{L}_V(g) + g\mathscr{L}_V(f),$$

and every derivation of $\mathcal{A}(U)$ arises in this way.

Definition 23. Given vector fields V, W, the **Lie bracket** $[V, W] : \mathcal{A}(U) \to \mathcal{A}(U)$ of them is given by

$$[V, W](f) = \mathscr{L}_V(\mathscr{L}_W(f)) - \mathscr{L}_W(\mathscr{L}_V(f)).$$

Exercise 4. Prove that the Lie bracket is a derivation. Hence the Lie bracket is a Lie derivative of a vector field.

In local coordinates, if $V = \sum a_j \partial_{\xi_j}$, $W = \sum b_j \partial_{\xi_j}$, then

$$[V, W] = \sum_j \sum_k (a_k \partial_{\xi_k} b_j - b_k \partial_{\xi_k} a_j) \partial_{\xi_j}$$

4.4 Covariant Derivatives

Definition 24. Given a smooth (or holomorphic) map $F : M \to N$ between smooth (or holomorphic) manifolds and a smooth (or holomorphic) vector bundle $E \xrightarrow{\pi} N$. We can obtain a smooth (or holomorphic) vector bundle $F^*E \to M$ along the diagram

$$\begin{array}{ccc} F^*E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ M & \xrightarrow{F} & N \end{array}$$

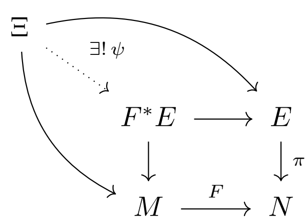
This bundle is called the pullback of $E \xrightarrow{\pi} N$, or a **pullback bundle** in short. In fact, we can write

$$F^*E = \{(m, v) \in M \times E : F(m) = \pi(v)\}$$

This "pullback" lives up to its name. It satisfies the universal property of pullback in category of corresponding manifolds. Namely, given the pullback bundle

$$\begin{array}{ccc} \Xi & \longrightarrow & E \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

there exists a unique bundle morphism $\psi : \Xi \to F^*E$ such that the following diagram commutes:



Note that the pullback of a trivial bundle is the trivial bundle. So we can obtain local trivialization of F^*E by pulling back local trivialization of E.

A vector bundle $E \to [0, 1] \times M$ gives us a family of vector bundles over M, namely $j_t^*E \to M$, where $j_t : M \hookrightarrow [0, 1] \times M$ being the inclusion into $(t, m) \in [0, 1] \times M$. We want to know if these are all isomorphic.

Note 1. This is not true for holomorphic bundles in general. (Why?)

In order to find the answer, we first need the tool called partition of unity.

Definition 25. Let M be a smooth manifold and $\{U_\alpha\}$ be an open cover of M. A **partition of unity** (subordinate to $\{U_\alpha\}$) is a collection of smooth functions $\{x_i\}$ such that

1. Each $x_i : M \to [0, 1]$ has compact support contained in some U_α.
2. For each $\xi \in M$, there are only finitely many nonzern x_i at ξ.
3. $\sum_i x_i(\xi) = 1$ for every $\xi \in M$.

Lemma 3. For every smooth manifold M, and any open cover $\{U_\alpha\}$ of M, there is a partition of unity subordinate to $\{U_\alpha\}$.

Proof. Exercise. Also, does there exist any similar results holding for holomorphic manifolds? □

Recall that given a $\mathbb{F}$-vector bundle $\xi \to M$, there is a dual vector bundle $\xi^* = \mathrm{Hom}(\xi, \mathbb{F}) \to M$. The dual of the tangent bundle is called the **cotangent bundle**, denoted by $T^*M \to M$. At each point $\xi \in M$, elements of $T^*_\xi M$ are known as **covectors**. Sections of T^*M are known as **1-forms**, and sections of $\bigwedge^k T^*M$ are known as k**-forms**.

Note 2. We use the notation $\Omega^1(M) := C^\infty(M, T^*M)$ to denote the set of 1-forms, and $\Omega^k(M) := C^\infty(M, \bigwedge^k T^*M)$ to denote the set of k-forms. When $k = 0$, 0-forms are the same as functions on M, i.e. $\Omega^0(M) = C^\infty(M)$.

Now thinking of M as a geometric space $(M, \mathcal{A})$. Then $T^*_\xi M = J_\xi / J_\xi^2$ is the stalk of functions vanishing at ξ modulo stalks vanishing to second order at ξ. If $U \subset M$ is open, then any $f \in \mathcal{A}(U)$ defines a section of $T^*M\mid_U$, denoted by $df \in \Omega^1(U)$, $df(\xi) = [f - f(\xi)]_\xi \in J_\xi/J_\xi^2$. In local coordinates $x_1, \cdots, x_n$, dx_i's form a local frame for $T^*M\mid_U$, and

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$

If $V \in C^\infty(M, TM)$ is a vector field, then $df(V) = \mathscr{L}_V(f) \in C^\infty(M)$. However, considering a section s of a vector bundle $E \to M$, there is no canonical way to differentiate s. So we need to construct a "covariant derivative" ∇ which, for each of vector field $V \in C^\infty(M, TM)$, gives us a $\mathbb{F}$-linear map $\nabla_V : C^\infty(M, E) \to C^\infty(M, E)$. Alternately, we ought to find ∇ satisfying

- $\nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E) = \Omega^1(M, E)$ with Leibniz's rule

$$\nabla_V(fs) = \mathscr{L}_V(f)s + f\nabla_V s \tag{1}$$

  Or:

- $$\nabla(fs) = df \otimes s + f\nabla s \in \Omega^1(M, E). \tag{2}$$

To construct such covariant derivatives, first cover the manifold M with coordinate charts that trivialize E and let $\{x_i\}$ be a subordinate partition of unity. On the support of each x_i, E is trivial. Thus it is natural that we choose to work with $U \times \mathbb{F}^k \to U$, and a section is a smooth map $s : U \to \mathbb{F}^k$ i.e. a vector with k smooth functions as its entries. We could take ds by its first differentiation for each entry. More generally, we could take $(d + A_j)(s)$, where A_j is a function associated to each vector field V (resulting in a family of $k \times k$-matrices $A_j(V)$). One can see

$$\begin{aligned} A_j &\in C^\infty(U, T^*M\mid_U \otimes M_{k\times k}(\mathbb{F})) \\ &= C^\infty(U, T^*M\mid_U \otimes \mathrm{End}(E\mid_U)) \\ &= \Omega^1(U, \mathrm{End}(E)\mid_U). \end{aligned}$$

The Leibniz's rule is satisfied by $d + A_j$ (Check!) and this is the most general solution.

Choosing $d + A_j$ on the support of x_j for each j, we can put them together and define

$$\nabla_V : C^\infty(M, E) \to C^\infty(M, E)$$

by

$$s \mapsto \sum (d + A_j)(x_j \cdot s). \tag{3}$$

Exercise 5. Check that the definition (3) satisfies properties (1) and (2).

If ∇ and ∇' are two covariant derivations, then $\nabla - \nabla' \in \Omega^1(M, \mathrm{End}(E))$ because

$$(\nabla - \nabla')(fs) = (df) \otimes s + f\nabla s - ((df) \otimes s + f\nabla' s) = f(\nabla - \nabla')s,$$

which is in $\Omega^1(M, \mathrm{End}(E))$.

4.5 Parallel Transport

Given a covariant derivative ∇ on sections of a vector bundle $E \to N$ and a map between base spaces $F : M \to N$, we have a pullback covariant derivative $F^*\nabla$ on the pullback bundle $F^*E \to M$ determined by $(F^*\nabla)_V(F^*s) = \nabla_{DF(V)}s$. In particular, if $E \to M$ is a vector bundle and $\gamma : [0,1]_t \to M$ is a path on M, then any covariant derivative ∇ on sections of E induces a covariant derivative on $F = \gamma^*E \to [0,1]_t$. We can use this to construct an isomorphism $\mathscr{P}_\gamma$ from the fiber over 0 to the fiber over 1. This is called the **parallel transport**. A section s of F is called **parallel** if $\nabla_{\partial_k} s = 0$.

Lemma 4. Let $\mathrm{Par}(F)$ denote the vector space of parallel sections of F. Evaluation at $t = 0$, i.e. $\mathrm{Par}(F) \to F_0$ sending s to $s(0)$ is an isomorphism of $\mathbb{F}$-vector spaces.

To prove the lemma, we need to recall the fundamental theorem of ODEs:

Theorem 14 (Uniqueness and Existence Theorem). The initial value problem

$$\begin{cases} \frac{\partial y}{\partial t} = F(t, y) & \text{on } (a, b) \\ y(0) = y_0 \end{cases}$$

where F is continuous with respect to t and Lipschitz with respect to y (in some region containing (a, b)). Then there exists a unique solution to this problem on a smaller open set, which depends continuously on y_0.

We will not prove this theorem and the reader can find it in any books of ODEs. Now we are ready to prove the previous lemma.

Proof of Lemma 4. Assume F is trivializable, and pick a local frame $(e_1, \cdots, e_n)$ for it. Define A_j^k :

$[0,1] \to \mathbb{F}$ by $\nabla_{\partial_k} e_j = \sum A_j^k e_k$. Then a section $s = \sum f^j e_j$ is parallel iff

$$\begin{aligned} 0 = \ & \nabla_{\partial_k} s = \nabla_{\partial_k} \sum f^j e_j \\ = & \sum [(f^j \nabla_{\partial_k} e_j) + e_j(\partial_t f^j)] \\ = & \sum (\partial_t f^j) e_j + f^j \sum A_j^k e_k \\ = & \sum_{j,\ell} (\partial_t f^j + f^\ell A_\ell^j) e_j. \end{aligned}$$

Hence s is parallel iff $(\partial_t f^j + f^\ell A_\ell^j) = 0$ for any j. By Theorem 14, regarding the equation as a first order ODE of f^j with respect to t, we are done with this case. In general, using local triviality and compactness of $[0,1]$, we can find $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = 1$ such that $F\mid_{[t_i,t_{i+1}]}$ is trivializable (over corresponding interval). Composing the parallel transport gives the conclusion. □

Theorem 15. Let $E \xrightarrow{\pi} [0,1] \times M$ be a vector bundle and let $j_t : M \to [0,1] \times M$ be the inclusion of $\{t\} \times M$, then $j_0^* E \simeq j_1^* E$.

We present two ways to prove this theorem.

*Proof of Theorem 15, **Method I**.* Let ∇ be a covariant derivative on sections of $E \to [0,1] \times M$. Use parallel transport along the family of paths $[0,1] \times \{\xi\}$ for a chosen $\xi \in M$. Since the coefficients A_j^k of the ODEs vary smoothly with respect to ξ, so do the solutions of the ODEs. That is to say, the various isomorphisms of the fibers over $\{0\} \times M$ and $\{1\} \times M$ fit together into a vector bundle isomorphism. □

*Proof of Theorem 15, **Method II**.* This method is in a more topological view. Here we add an assumption that M is compact. Start with two observations:

1. Any trivialization $U \times \mathbb{F}^k \to U$ gives isomorphisms between the fibers $E_p \to E_q$, varying smoothly with respect to $p, q \in U$.

2. If we have trivializations $\varphi_j : \pi^{-1}(U \times (a_j, b_j)) \to U \times (a_j, b_j) \times \mathbb{F}^k$ with $a_1 < a_2 < b_1 < b_2$, $j = 1, 2$, then there is a trivialization $\psi : \pi^{-1}(U \times (a_1, b_2)) \to U \times (a_1, b_2) \times \mathbb{F}^k$. Indeed, $\varphi_2(u,t) = G(u,t)\varphi_1(u,t)$, where $G(u,t) \in \mathrm{GL}_k(\mathbb{F})$ for any $(u,t) \in U \times (a_2, b_1)$. Define $\Phi(u,t) = \tilde{G}(u,t)\varphi_1(u,t)$, where
$$\tilde{G}(u,t) = \begin{cases} G(u,t) & \text{if } t > b_1 - \epsilon \\ \pm \operatorname{id} & \text{if } t < a_2 + \epsilon \end{cases}$$

Cover $[0,1] \times M$ with open sets over which E is trivial and use compactness to find a finite subcover. Through the second observation we can find a finite over $\{U_j\}$ of M such that $E\mid_{[0,1]\times U_j}$ is trivializable. From the first observation, we have continuous (actually smooth) isomorphisms $E\mid_{\{a\}\times U_j} \simeq E\mid_{\{b\}\times U_j}$ for any $a, b \in [0,1]$.

Choose a partition of unity $\{x_j\}$ subordinate to $\{U_\alpha\}$, and define $\phi_0 = 0$, $\phi_\ell = x_1 + \cdots + x_\ell$ for any $\ell \in \mathrm{N}$. Let $\Gamma_\ell \subset [0,1] \times M$ be the graph of ϕ_ℓ. The trivialization on $[0,1] \times U_\ell$ gives an isomorphism

$$\rho_\ell : E\mid_{\Gamma_{\ell-1}} \xrightarrow{\simeq} E\mid_{\Gamma_\ell}.$$

The composition $\cdots \circ \rho_2 \circ \rho_1$ is well-defined by the local finiteness of $\{x_j\}$. This gives the desired isomorphism. □

Corollary 2. Given a vector bundle $E \to N$ and maps $f_0, f_1 : M \to N$. If f_0 is homotopic to f_1, then $f_0^*E \simeq f_1^*E$.

Proof. Let $H : [0,1] \times M \to N$ be the homotopy between f_0 and f_1 such that $H\,|_{\{0\}\times M} = f_0$, $H\,|_{\{1\}\times M} = f_1$. Let $j_t : M \times [0,1] \to M$ be the inclusion of $M \times \{t\}$ into $M \times [0,1]$. Then

$$f_0^*E = (H \circ j_0)^*E = j_0^*H^*E \simeq j_1^*H^*E = (H \circ j_1)^*E = f_1^*E.$$

□

This lead to the following direct corollary.

Corollary 3. If M is contractible, then every smooth bundle over M is smoothly trivializable.

4.6 Line Bundle

Consider the $\mathbb{C}$-line bundles over $\mathbb{C}P^1 \simeq S^2$. We can choose two charts on S^2, namely $S^2\backslash\{\mathrm{N}\} \simeq \mathbb{C}$ and $S^2\backslash\{\mathrm{S}\} \simeq \mathbb{C}$. There is a transition map $\mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$, sending z to $\frac{1}{z}$. The standard atlas on $\mathbb{C}P^1$ is similar. We have $U_1 = \{(z_1 : z_2) \in \mathbb{C}P^1 : z_1 \neq 0\} \simeq \mathbb{C}$ (ismorphism through $(z_1 : z_2) \mapsto \frac{Z_2}{z_1}$, similar for the second chart), and $U_1 = \{(z_1 : z_2) \in \mathbb{C}P^1 : z_2 \neq 0\} \simeq \mathbb{C}$. There is a transition map $\omega \mapsto \frac{1}{\omega}$. Since a $\mathbb{C}$-line bundle will be trivial over these charts, a rank k bundle will be determined by a map from the overlap to $\mathrm{GL}_k(\mathbb{C})$.

In particular, this is true for the tangent bundle. $T\mathbb{C}P^1$ is trivial on each of these charts and its transition map

$$\begin{aligned} \mathbb{C}\backslash\{0\} \times \mathbb{C} &\to \mathbb{C}\backslash\{0\} \times \mathbb{C} \\ (z, v) &\mapsto \left(-\frac{1}{z}, \frac{1}{z^2}v\right). \end{aligned}$$

This line bundle is denoted $\mathcal{O}_{\mathbb{C}P^1}(2)$. For all $k \in \mathbb{Z}$, we denote by $\mathcal{O}_{\mathbb{C}P^1}(k)$ the line bundle with transition map

$$(z, v) \mapsto \left(-\frac{1}{z}, \frac{1}{z^k}v\right).$$

These form a group $\mathscr{G}(\mathbb{C}P^1)$ with $\otimes$ as the operation: $\mathcal{O}_{\mathbb{C}P^1}(k) \otimes \mathcal{O}_{\mathbb{C}P^1}(\ell) = \mathcal{O}_{\mathbb{C}P^1}(k+\ell)$.

Consider a holomorphic section s of $\mathcal{O}_{\mathbb{C}P^1}(k)$. Let $(z_+, s_+(z_+))$ and $(z_-, s_-(z_-))$ denote the representations of s in the coordinate charts. Since s is holomorphic, we have

$$s_+(z_+) = \sum_{j=0}^{\infty} a_{+,j} z_+^j.$$

Writing this in the other coordinates, it becomes

$$\varphi(z_+, s_+(z_+)) = \left(-z_+^{-1}, \sum_{j=0}^{\infty} a_{+,j} z_+^{j-k}\right).$$

So the function $z_- \to \sum_{j\geq 0} a_{+,j} z_-^{k-j}$ needs to be analytic, implying $a_{+,j} = 0$ if $j > k$. Hence we conclude that the $\mathbb{C}$-dimension of the space of holomorphic section of $\mathcal{O}_{\mathbb{C}P^1}(k)$ is $k+1$ for $k \geq 0$ and 0 for $k < 0$.

For $k = 0$, $\mathcal{O}_{\mathbb{C}P^1}(0)$ is the trivial line bundle $\underline{\mathbb{C}} : \mathbb{C}P^1 \times \mathbb{C} \to \mathbb{C}P^1$, so a holomorphic section is just a holomorphic function $\mathbb{C}P^1 \to \mathbb{C}$, hence a constant function by Liouville's theorem.

Definition 26. For each point $[z] \in \mathbb{C}P^1$, let $\ell_{[z]}$ be the $\mathbb{C}$-line in $\mathbb{C}^2$ represented by this point. Define

$$L = \{([z], v) \in \mathbb{C}P^1 \times \mathbb{C}^2 : v \in \ell_{[z]}\} \xrightarrow{\pi} \mathbb{C}P^1.$$

This is a line bundle. (Check!) We denote it by $\mathcal{O}_{\mathbb{C}P^1}(-1)$, and call it a **tautological line bundle** over $\mathbb{C}P^1$.

It is worth considering the other projection $L \xrightarrow{\beta} \mathbb{C}^2$ sending $([z], v)$ to v. β maps the "zero section" to the origin $\mathbf{0} \in \mathbb{C}^2$ and maps the complement biholomorphically onto $\mathbb{C}^2 \backslash \{0\}$. Thus the total space of L is obtained from $\mathbb{C}^2$ by removing the origin and gluing in a $\mathbb{C}P^1$. The geometric effect is to give each line through the origin its own distinct origin. The total space of L is called the **blow-up** of $\mathbb{C}^2$ at the origin.

Proposition 3. On $\mathbb{C}P^n$ with $\{U_\alpha\}$ the standard atlas, the line bundle $\mathcal{O}_{\mathbb{C}P^n}(k)$ is defined by the transition functions $g_{\alpha_k}([z]) = \left(\frac{z_\beta}{z_\alpha}\right)^k$.

Tautological bundles are more interesting over the Grassmannian.

Definition 27. The **Grassmannian**, or **Grassmann manifold**, $\mathrm{Gr}_n(\mathbb{F}^{n+k})$ is the set of n-dimensional subspaces in $\mathbb{F}^{n+k}$. The collection $V_n(\mathbb{F}^{n+k}) = \{(v_1, \cdots, v_n) \in (\mathbb{F}^{n+k})^n : (v_1, \cdots, v_n) \text{ orthonormal}\}$ is an open subset of $(\mathbb{F}^{n+k})^n$, called the **Stiefel manifold**, where $(v_1, \cdots, v_n)$ is called an ***n*-frame**.

Remark 6. The terminology "manifold" lives up to its name. In fact, Stiefel manifold $V_n(\mathbb{F}^{n+k})$ can actually be viewed as $\{M \in \mathbb{F}^{n+k} : M^*M = I_n\}$, where M^* is the conjugate transpose of M. It can be topologized as a subspace of the product of n copies of the unit sphere in $\mathbb{F}^{n+k}$, and there is a natural surjection $p : V_n(\mathbb{F}^{n+k}) \to \mathrm{Gr}_n(\mathbb{F}^{n+k})$ sending an n-frame to the subspace it spans. This topologizes the Grassmannian as a quotient space of $V_n(\mathbb{F}^{n+k})$ via p. Moreover, $V_n(\mathbb{F}^{n+k})$ is closed and bounded, thus compact.

To see these two "manifolds" are truly manifolds, first note that these two topological spaces are Hausdorff and second countable since they are the subspaces of some Hausdorff and second countable spaces. Consider the map

$$p : V_n(\mathbb{F}^{n+k}) \to \mathrm{Gr}_n(\mathbb{F}^{n+k})$$
$$(v_1, \cdots, v_n) \mapsto \mathrm{Span}\{v_1, \cdots, v_n\}$$

Any n-space W in the neighborhood U of $\mathrm{Gr}_n(\mathbb{F}^{n+k})$ can be uniquely represented by graph of function $W \mapsto W^\perp$, so one can find a one-to-one correspondence $U \mapsto \mathbb{R}^{nk}$, which implies a natural atlas on $\mathrm{Gr}_n(\mathbb{F}^{n+k})$. Pulling back along p, one obtains the atlas for $V_n(\mathbb{F}^{n+k})$.

Another way of seeing Stiefel manifolds is as follows: consider the function $f : \mathbb{F}^{nk} \to \mathrm{Sys}(n)$ sending A to $A^*A - I_n$, where $\mathrm{Sys}(n)$ is the set of symmetric $n \times n$-matrices, which is a vector space. One can show that f is a submersion. Then $V_n(\mathbb{F}^{n+k}) = f^{-1}(\mathbf{0}_n)$, an embedded submanifold of $\mathbb{R}^{nk}$.

Proposition 4. If $\mathbb{F} = \mathbb{R}$, then

$$V_n(\mathbb{R}^{n+k}) \cong \frac{O(n+k)}{O(k)}, \qquad \mathrm{Gr}_n(\mathbb{R}^{n+k}) \cong \frac{O(n+k)}{O(n) \times O(k)};$$

if $\mathbb{F} = \mathbb{C}$, then

$$V_n(\mathbb{C}^{n+k}) \cong \frac{U(n+k)}{U(k)}, \qquad \mathrm{Gr}_n(\mathbb{C}^{n+k}) \cong \frac{U(n+k)}{U(n) \times U(k)}.$$

Let $\gamma^n(\mathbb{F}^{n+k}) \xrightarrow{\pi} \mathrm{Gr}_n(\mathbb{F}^{n+k})$ be a tautological bundle over a Grassmannian, $\gamma^n(\mathbb{F}^{n+k}) = \{(W, v) \in \mathrm{Gr}_n(\mathbb{F}^{n+k}) \times \mathbb{F}^{n+k} : v \in W\}$. Choose $W_0 \in \mathrm{Gr}_n(\mathbb{F}^{n+k})$ and let $p_0 : \mathbb{F}^{n+k} \to W_0$ be the projection. Write $U_0 = \{W \in \mathrm{Gr}_n(\mathbb{F}^{n+k}) : p_0(W) = W_0\}$. Then $W \in U_0$ is the graph of a linear map $T_W : W_0 \mapsto W_0^\perp$, and $T : U_0 \to \mathrm{Hom}(W_0, W_0^\perp) \cong \mathbb{F}^{nk}$ is a coordinate chart. γ^n is trivial in each of these charts

$$\begin{aligned} h : \pi^{-1}(U_0) &\to U_0 \times \mathbb{F}^n \\ (W, v) &\mapsto (W, p_0(v)). \end{aligned}$$

Consider the tangent bundle of a manifold M embedded in $\mathbb{F}^N$. If $\dim_{\mathbb{F}} M = n$, at each point $p \in M$, we have $T_pM \subset T_p\mathbb{F}^N = \{p\} \times \mathbb{F}^N$. So $TM = \cup_p (p, T_pM) \subset \mathbb{F}^N \times \mathbb{F}^N$ is the graph of a map $M \to \mathrm{Gr}_n(\mathbb{F}^N)$. The pullback of $\gamma^n(\mathbb{F}^N) \to \mathrm{Gr}_n(\mathbb{F}^{n+k})$ to M gives a bundle isomorphic to $TM \to M$. This phenomenon can be generalized in the following sense.

Theorem 16. Let M be a compact $\mathbb{F}$-manifold and $E \to M$ be a rank n $\mathbb{F}$-bundle. For large $k > 0$, there is a vector bundle map $F : E \to \gamma^n(\mathbb{F}^{n+k})$ that is an isomorphism on fibers. Hence, E is the pullback of tautological bundle $\gamma^n(\mathbb{F}^{n+k}) \xrightarrow{\overline{F}} \mathrm{Gr}_n(\mathbb{F}^{n+k})$, i.e. the diagram below commutes.

$$\begin{array}{ccc} E & \xrightarrow{F} & \gamma^n(\mathbb{F}^{n+k}) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\overline{F}} & \mathrm{Gr}_n(\mathbb{F}^{n+k}) \end{array}$$

Proof. It is sufficient to construct a map $\hat{F} : E \to \mathbb{F}^m$ for some m which is linearly injective on each fiber of $E \to M$. Having $\hat{F}$ in hand, we can take $F(v) = (\hat{F}(\mathscr{F}_v), \hat{F}(v))$, where $\mathscr{F}_v$ is the fibers through v. F is well-defined, following from the local triviality of E. Similar to the tangent bundle case we discussed above, we are done with the proof.

It remains to construct the map $\hat{F}$. First choose finite cover $\{U_j\}_{1 \le j \le r}$ of M, trivializing E. Choose another covers $\{V_j\}$, $\{W_j\}$ of M, $1 \le j \le r$, such that $\overline{V_j} \subset U_j$ and $\overline{W_j} \subset V_j$. Let $\lambda_j : M \to \mathbb{R}$ denote a smooth function equal to 1 on $\overline{W_j}$ and equal to 0 outside V_j. Since $E\mid_{U_j}$ is trivial, there is a map $h_j : \pi^{-1}(U_j) \to \mathbb{F}^n$ linear on the fibers. Define $h'_j : E \to \mathbb{F}^n$ by

$$h'_j(v) = \begin{cases} 0 & \text{if } \pi(v) \notin V_j \\ \lambda_j(\pi(v)) h_j(v) & \text{if } \pi(v) \in U_j \end{cases}$$

Clearly h'_j is linear on fibers. Define $\hat{F} : E \to (\mathbb{F}^n)^r$ by $\hat{F}(v) = (h'_1(v), h'_2(v), \cdots, h'_r(v))$. This $\hat{F}$ is the desired one. □

Corollary 4. Every line bundle $E \to M$ is the pullback of the tautological line bundle over $\mathbb{F}P^N$ for some N.

Note that as soon as we find some k that works in Theorem 16, then any $K > k$ will also work in the same setting. This is true because we have the inclusions $\mathbb{F}^{n+k} \cong \mathbb{F}^{n+k} \times \{0\} \hookrightarrow \mathbb{F}^{n+k+1} \hookrightarrow \cdots$, inducing inclusions $\mathrm{Gr}_n(\mathbb{F}^{n+k}) \hookrightarrow \mathrm{Gr}_n(\mathbb{F}^{n+k+1}) \hookrightarrow \cdots$. The tautological bundle $\gamma^n(\mathbb{F}^{n+k+1})$ then restricts to $\gamma^n(\mathbb{F}^{n+k})$, i.e.

$$\begin{array}{ccc} \gamma^n(\mathbb{F}^{n+k}) & \longrightarrow & \gamma^n(\mathbb{F}^{n+k+1}) \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Gr}_n(\mathbb{F}^{n+k}) & \longrightarrow & \mathrm{Gr}_n(\mathbb{F}^{n+k+1}) \end{array}$$

as a pullback bundle. To get a single n-bundle that works for all situations, we can take the direct limit

$$\mathbb{F}^\infty := \varinjlim_k \mathbb{F}^{n+k} = \bigcup_k \mathbb{F}^{n+k} \Big/ \sim,$$

where "$\sim$" is given by $\mathbb{F}^s \ni u \sim v \in \mathbb{F}^{s'}$ iff $(u, 0) = (v, 0)$ in $\mathbb{F}^N$ for some $N > 0$. Similarly we define

$$\mathrm{Gr}_n := \mathrm{Gr}_n(\mathbb{F}^\infty) = \varinjlim_k \mathrm{Gr}_n(\mathbb{F}^{n+k}),$$
$$\gamma^n := \gamma^n(\mathbb{F}^\infty) = \varinjlim_k \gamma^n(\mathbb{F}^{n+k}).$$

Since the maps $\gamma^n(\mathbb{F}^{n+k}) \to \mathrm{Gr}_n(\mathbb{F}^{n+k})$ are compatible with the maps $\mathrm{Gr}_n(\mathbb{F}^{n+k}) \to \mathrm{Gr}_n(\mathbb{F}^{n+k+1})$ and the maps $\gamma^n(\mathbb{F}^{n+k}) \to \gamma^n(\mathbb{F}^{n+k+1})$, they induce a map $\gamma^n \to \mathrm{Gr}_n$. One can check this is locally trivial. The bundle $\gamma^n \to \mathrm{Gr}_n$ is the **universal bundle**. Every vector bundle $E \to M$ of $\mathbb{F}$-rank n is the pullback of $\gamma^n \to \mathrm{Gr}_n$ along some map $M \to \mathrm{Gr}_n$, and that map is called the **classifying map** of E.

Note 3. In Theorem 16, we ask the manifold M to be compact since we need the finite covers of M. However, this restriction does **NOT** apply once we pass to the universal bundle. Even if M is not compact, we can find a countable cover of M by local trivialization of E with each point of M contained in only finitely many of the open sets. The proof of Theorem 16 still works but with $\hat{F}$ mapping into $\mathbb{F}^\infty$, one copy for each of the open sets in the cover.

Proposition 5. Any two classifying maps of $E \to M$ are homotopic.

Proof. As we discussed above, the bundle map $E \xrightarrow{F} \gamma^n$ is equivalent to a map $\hat{F} : E \to \mathbb{F}^\infty$ whose restriction to each fiber is injective and linear. Let $F, G : E \to \gamma^n$ be two bundle maps, fiberwise isomorphisms.

Case 1 :

Assume that for each nonzero vector $v \in E$, $\hat{F}(v)$ is never equal to a negative multiple of $\hat{G}(v)$. Then $\hat{H}_t(v) = (1-t)\hat{F}(v) + t\hat{G}(v)$, $t \in [0, 1]$ is a continuous homotopy between them with $\hat{H}_t$ injective and linear on fibers.

Case 2 :

Consider the bundle maps $s_o, s_e : \gamma^n \to \gamma^n$, sending (W, v) to $(W, \mathscr{L}_o(v))$ and $(W, \mathscr{L}_e(v))$, respectively, where

$$\mathscr{L}_o(a, b, c, d, \cdots) = (a, 0, b, 0, c, 0, d, 0, \cdots),$$
$$\mathscr{L}_e(a, b, c, d, \cdots) = (0, a, 0, b, 0, c, 0, d, \cdots).$$

Now $\hat{F}(v)$ and $s_o \circ \hat{F}(v)$ are always homotopic, so do $s_o \circ \hat{F}(v)$ and $s_e \circ \hat{G}(v)$, $s_e \circ \hat{G}(v)$ and $\hat{G}(v)$. Therefore, we can find $\hat{F} \sim s_o \circ \hat{F} \sim s_e \circ \hat{G} \sim \hat{G}$ by Case 1.

□

4.7 Almost Complex Structures

Notice that a complex vector space is the same as a real vector space together with a "complex structure" in the form of an endomorphism J such that $J^2 = -\operatorname{id}$. Indeed, given J, we can define $(a+ib)v = av + bJ(v)$, making $\mathbb{R}$-vector spaces the $\mathbb{C}$-vector spaces. If we start with a $\mathbb{C}$-vector space, then multiplication by i is an $\mathbb{R}$-linear endomorphism of the underlying $\mathbb{R}$-vector space that squares to $-\operatorname{id}$. In general, if we have V which is a $\mathbb{R}$-vector space of dimension $2n$ with a $\mathbb{C}$-structure J, then $V \otimes \mathbb{C}$ is a $\mathbb{C}$-vector space of dimension $2n$ with two complex structures:

1. J extends to $V \otimes \mathbb{C}$ by $J(v \otimes \alpha) = J(v) \otimes \alpha$
2. i from the $\mathbb{C}$-factor acts on $V \otimes \mathbb{C}$ by $i(v \otimes \alpha) = v \otimes i\alpha$.

Since $J^2 = -\operatorname{id}$, J is diagonalizable with two eigenvalues i and $-i$. Denote the eigenspaces by $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$, where

$$V^{1,0} = \{\omega \in V \otimes \mathbb{C} : J(\omega) = i\omega\},$$
$$V^{0,1} = \{\omega \in V \otimes \mathbb{C} : J(\omega) = -i\omega\}.$$

Thus $V^{1,0}$ is the subspace where the two complex structures coincide and $V^{0,1}$ is the subspace where they don't.

Notice that we can define conjugation on $V \otimes \mathbb{C}$ by $\overline{v \otimes \alpha} = v \otimes \overline{\alpha}$. Then $\overline{V^{1,0}} = \overline{V^{0,1}}$. So

$$\dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1} = \frac{1}{2} \dim_{\mathbb{C}} V \otimes \mathbb{C} = n.$$

We have an isomorphism of $\mathbb{C}$-vector spaces:

$$(V, J) \xrightarrow{\simeq} (V^{1,0}, i)$$
$$v \mapsto \frac{1}{2}(v - iJv).$$

Example 16. If we start with $\mathbb{C}^n = \{(\omega_1, \cdots, \omega_n) : \omega_j \in \mathbb{C}\}$ and decompose $\omega_j = a_j + ib_j$ with $a_j, b_j \in \mathbb{R}$, then we have the natural identification with $\mathbb{R}^{2n} = \{(a_1, b_1, \cdots, a_n, b_n) : a_j, b_j \in \mathbb{R}\}$. Scalar multiplication by i in $\mathbb{C}^n$ induces the complex structure $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ sending $(a_1, b_1, \cdots, a_n, b_n)$ to $(-b_1, a_1, \cdots, -b_n, a_n)$. Now consider the complexification $\mathbb{R}^{2n} \otimes \mathbb{C} \simeq \mathbb{C}^{2n}$. The standard basis of $\mathbb{R}^{2n} : (x_1, y_1, \cdots, x_n, y_n)$ induces the standard basis of $\mathbb{C}^{2n} : (X_1, Y_1, \cdots, X_n, Y_n)$, where $X_j = x_j \otimes 1$ and $Y_j = y_j \otimes 1$. J extends to $\mathbb{C}^{2n}$: $J(X_k) = Y_k$, $J(Y_k) = -X_k$. Hence its i-eigenspace is $\mathrm{Span}\{X_k - iY_k (= x_k \otimes 1 - y_k \otimes i)\}$, and its $(-i)$-eigenspace is $\mathrm{Span}\{X_k + iY_k (= x_k \otimes 1 + y_k \otimes i)\}$. Note $\overline{X_k - iY_k} = X_k + iY_k$, so these two eigenspaces are conjugate. Every $\mathbb{R}$-vector space of even dimension can be given a complex structure.

Definition 28. An **almost complex structure** on a smooth manifold M is a vector bundle isomorphism $J : TM \to TM$ such that $J^2 = -\,\mathrm{id}$.

This turns TM into a $\mathbb{C}$-vector bundle, but does not turn M into a $\mathbb{C}$-manifold. Clearly if M admits an almost $\mathbb{C}$-structure, it must be even dimension and orientable.

Theorem 17 (Borel-Serre)**.** The only spheres that admit an almost complex structure are S^2 and S^6.

Definition 29. A map between smooth manifolds with almost complex structures $F : (M, J) \to (M', J')$ is called **almost complex**, or **pseudo-holomophic**, if $DF \circ F = J' \circ DF$.

Proposition 6. If $U \subset \mathbb{C}^m$, $V \subset \mathbb{C}^n$ are open sets, a map $F : U \to V$ is pseudo-holomophic iff it is holomorphic.

In particular, a complex manifold M induces an almost complex structure on its underlying smooth manifold, i.e. a complex structure implies an almost complex structure. We say that an almost complex structure is **integrable** iff it comes from a complex structure.

Example 17. If M is a $\mathbb{C}$-manifold, then $T^{\mathrm{hol}}M \cong T^{1,0}M = (TM \otimes \mathbb{C})^{1,0}$.

What it means to be integrable? On $\mathbb{R}^2$, a complex structure is a way to decide which functions are holomorphic:

$$f \text{ is holomorphic } \iff \partial_{\bar{z}} f = (\partial_x + i\partial_y) f = 0.$$

Suppose we are given two real vector fields

$$Q_j = a_j(x, y)\partial_x + b_j(x, y)\partial_y, \quad j \in \{1, 2\},$$

and let $Pf = (Q_1 + iQ_2)f$. Can we find coordinates $u = u(x, y), v = v(x, y)$ such that in these coordinates, $Pf = 0$ is equivalent to $(\partial_u + i\partial_v)f = 0$?

A necessary condition is that Q_1, Q_2 are linearly independent. It turns out that this is sufficient (hence $T^{1,0}M$ is always closed under Lie bracket). Suppose we can solve $P\omega = 0$ with $\omega = u + iv$, where u, v are $\mathbb{R}$-valued and $\nabla u, \nabla v$ are linearly independent. Then we will use u and v as the new coordinates. On the one hand, by the chain rule

$$P = \alpha(u, v)\partial_u + \beta(u, v)\partial_v$$

for some $\mathbb{C}$-functions α, β, through

$$\partial_x \mapsto \frac{\partial u}{\partial x}\partial_u + \frac{\partial v}{\partial x}\partial v,$$
$$\partial_y \mapsto \frac{\partial u}{\partial y}\partial_u + \frac{\partial v}{\partial y}\partial v.$$

On the other hand, since $P\omega = 0 = P(u+iv) = \alpha + i\beta$, $P = -i\beta(\partial_u + i\partial_v)$, and so

$$Pf = 0 \Longleftrightarrow (\partial_u + i\partial_v)f = 0,$$

by the fact that $\beta \neq 0$ by linear independence. To solve $P\omega = 0$, we will need the elliptic equations (See 6.1).

Not every almost complex structure is integrable. There is a nice characterization. Recall that we have Lie bracket for vector fields: for $V, W \in C^\infty(M, TM)$, $[V, W]$ is the vector field satisfying $\mathscr{L}_{[V,W]} = [\mathscr{L}_V, \mathscr{L}_W]$. This easily extends to sections of $TM \otimes \mathbb{C}$. The Newlander-Nirenberg Theorem says that an almost complex structure is integrable iff Lie bracket of two sections of $T^{0,1}M$ is another section of $T^{0,1}M$. In order to state the theorem, we need to review some knowledge in differentiable manifolds.

Definition 30. Suppose M is a smooth manifold and $E \subset TM$ is a subbundle of rank k. We say

1. E is **involutive** if the Lie bracket of two sections of E is a section of E.
2. E is **integrable** if each point of M has a neighborhood U and a map $\phi_U : U \to \mathbb{R}^{n-k}$ such that $E\mid_U = \ker D\phi_U$. That is to say, each fiber $\phi_U^{-1}(v)$ is a submanifold of U with target space $E\mid_{\phi_U^{-1}(v)}$.

Theorem 18 (Frobenius, Smooth Manifolds)**.** E is involutive iff E is integrable.

Given the Frobenius Theorem for smooth manifolds, we can deduce a Frobenius Theorem for complex manifolds.

Theorem 19 (Frobenius, Complex Manifolds)**.** Let M be a complex manifold of $\mathbb{C}$-dimension n, $E \subset T^{\text{hol}}M$ be a holomorphic subbundle of $\mathbb{C}$-rank k. Then E is involutive iff E is holomorphically integrable (i.e. we have local holomorphic maps $\phi_U : U \to \mathbb{C}^{n-k}$ such that $E_\xi = \ker(D_\xi \phi_U)$ for every $\xi \in M$).

Proof. Note that $\Re : T^{1,0}M = T^{\text{hol}}M \xrightarrow{\simeq} TM$ sending $\omega - iJ(\omega)$ to ω is an isomorphism of $\mathbb{C}$-vector spaces. If E is involutive, then so is $\Re E$. Then the smooth Frobenius Theorem gives us smooth local map $\phi_U : U \to \mathbb{R}^{2n-2k}$ with $\Re E\mid_U = \ker(D_\xi \phi_U)$. Next we want to put a complex structure on $\operatorname{im}(\phi_U) = V \subset \mathbb{R}^{2n-2k}$ for which ϕ_U is holomorphic. We can identify $T_{\phi_U(\xi)}V = T_\xi U \big/ \Re E_\xi$. Since E is a holomorphic subbundle, the integrable almost $\mathbb{C}$-structure on TM preserves $\Re E$ because J descends to TV. Thus $T_{\phi_U(\xi)}V$ inherits a complex structure. By construction, $D\phi_U$ commutes with J, so it is holomorphic. □

Definition 31. (M, J)dimension is **real analytic** if M has an atlas whose transition maps are real analytic, and in each of these coordinate charts, J is a real analytic family of matrices.

Now we can state the Newlander-Nirenberg Theorem as follows:

Theorem 20 (Newlander-Nirenberg)**.** If J is an almost complex structure on M and (M, J) is real analytic, then J is integrable iff $T^{0,1}M$ (or $T^{1,0}M$) is involutive.

Proof. (Weil) It is enough to work locally. Assume $M = U \subset \mathbb{R}^{2n}$ is open, $0 \in U$, and J is a local analytic matrix-valued map satisfying $J^{-2} = -\,\mathrm{id}$ given by a convergent power series. Hence there is a neighborhood of the origin $\tilde{U} \subset \mathbb{C}^{2n}$ on which this power series converges. Denote the extension by $\tilde{J}$. Let $\tilde{E}$ be the $-i$ eigen-bundle of $\tilde{J}$, so

$$E = \tilde{E}\mid_U = T^{0,1}U \subset TU \otimes \mathbb{C} \simeq \mathbb{C}^{2n}.$$

Sections of $\tilde{E}$ over $\tilde{U}$ are vector fields of the form $v + iJ(v)$, where V is a $\mathbb{C}$-vector field over $\tilde{U}$. Hence the involutivity of $T^{0,1}U$ implies the involutivity of $\tilde{E}$. Thus, up to shrinking the neighborhood, we know that there exists a holomorphic function $\phi_{\tilde{U}} : \tilde{U} \to \mathbb{C}^n$ where fibers are the integrable submanifolds of $\tilde{E}$, such that $\tilde{E} = \ker(D\phi_{\tilde{U}})$. Now note that U sits in $\tilde{U}$ like $(\Re z_1, \Re z_2, \cdots, \Re z_n)$ in $(z_1, z_2, \cdots, z_n)$, and this space TU is transverse to $\tilde{E}$. Therefore, the restriction $\phi_{\tilde{U}}\mid_U = \phi : U \to \mathbb{C}^n$ is a diffeomorphism. Finally, note that the derivative of ϕ, $D\phi : T_\xi U \to T_{\phi(\xi)}\mathbb{C}^n$ identifies J with the complex structure on $\mathbb{C}^n$. This follows from $\mathbb{C}$-linearity of $T_\xi U \hookrightarrow T_\xi \tilde{U} \to T_\xi \tilde{U}/\Re E_\xi$. In the quotient $T_\xi \tilde{U}/\Re E_\xi$, we have $V = -iJ(V)$, i.e. $iV = J(V)$. $\square$

5 Complex and Cohomology

We will talk about the complexes in this chapter. They are useful in the computation, and they are important tools to observe the topological and algebraic properties associated to various manifolds and bundles.

5.1 de Rham Complex

Recall that $\Omega^k(M) = C^\infty(M, \bigwedge^k T^*M)$. We have a ma $d : \Omega^k \to \Omega^{k+1}$ called differential. When $k = 0$, $\Omega^k(M) = C^\infty(M)$. In this case, for V a vector field, $Df : TM \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$, at each point $\xi \in M$, Df sends V to $(f(\xi), \mathscr{L}_V f(\xi))$. Thus $d : C^\infty(M) \to \Omega^1(M)$ is given by

$$df(V) = \mathscr{L}_V(f) = Vf = \mathrm{Proj}_2 Df(V).$$

In local coordinates $(x_1, \cdots, x_n)$, $df = \sum \partial_{x_j} f dx_j$. When $k > 0$, d extends to a map

$$\begin{aligned} d : \Omega^k(M) &\to \Omega^{k+1}(M) \\ \sum_{j=1}^k a_j dx_j &\mapsto \sum_{j=1}^k da_j \wedge dx_j \end{aligned}$$

in local coordinates. In general, the coordinate-free description for d is given as follows: let $\omega \in \Omega^k(M)$, then $d\omega \in \Omega^{k+1}(M)$, and

$$\begin{aligned} d\omega(V_0, V_1, \cdots, V_k) = &\sum_{j=0}^k (-1)^j \mathscr{L}_{V_j}(\omega(V_0, \cdots, \hat{V}_j, \cdots V_k)) \\ &+ \sum_{j<\ell} (-1)^{j+\ell} \omega([V_j, V_\ell], V_0, \cdots, \hat{V}_j, \cdots \hat{V}_k, \cdots V_k). \end{aligned}$$

If $k = 1$,

$$d\omega(V_0, V_1) = \mathscr{L}_{V_0}(\omega(V_1)) - \mathscr{L}_{V_1}(\omega(V_0)) - \omega([V_0, V_1]).$$

Proposition 7. *d satisfies the following properties:*

1. $d^2\omega = 0$ for any $\omega \in \Omega^\bullet(M)$.
2. (Leibniz's Rule) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, then
$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta.$$
3. (Chain Rule) If $F : M \to N$ is smooth, then
   - there is a natural map $F^* : \Omega^k(N) \to \Omega^k(M)$ with
   $$(F^*\omega)_\xi(V_1, \cdots, V_k) = \omega_{F(\xi)}(D_\xi F(V_1), \cdots, D_\xi F(V_k));$$
   - for any $\omega \in \Omega^\bullet(N)$, $d_M F^* = F^* d_N$, and
   $$d(F^*\omega) = F^*(d\omega);$$

on functions,

$$\begin{aligned} d(F^* f)(V) &= d(f \circ F)(V) = \mathrm{Proj}_2 D(f \circ F)(V) \\ &= \mathrm{Proj}_2 Df(DF(V)) = df(DF(V)) \\ &= F^*(df)(V). \end{aligned}$$

Exercise 6. Check the previous properties.

Definition 32. A k-form ω is **closed** if $d\omega = 0$ and is **exact** if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$.

Remark 7. Since $d^2\omega = 0$, all exact forms are closed. If $\dim M > 0$, then $\Omega^k(M)$, sets of exact forms and sets of closed forms are all infinite dimensional $\mathbb{R}$-vector spaces. However, if M is a closed manifold (i.e. compact and without boundary), then the quotient vector space (closed k-forms)/(exact k-forms) is finite dimensional.

Definition 33. The above quotient vector space defined in a closed manifold M, i.e.

$$H^k_{\mathrm{dR}}(M) := \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}} = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}\,(d : \Omega^{k-1}(M) \to \Omega^k(M))}$$

is called the k^{th} **de Rham cohomology group** of M.

The de Rham Theorem discussed later identifies these groups with the topological cohomology of M with $\mathbb{R}$-coefficients. Given $F : M \to N$, since the maps $F^\bullet : \Omega^\bullet(N) \to \Omega^\bullet(M)$ commutes with the exterior derivative d, they induce maps $F^* : H^*_{\mathrm{dR}}(N) \to H^*_{\mathrm{dR}}(M)$. It turns out that this map only depends on the homotopy class of F.

Proposition 8. If $H : M \times [0,1] \to N$ is smooth and $F_t = H\mid_{M\times\{t\}}$, then the induced map F_t^* on de Rham cohomology is independent of t.

Proof. Abusing the notation, we can write

$$\Omega^k(M \times [0,1]) = \Omega^k(M) + dt \wedge \Omega^{k-1}(M).$$

In fact, this is short for the following process:

$$\begin{array}{ccc} & M \times [0,1] & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ M & & [0,1] \end{array}$$

and

$$\Omega^k(M \times [0,1]) = C^\infty\left(M \times [0,1], p_1^* \bigwedge^k T^*M\right) + p_2^* dt \wedge C^\infty\left(M \times [0,1], p_1^* \bigwedge^{k-1} T^*M\right).$$

Consider $F_t^*[\omega] = [F_t^*\omega] \in H^k(M)$, where $\omega \in \Omega^k(M)$ and $d\omega = 0$. Write $H^*\omega = \omega_0 + dt \wedge \omega_1$, so $F_s^*\omega = \omega_0\mid_{t=s}$. Note

$$H^* d\omega = 0 = dH^*\omega = d_M\omega_0 + dt \wedge (\partial_t\omega_0 - d_M\omega_1),$$

so $d_M\omega_0 = 0$ and $\partial_t\omega_0 = d_M\omega_1$. Hence

$$\begin{aligned} F_1^*\omega - F_0^*\omega = \omega_0(1) - \omega_0(0) &= \int_0^1 \frac{\partial\omega_0}{\partial t}dt \\ &= \int_0^1 d_M\omega_1 dt = d_M\int_0^1 \omega_1 dt, \end{aligned}$$

and

$$[F_0^*\omega] = \left[F_0^*\omega + d_M\int_0^1 \omega_1 dt\right] = [F_1^*\omega].$$

□

Theorem 21 (Poincaré's Lemma)**.** If $U \subset \mathbb{R}^n$ is smoothly contractible, then $H^k_{\mathrm{dR}}(U) = 0$ for any $k > 0$.

Proof. Let $u_0 \in U$ and $H : U \times [0,1] \to U$ be such that for any $u \in U$, $H(u,0) = u_0$ and $H(u,1) = u$. Then $F_1^* : H^k_{\mathrm{dR}}(U) \to H^k_{\mathrm{dR}}(U)$ is the identity, hence so is F_0^* by Property 8. But $F_0^*\omega = 0$ on $\Omega^k(U)$ for $k \neq 0$, implying the theorem. □

Corollary 5. Let N be the number of connective components of M. Then

$$H^0_{\mathrm{dR}}(M) \cong \mathbb{R}^N.$$

Given a vector bundle $E \to M$, we can define differential forms with coefficients in E, i.e.

$$\Omega^k(M,E) = C^\infty\left(M, \bigwedge^k T^*M \otimes E\right).$$

Recall that a covariant derivative on E is a map

$$\nabla^E : C^\infty(M,E) \to C^\infty(M, T^*M \otimes E),$$

which is equivalent to

$$\nabla^E : \Omega^0(M,E) \to \Omega^1(M,E),$$

satisfying the Leibniz's rule: for any $f \in C^\infty(M)$,

$$\nabla^E(fs) = df \otimes s + f\nabla^E s.$$

We can extend ∇^E to the "E-valued exterior derivative"

$$d^E : \Omega^k(M,E) \to \Omega^{k+1}(M,E)$$

by declaring that, on elementary tensors $\omega \otimes s$ where $\omega \in \Omega^k(M)$ and $s \in C^\infty(M,E)$,

$$d^E(\omega \otimes s) = d\omega \otimes s + (-1)^k\omega \wedge \nabla^E s.$$

However, this is generally not a differential, since $(d^E)^2 \neq 0$.

Definition 34. Let M be a smooth manifold, $E \to M$ be a smooth vector bundle with covariant derivative ∇^E. Give $s \in C^\infty(M, E)$ the value of the section $R^\nabla(s) = d^E(d^E s) = d^E(\nabla^E s)$ at the point $\xi \in M$. Hence the correspondence $s(\xi) \to R^\nabla(s)(\xi)$ defines a smooth section of the vector bundle $\mathrm{Hom}(E, \bigwedge^2 T^*M \otimes E)$. R^∇ is called the **curvature** of ∇^E, and ∇^E is called **flat** if its curvature vanishes.

Proposition 9. The above definition is well-defined, i.e. independent of the choice of the point ξ.

Proof. For any $f \in C^\infty(M)$,

$$\begin{aligned} d^E(d^E(fs)) &= d^E(df \otimes s + f\nabla^E s) = d^2 f \otimes s - df \wedge \nabla^E s + df \wedge \nabla^E s + f d^E(\nabla^E s) \\ &= f d^E(\nabla^E s). \end{aligned}$$

□

Exercise 7. Prove that d^E is actually a differential, provided ∇^E is flat.

If we pick a local frame $s_1, \cdots, s_n$ for $E|_U$, where $U \subset M$ is open, then

$$\begin{aligned} \nabla^E s_j &= \sum \omega_{jk} \otimes s_k, \quad \omega_{jk} \in \Omega^1(U) \\ R^\nabla s_j &= d^E\left(\sum \omega_{jk} \otimes s_k\right) = \sum \Omega_{jk} \otimes s_k, \end{aligned}$$

where $\Omega_{jk} = d\omega_{jk} - \sum_\ell \omega_{j\ell} \wedge \omega_{\ell k}$. In matrix notation,

$$\Omega = d\omega - \omega \wedge \omega.$$

Exercise 8. ∇^E is flat iff there are local frames that are parallel, i.e. $\nabla^E s_j = 0$.

5.2 Dolbeault Complex

If (M, J) is a manifold with an almost complex structure, then $TM \otimes \mathbb{C} = \underbrace{T^{1,0}M}_{J=i} \oplus \underbrace{T^{0,1}M}_{J=-i}$. J induces a bundle map $T^*M \to T^*M$ by $J(\omega)(V) = \omega(J(V))$ and a decomposition $T^*M \otimes \mathbb{C} = T^*M^{1,0} \oplus T^*M^{0,1}$. In local coordinates $x_1, y_1, \cdots, x_n, y_n$ on M, $T^{1,0}$ is spanned by $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$, and $T^{0,1}$ is spanned by $\partial_{\overline{z_j}} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$. Correspondingly, $T^*M^{1,0}$ is spanned by $dz_j = dx_j + idy_j$, and $T^*M^{0,1}$ is spanned by $dz_j = dx_j - idy_j$. We have

$$\begin{aligned} dz_j(\partial_{z_k}) &= \delta_{jk}, \quad dz_j(\partial_{\overline{z_k}}) = 0, \\ d\overline{z_j}(\partial_{z_k}) &= 0, \quad d\overline{z_j}(\partial_{\overline{z_k}}) = \delta_{jk}, \end{aligned}$$

where δ is the Kronecker symbol. Similarly,

$$\begin{aligned} \bigwedge^k T^*M \otimes \mathbb{C} &= \bigoplus_{p+q=k} \bigwedge^{p,q} T^*M \\ &= \bigoplus_{p+q=k} \bigwedge^p (T^*M)^{1,0} \wedge \bigwedge^q (T^*M)^{0,1}. \end{aligned}$$

A form of type (p,q) in local holomorphic coordinates can be written in the form

$$\sum a_{j_1,\cdots,j_p,\ell_1,\cdots,\ell_q} d_{z_{j_1}} \wedge \cdots \wedge d_{z_{j_p}} \wedge d\overline{z_{\ell_1}} \wedge \cdots \wedge d\overline{z_{\ell_q}} = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} a_{\alpha,\beta} dz_\alpha \wedge d\overline{z_\beta},$$

where $\alpha, \beta \in \mathrm{N}^m$, $m = \dim M$.

Remark 8. If M is a complex manifold, then $T^{1,0}M$ is a holomorphic vector bundle, so are $\bigotimes^p T^{1,0}M$ and $\bigwedge^p T^{1,0}M$. On the other hand, $T^{0,1}M$ is just a smooth vector bundle, the composition of anti-holomorphic maps is not anti-holomorphic, so there is no notion of "anti-holomorphic" vector bundle.

Write

$$\Omega^{p,q}(M) \coloneqq C^\infty\left(M, \bigwedge^{p,q} T^*M\right).$$

On an arbitrary manifold with an almost complex structure, the exterior derivative has four types of components

$$d : \Omega^{p,q}(M) \to \Omega^{p-1,q+2}(M) \oplus \Omega^{p,q+1}(M) \oplus \Omega^{p+1,q}(M) \oplus \Omega^{p+2,q-1}(M).$$

Example 18. Consider a 1-form $\omega \in \Omega^1(M)$. ω has type $(1,0)$ if it vanishes on vector fields V of type $(0,1)$, i.e.

$$\omega(V) = \omega(\pi_{1,0}V), \quad (\pi_{1,0}\omega)(V) = \omega(\pi_{1,0}V),$$

where $\pi_{i,j}$ denotes the projection onto (i,j)-type. Then

$$d\omega(V_1,V_2) = \mathscr{L}_{V_1}(\omega(V_2)) - \mathscr{L}_{V_2}(\omega(V_1)) - \omega([V_1,V_2]).$$

Note that

$$\begin{aligned}
\pi_{2,0}d\omega(V_1,V_2) &= d\omega(\pi_{1,0}V_1, \pi_{1,0}V_2), \\
\pi_{1,1}d\omega(V_1,V_2) &= d\omega(\pi_{1,0}V_1, \pi_{0,1}V_2) + d\omega(\pi_{0,1}V_1, \pi_{1,0}V_2), \\
\pi_{0,2}d\omega(V_1,V_2) &= d\omega(\pi_{0,1}V_1, \pi_{0,1}V_2),
\end{aligned}$$

none of these can be guaranteed to vanish.

Remark 9. If ω has type $(1,0)$, then $\pi_{0,2}d\omega(V_1,V_2) = -\omega([\pi_{0,1}V_1, \pi_{0,1}V_2])$. So if $T^{0,1}$ is involutive, i.e. J is integrable, then $\pi_{0,2}d\omega = 0$. On the other hand, if $\pi_{0,2}d\omega = 0$ for all ω of type $(1,0)$, then $\pi_{1,0}[\pi_{0,1}V_1, \pi_{0,1}V_2] = 0$ for any V_1, V_2, i.e. $T^{0,1}M$ is involutive.

Theorem 22. Let (M,J) be a manifold with an almost complex structure. The following are equivalent:

1. M has a complex structure inducing J.

2. $T^{1,0}M$ is involutive.

3. $T^{1,0}M$ is involutive.

4. $d : \Omega^{1,0}(M) \to \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$.

5. $d : \Omega^{0,1}(M) \to \Omega^{0,2}(M) \oplus \Omega^{1,1}(M)$.

6. $d : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$ for any p, q.

In the preceding setting, we write $d = \partial + \overline{\partial}$, where

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M),$$
$$\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).$$

It is easy to see $\partial\omega = \overline{\overline{\partial}\overline{\omega}}$. These satisfy their own Leibniz's rule: if $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$, then

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^k \omega \wedge \partial\eta,$$
$$\overline{\partial}(\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^k \omega \wedge \overline{\partial}\eta.$$

Moreover, since $d^2 = 0$, we have $(\partial + \overline{\partial})^2 = 0 = \partial^2 + \overline{\partial}^2 + \partial\overline{\partial} + \overline{\partial}\partial$. Comparing types, we obtain

$$\overline{\partial}^2 = 0, \;\; \partial^2 = 0, \;\; \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

Definition 35. The **Dolbeault complex** of holomorphic p-forms is

$$0 \to \Omega^{p,0}(M) \xrightarrow{\overline{\partial}} \Omega^{p,1}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,m-p}(M) \to 0.$$

The **Dolbeault cohomology groups** are

$$H^{p,q}_{\overline{\partial}}(M) := \frac{\overline{\partial}\text{-closed } (p,q)\text{-forms}}{\overline{\partial}\text{-exact } (p,q)\text{-forms}} = \frac{\ker(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\operatorname{im}(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

In particular,

$$H^{0,0}_{\overline{\partial}}(M) = \text{ holomorphic sections of } \bigwedge^p (T^*M)^{1,0}.$$

Proposition 10. If M is a closed complex manifold, then $H^{p,q}_{\overline{\partial}}(M)$ is a finite dimensional complex vector space.

Remark 10. There is **NO** natural map between Dolbeault cohomology groups and de Rham cohomology groups on general complex manifolds. However, we can relate them through other theories. From $\partial\overline{\partial} + \overline{\partial}\partial = 0$, we notice that

1. $\partial\overline{\partial}(\overline{\partial} + \partial) = \partial\overline{\partial}\partial = -\overline{\partial}\partial^2 = 0$. So we define **Bott-Chern cohomology groups**:

$$H^{p,q}_{\mathrm{BC}}(M) := \frac{\ker(\partial + \overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M))}{\operatorname{im}(\partial\overline{\partial} : \Omega^{p-1,q-1}(M) \to \Omega^{p,q}(M))} = \frac{\ker(\partial + \overline{\partial})}{\operatorname{im}(\partial\overline{\partial})}.$$

2. $(\partial + \overline{\partial})\partial\overline{\partial} = 0$. So we define **Aeppli cohomology groups**:

$$H^{p,q}_{\mathrm{A}}(M) := \frac{\ker(\partial\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p+1,q+1}(M))}{\operatorname{im}(\partial + \overline{\partial}) \cap \Omega^{p,q}(M)} = \frac{\ker(\partial\overline{\partial})}{\operatorname{im}(\partial + \overline{\partial})}.$$

There are natural maps

$$\begin{array}{ccccc} & & \bigoplus H^{p',q'}_{\mathrm{BC}}(M) & & \\ & \swarrow & \downarrow \vartheta & \searrow & \\ \bigoplus H^{p',q'}_{\partial}(M) & & H^{p+q}_{\mathrm{dR}}(M) & & \bigoplus H^{p',q'}_{\overline{\partial}}(M) \\ & \searrow & \downarrow & \swarrow & \\ & & \bigoplus_{p'+q'=p+q} H^{p',q'}_{\mathrm{A}}(M) & & \end{array}$$

There is a theorem that if ϑ is injective, then all maps in the diagram are isomorphism. It used some cohomological algebra to prove. The readers can see [1] for a reference.

Lemma 5. The map ϑ is injective if $(\ker \partial \cap \ker \overline{\partial} \cap \operatorname{im} d) \subset \operatorname{im} \partial\overline{\partial}$.

We say that a manifold satisfies the $\partial\overline{\partial}$-lemma (Lemma 12, see Lecture 6.4) if this is true. Hence, if a manifold satisfies the $\partial\overline{\partial}$-lemma, then

$$H^k_{\mathrm{dR}}(M) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M),$$

which is the relation between de Rham cohomology and Dolbeault cohomology. This is called the **Hodge-Dolbeault decomposition**.

Similar to de Rham cohomology, we have a Poincaré's lemma for Dolbeault cohomology. In order to get the theorem, we first prove a useful theorem.

Theorem 23. Let $D \subset \mathbb{C}^n$ be a polydisc, $f \in \Omega^{p,q+1}(D)$ such that $\overline{\partial} f = 0$. If D' is another bounded polydisc with $\overline{D'} \subset D$, then there is a form $u \in \Omega^{p,q}(D')$ satisfying $\overline{\partial} u = f$ in D'.

Proof. We shall prove inductively that the theorem is true if f does not involve $d\overline{z_{k+1}}, \cdots, d\overline{z_n}$.

For $k = 0$, clearly $f \equiv 0$ and the theorem is trivial. Assume we know the theorem for $k-1$. Let $f = d\overline{z_k} \wedge g + h$ with $g \in \Omega^{p,q}(M)$ and $h \in \Omega^{p,q+1}(M)$. Both g and h don't involve $d\overline{z_{k+1}}, \cdots, d\overline{z_n}$. Write

$$g = \sum_{|\alpha|=p}{}' \sum_{|\beta|=q}{}' g_{\alpha\beta} dz^\alpha \wedge d\overline{z}^\beta.$$

Since $\overline{\partial} f = 0$, we know that $\partial_{\overline{z_j}} g_{\alpha\beta} = 0$ if $j > k$, i.e. $g_{\alpha\beta}$ is holomorphic in $z_{k+1}, \cdots, z_n$. Next we will find $G_{\alpha\beta}$ such that $\partial_{\overline{z_k}} G_{\alpha\beta} = g_{\alpha\beta}$. Pick $\phi \in C^\infty_c(\mathbb{C})$ such that $\phi(z_k) \equiv 1$ in a neighborhood D'' of $\overline{D'}$ in D, and take

$$G_{\alpha\beta} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\phi(\xi)}{\xi - z_k} g_{\alpha\beta}(z_1, \cdots, z_{k-1}, \xi, z_{k+1}, \cdots, z_n) d\xi d\overline{\xi}.$$

This satisfies $G_{\alpha\beta} \in C^\infty(D)$ satisfying $\partial_{\overline{z_k}} G_{\alpha\beta} = g_{\alpha\beta}$ in D'' and satisfying $\partial_{\overline{z_j}} G_{\alpha\beta} = 0$ for any $j > k$. Set

$$G = \sum_{|\alpha|=p}{}' \sum_{|\beta|=q}{}' G_{\alpha\beta} dz^\alpha \wedge d\overline{z}^\beta.$$

Note that in D'',

$$\overline{\partial} G = \sum{}' \partial_{\overline{z_j}} G_{\alpha\beta} d\overline{z_j} \wedge dz^\alpha \wedge d\overline{z}^\beta = d\overline{z_k} \wedge g + h_1,$$

where h_1 does not involve $d\overline{z_{k+1}}, \cdots, d\overline{z_n}$. Hence $h - h_1 = f - \overline{\partial} G$ does not involve $d\overline{z_k}, d\overline{z_{k+1}}, \cdots, d\overline{z_n}$. By inductive hypothesis, since $\overline{\partial}(h - h_1) = 0$, there exists $v \in \Omega^{p,q}(D')$ such that $\overline{\partial} v = f - \overline{\partial} G$ on D'. Thus $f = \overline{\partial}(v + G)$ on D' as required. □

Theorem 24 ($\overline{\partial}$-Poincaré's Lemma)**.** If $D = D(\varepsilon_1, \cdots, \varepsilon_n) \subset \mathbb{C}^n$ is a polydisc (possibly unbounded), then every $\omega \in \Omega^{p,q}(D)$, $q > 0$, is $\overline{\partial}$-closed and $\overline{\partial}$-exact.

Proof. For each j, choose strictly monotonic sequence $\varepsilon_j(m)$ with $\varepsilon_j(m) \to \varepsilon$ when $m \to \infty$. Let $D_m = D(\varepsilon_1(m), \cdots, \varepsilon_n(m))$, so $D_1 \subset D_2 \subset \cdots \subset \cup D_m = D$. From Theorem 23, for every m, there is an $\eta'_m \in \Omega^{p,q-1}(D_{m+1})$ such that $\overline{\partial}\eta'_m = \omega$ on D_m. Pick $\phi_m \in C_c^\infty(D)$ for each m, with $\phi_m \equiv 1$ on D_m and vanishing outside D_{m+1}. Then $\eta_m = \phi_m \eta'_m \in \Omega^{p,q-1}(D)$ such that $\overline{\partial}\eta_m = \omega$ on D_m. It is easy to see $\overline{\partial}\eta_{m+1} = \omega$ on $D_m \subset D_{m+1}$ by definition.

Case 1 : $q > 1$.

We claim that there exists $(\beta_m) \subset \Omega^{p,q-1}(D)$ satisfying $\overline{\partial}\beta_m = \omega$ on D_m and $\beta_{m+1} = \beta_m$ on D_{m-1}. Assume we have constructed $\beta_1, \cdots, \beta_m$. Since $\overline{\partial}(\beta_m - \eta_{m+1}) = 0$ on D_m, we can apply Theorem 23 to find $\gamma_m \in \Omega^{p,q-2}(D_m)$ such that $\overline{\partial}\gamma_m = \beta_m - \eta_{m+1}$ on D_{m-1}. Define $\beta_{m+1} = \eta_{m+1} + \overline{\partial}(\phi_{m-1}\gamma_m)$ to establish the claim. The sequence β_m converges to $\beta \in \Omega^{p,q-1}(D)$ such that $\overline{\partial}\beta = \omega$.

Case 2 : $q = 1$.

We claim that there exists $(\beta_m) \subset \Omega^{p,0}(D)$ satisfying $\overline{\partial}\beta_m = \omega$ on D_m and $|\beta_{m+1} - \beta_m| < 2^{-m}$ on D_{m-1}. Assume we have constructed $\beta_1, \cdots, \beta_m$. Since $\overline{\partial}(\beta_m - \eta_{m+1}) = 0$ on D_m, we can write

$$\beta_m - \eta_{m+1} = \sum_{|\alpha|=p}{}' (\gamma_\alpha + r_\alpha) dz^\alpha,$$

where γ_α are polynomials and $\sup_{D_m} |r_\alpha(\xi)| < 2^{-m}$. Take

$$\beta_{m+1} = \eta_{m+1} + \sum_{|\alpha|=p}{}' \gamma_\alpha dz^\alpha \in \Omega^{p,0}(D).$$

Since γ_α are holomorphic, $\overline{\partial}\beta_{m+1} = \overline{\partial}\eta_{m+1}$. The sequence β_m converges (locally uniformly) to $\beta \in \Omega^{p,0}(D)$ such that $\overline{\partial}\beta = \omega$.

□

5.3 Chern Connection

If M is a complex manifold, $E \to M$ is a complex vector bundle, then a covariant derivative

$$\begin{aligned} \nabla^E : & C^\infty(M, E) \to C^\infty(M, T^*M \otimes E) \\ & \Omega^0(M, E) \mapsto \Omega^1(M, E) \end{aligned}$$

extends to complexified differential forms, which is the sum of

$$(\nabla^E)^{1,0} : \Omega^0(M,E) \to \Omega^{1,0}(M,E)$$

and

$$(\nabla^E)^{0,1} : \Omega^0(M,E) \to \Omega^{0,1}(M,E),$$

with Leibniz's rules

$$(\nabla^E)^{1,0}(fs) = \partial f \otimes s + f(\nabla^E)^{1,0}s,$$
$$(\nabla^E)^{0,1}(fs) = \overline{\partial} f \otimes s + f(\nabla^E)^{1,0}s.$$

The curvature $R^\nabla \in \Omega^2(M, \mathrm{End}(E))$ can be written as

$$R^\nabla = \underbrace{\left((d^E)^{1,0}\right)^2}_{\text{type (2,0)}} + \underbrace{(d^E)^{1,0}(d^E)^{0,1} + (d^E)^{0,1}(d^E)^{1,0}}_{\text{type (1,1)}} + \underbrace{\left((d^E)^{0,1}\right)^2}_{\text{type (0,2)}}.$$

Proposition 11. If M is a complex manifold and $E \to M$ is a holomorphic vector bundle, then there is a canonical differential operator

$$\overline{\partial}^E : C^\infty(M,E) \to \Omega^{0,1}(M,E)$$

that vanishes on holomorphic sections.

Proof. Let U be any trivializing neighborhood for E and $\{e_j\}_{j=1}^r$ be a local holomorphic frame. Any other local holomorphic frame $\{\tilde{e_j}\}$ on U is of the form $\tilde{e_j} = \sum g_j^k e_k$ for a non-singular matrix (g_j^k) of holomorphic functions on U. Let $s = \sum s_j e_j = \sum \tilde{s_k} g_k^j e_j = \sum \tilde{s_k}\tilde{e_k}$ be a local holomorphic section of E. Then the expression $\overline{\partial}^E s = \sum \overline{\partial} s_j \otimes e_j$ is independent of the choice of local frame. Hence $\overline{\partial}^E$ is independent of the choice of charts of the vector bundle. □

It is natural to extend $\overline{\partial}^E : \Omega^{p,q}(M,E) \to \Omega^{p,q+1}(M,E)$ by requiring the Leibniz's rule

$$\overline{\partial}^E(\omega \otimes s) = \overline{\partial}\omega \otimes s + (-1)^{p+q}\omega \otimes \overline{\partial}^E s.$$

It is straightforward to check $(\overline{\partial}^E)^2 = 0$. So we can define

$$H^{p,q}_{\overline{\partial}}(M,E) = \frac{\ker \overline{\partial}^E}{\operatorname{im} \overline{\partial}^E}.$$

This is the complex analogue of the de Rham cohomology of flat vector bundles. If $E \to M$ is a holomorphic vector bundle and ∇^E is a covariant derivative, then we say that it is **holomorphic** if $(\nabla^E)^{0,1} = \overline{\partial}^E$. Equivalently, ∇^E is holomorphic if whenever s is a local holomorphic section of E, $\nabla^E s$ has type $(1,0)$.

Lemma 6. Any holomorphic vector bundle admits holomorphic covariant derivatives.

Proof. Let $\{U_\alpha\}$ be a locally finite cover of M by trivializing charts of E. Let $\{x_\alpha\}$ be a partition of unity subordinate to the cover. Identify $E\mid_{U_\alpha} = U_\alpha \times \mathbb{C}^r$ using a holomorphic frame and let $\nabla_\alpha = d$ (acting on $\mathbb{C}^r$-valued functions). Then set $\nabla^E = \sum x_\alpha \nabla_\alpha$. □

Conversely, we have:

Proposition 12. Let $E \to M$ be a complex vector bundle over a complex manifold M. If ∇^E is a covariant derivative on sections of E such that $((\nabla^E)^{0,1})^2 = 0$, then there is a unique holomorphic vector bundle structure on E with $(\nabla^E)^{0,1} = \overline{\partial}^E$.

Proof. We will define an almost complex structure on E by specifying a splitting of the $\mathbb{C}$-cotangent spaces into $(1,0)$ and $(0,1)$-forms, and then verify their integrability.

Fix a local trivialization $E\mid_U = U \times \mathbb{C}^r$. Let $z_1, \cdots, z_n$ be coordinates on U and $\omega_1, \cdots, \omega_r$ be the natural coordinate system on $\mathbb{C}^r$. Let $A = (A^j_k)$ be the connection form in this trivialization and $A^j_k = (A')^j_k + (A'')^j_k$ be the splitting into $(1,0)$ and $(0,1)$-types. Then $((\nabla^E)^{0,1})^2 = 0$ is equivalent to $\overline{\partial}(A'')^j_k = \sum (A'')^j_\ell \wedge (A'')^\ell_k$. Now define an almost complex structure on E by taking $\{dz_\alpha . d\omega_j + \sum (A'')^k_j \omega_k\}$ as a basis for the space of $(1,0)$-forms on E. We will check that d sends $(1,0)$-forms to $(2,0)+(1,1)$-forms. Note

$$\begin{aligned}
d(dz_\alpha) &= 0,\\
d\left(d\omega_j + \sum (A'')^k_j \omega_k\right) &= \sum d(A'')^k_j \omega_k - \sum (A'')^k_j d\omega_k\\
&= \sum \partial (A'')^k_j \omega_k + \sum \overline{\partial}(A'')^k_j \omega_k - \sum (A'')^k_j d\omega_k\\
&= \sum \partial (A'')^k_j \omega_k + \sum\sum (A'')^j_\ell \wedge (A'')^\ell_k \omega_k - \sum (A'')^k_j d\omega_k\\
&= \sum \partial (A'')^k_j \omega_k - \sum (A'')^j_\ell \wedge \left[d\omega_\ell - \sum (A'')^k_\ell \omega_k\right] \equiv 0,
\end{aligned}$$

where the last step uses the basis defined before. Hence the almost complex structure is integrable, i.e. it is a complex structure. To see that this has the desired property, we will check that if $s : U \to E$ is a local section with $(\nabla^E)^{0,1} s = 0$, then it pulls back every $(1,0)$-form on E to a $(1,0)$-form on M. Indeed, if locally s is generated by

$$\begin{aligned}
U &\to U \times \mathbb{C}^r\\
z &\mapsto (z, \xi(z)),
\end{aligned}$$

then the condition $(\nabla^E)^{0,1} s = 0$ is given by $\overline{\partial}\xi_j + \sum (A'')^k_j \xi_k = 0$. Pulling back the $(1,0)$-forms in the basis defined before, we obtain

$$\begin{aligned}
s^*(dz_\alpha) &= dz_\alpha,\\
s^*\left(d\omega_j + \sum (A'')^k_j \omega_k\right) &= d\xi_j + \sum (A'')^k_j \xi_k = \partial \xi_j.
\end{aligned}$$

These are all of type $(1,0)$ on M. □

Definition 36. The operator $\overline{\partial}^E$ is called a **holomorphic structure** on $E \to M$.

Note 4. There are many holomorphic covariant derivatives on $E \to M$. We will usually choose one by adding the requirement that it preserves lengths of vectors.

Definition 37. Let $E \to M$ be a complex vector bundle over a smooth manifold M. A **Hermitian metric** h^E on E is a smooth family of Hermitian inner products on the fibers of E. That is, for each $\xi \in M$, $h^E_\xi : E_\xi \times E_\xi \to \mathbb{C}$ satisfies

1. $h^E(u, v)$ is $\mathbb{C}$-linear in u for each $v \in E_\xi$.
2. $h^E(u, v) = \overline{h^E(v, u)}$.
3. $h^E(u, u) \geq 0$ and $h^E(u, u) = 0$ iff $u = 0$.
4. If $s_1, s_2 \in C^\infty(M, E)$, then $h^E(s_1, s_2) \in C^\infty(M)$.

Actually, h^E is equivalent to a $\mathbb{C}$-anti-linear bundle isomorphism $h^\flat : E \to E^*$ with

$$h^\flat(u)(v) = h^E(v, u).$$

Lemma 7. Every complex vector bundle admits a Hermitian bundle metric.

Proof. On the trivial bundle $U \times \mathbb{C}^r$, we can take the inner product on $\mathbb{C}^r$: $(u, v) \mapsto v^* u$. Since convex linear combinations of Hermitian metric are again Hermitian metric (Why?), it suffices to patch together these local metrics using a partition of unity. □

Definition 38. A covariant derivative ∇^E and a Hermitian metric h^E are **compatible** if, for any sections $s_1, s_2 \in C^\infty(M, E)$, we have

$$d(h^E(s_1, s_2)) = h^E(\nabla^E s_1, s_2) + h^E(s_1, \nabla^E s_2),$$

i.e.

$$\mathscr{L}_V(h^E(s_1, s_2)) = h^E(\nabla^E_V s_1, s_2) + h^E(s_1, \nabla^E_V s_2)$$

for any vector field $V \in C^\infty(M, TM)$.

Theorem 25. If M is a complex manifold and $E \to M$ is a holomorphic vector bundle, then for every Hermitian metric h^E on E, there is a unique holomorphic connection ∇^E on E compatible with h^E. This connection is called the **Chern connection** of (E, h^E).

Proof. Recall ∇^E is a holomorphic connection iff $(\nabla^E)^{0,1} = \overline{\partial}^E$ (Proposition 12). If $E \to M$ is holomorphic, then so is $E^* \to M$. Given a connection ∇^E on E, we obtain one on E^* by demanding

$$d(\omega(s)) = (\nabla^{E^*}\omega)(s) + \omega(\nabla^E s)$$

for any $s \in C^\infty(M, E)$ and $\omega \in C^\infty(M, E^*)$. In particular, if ∇^E is a holomorphic, then ∇^{E^*} is a holomorphic connection on E^* (sending holomorphic sections to $(1, 0)$-forms). An equivalent way of expressing the compatibility of ∇^E or ∇^{E^*} is to say that $h^\flat(\nabla^E s) = \nabla^{E^*} h^\flat(s)$ for any section of E. But note that the $\mathbb{C}$-anti-linearity of $h^\flat$ implies that whenever $z \in C^\infty(M, TM \otimes \mathbb{C})$, we have $h^\flat(\nabla^E_z s) = \nabla^{E^*}_{\overline{z}} h^\flat(s)$. Hence

$$(\nabla^E)^{1,0} s = (h^\flat)^{-1}((\nabla^{E^*})^{0,1} h^\flat(s)) = (h^\flat)^{-1}(\overline{\partial}^{E^*} h^\flat(s)).$$

Thus a holomorphic connection compatible with h^E must be equal to

$$\nabla^E = \underbrace{(h^\flat)^{-1} \circ \overline{\partial}^{E^*} \circ h^\flat}_{\text{(0,1)-part}} + \underbrace{\overline{\partial}^E}_{\text{(1,0)-part}} .$$

Conversely, this formula gives existence. □

Remark 11. It follows from the formula for the Chern connection that R^∇ has type $(1,1)$. Indeed, the $(0,2)$-part vanishes since $(\overline{\partial}^E)^2 = 0$, and the $(2,0)$-part vanishes since $(\overline{\partial}^{E^*})^2 = 0$.

5.4 Kähler Manifolds

On the complex manifolds, previous discussion in Lecture 5.3 applies to the holomorphic tangent bundle $T^{\text{hol}}M$. We want to understand how it relates to the underlying smooth structure. In order to do that, we need to introduce some important definitions.

Definition 39. A **Riemannian metric** g on a smooth manifold M is a smoothly varying family of inner products on the fibers of the tangent bundle. Explicitly, for each $\xi \in M$, a map $g_\xi : T_\xi M \times T_\xi M \to \mathbb{R}$ satisfying

1. $g(u, v)$ is $\mathbb{R}$-linear in u for all v.
2. $g(u, v) = g(v, u)$ for any $u, v \in T_\xi M$.
3. $g(u, u) \geq 0$ and $g(u, u) = 0$ iff $u = 0$.
4. If $s_1, s_2 \in C^\infty(M, TM)$, then $g(s_1, s_2) \in C^\infty(M)$.

Remark 12. If (M, J) is an almost complex manifold and $h = h^{TM}$ is a Hermitian metric on TM (viewed as a $\mathbb{C}$-vector bundle), separating h into real and imaginary parts gives $h(u, v) = g(u, v) + i\omega(u, v)$, then g is a Riemannian metric on M and ω is a 2-form (Check!), i.e. $\omega \in \Omega^2(M)$. Since

$$h(J(u), J(v)) = i \cdot (-i) \cdot h(u, v) = h(u, v),$$

we have

$$g(J(u), J(v)) = g(u, v),$$
$$\omega(J(u), J(v)) = \omega(u, v).$$

Similarly, $h(J(u), v) = ih(u, v)$ implies $g(J(u), v) = \omega(u, v)$ and $\omega(J(u), v) = -g(u, v)$. In particular, having a Hermitian metric on TM as a $\mathbb{C}$-vector bundle is equivalent to having a Riemannian metric on TM compatible with J in that $g(J(u), J(v)) = g(u, v)$. In fact, if (g, J, ω) are compatible, then any two determine the third. We sometimes refer to (g, J, ω) as a **Hermitian structure**.

Exercise 9. Prove that both g and ω are non-degenerate.

Remark 13. The volume form of g is equal to $\frac{\omega^n}{n!}$. In particular,

$$\text{Vol}(M, g) = \int_M \frac{\omega^n}{n!}.$$

Definition 40. A **Kähler manifold** is a Hermitian manifold in which $d\omega = 0$. Here ω is called the **Kähler form** and $[\omega] \in H^2_{\mathrm{dR}}(M)$ is called the **Kähler class**.

Definition 41. A non-degenerate closed 2-form is called a **symplectic form**.

Thus, a Kähler manifold is a smooth manifold M together with (g, J, ω) a compatible choice of a Riemannian metric, a $\mathbb{C}$-structure and a symplectic form. In other words, a complex manifold M is Kähler if it is Hermitian with a compatible symplectic form.

Now to see a Hermitian structure in local holomorphic coordinates $\{z_j\}$. Let $H \in \mathrm{GL}_n(\mathbb{C})$ be the matrix with entries $h_{jk} = h(\partial_{z_j}, \partial_{z_k})$, then $H = H^*$ and H is positive definite. Recall that the natural $\mathbb{C}$-vector bundle isomorphism

$$\begin{aligned}(TM, J) &\xrightarrow{\xi} T^{1,0}M \\ v &\mapsto \frac{1}{2}(v - iJ(v)).\end{aligned}$$

To find the Riemannian metric, write $z_j = x_j + iy_j$. Note that $\xi(\partial_{x_j}) = \partial_{z_j}$ and $\xi(\partial_{y_j}) = \xi(J(\partial_{x_j})) = i\partial_{z_j}$. Thus we have, for instance,

$$\begin{aligned}g(\partial_{x_j}, \partial_{x_k}) &= \Re h(\partial_{z_j}, \partial_{z_k}) = \Re h_{jk}, \\ g(\partial_{x_j}, \partial_{y_k}) &= \Re h(\partial_{z_j}, i\partial_{z_k}) = \Re(-ih(\partial_{z_j}, \partial_{z_k})) = \Im h_{jk}.\end{aligned}$$

So in the basis $\partial_{x_1}, \cdots, \partial_{x_n}, \partial_{y_1}, \cdots, \partial_{y_n}$, g is the $2n \times 2n$-matrix

$$G = \begin{bmatrix} \Re H & \Im H \\ -\Im H & \Re H \end{bmatrix}.$$

Next consider the 2-form ω. It is not hard to find

$$\begin{aligned}\omega(\partial_{x_j}, \partial_{x_k}) &= -\Im h(\partial_{z_j}, \partial_{z_k}) = -\Im h_{jk}, \\ \omega(\partial_{x_j}, \partial_{y_k}) &= -\Im h(\partial_{z_j}, i\partial_{z_k}) = \Re h_{jk}, \\ \omega(\partial_{y_j}, \partial_{y_k}) &= -\Im h(i\partial_{z_j}, i\partial_{z_k}) = -\Im h_{jk}.\end{aligned}$$

This looks nicer if we view ω as a $\mathbb{C}$-valued 2-form by extending it bilinearly to the complexified tangent spaces $TM \otimes \mathbb{C}$ (be very carefully about the difference between J and i!) We want to express ω in terms of dz_j and $d\overline{z_k}$. Note that

$$\begin{aligned}\omega(\partial_{z_j}, \partial_{\overline{z_k}}) &= \omega(\partial_{x_j} - i\partial_{y_j}, \partial_{x_k} + i\partial_{y_k}) \\ &= \omega(\partial_{x_j}, \partial_{x_k}) - i\omega(\partial_{y_j}, \partial_{x_k}) + i\omega(\partial_{x_j}, \partial_{y_k}) + \omega(\partial_{y_j}, \partial_{y_k}) \\ &= -\Im h_{jk} + i\Re h_{jk} + i\Re h_{jk} - \Im h_{jk} \\ &= 2ih_{jk}.\end{aligned}$$

Similar computations show that $\omega(\partial_{z_j}, \partial_{z_k}) = 0$, $\omega(\partial_{\overline{z_j}}, \partial_{\overline{z_k}}) = 0$. So

$$\omega = \frac{i}{2} \sum h_{jk} dz_j \wedge d\overline{z_k}.$$

In particular, note that ω is of type $(1,1)$, $\omega \in \Omega^2(M) \cap \Omega^{1,1}(M)$, so we say ω is a **real form** of type $(1,1)$.

Example 19. $\mathbb{C}$ with the standard metric so that $\partial_{z_1}, \cdots \partial_{z_n}$ is a unitary basis. Then $H = \mathrm{id}_n$, $G = \left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{smallmatrix} \right)$ is the standard metric on $\mathbb{R}^{2n}$, and

$$\omega = \frac{i}{2} \sum dz_j \wedge d\overline{z_j} = \sum dx_j \wedge dy_j$$

is the standard symplectic form on $\mathbb{R}^{2n}$. Note $d\omega = 0$, so this is Kähler.

Example 20. The Kähler structure on $\mathbb{C}^n$ is translation invariant. So it descends to a Kähler structure on $\mathbb{C}$-tori.

Example 21. Any Hermitian structure on a Riemann surface is automatically Kähler.

Example 22. If $N \subset M$ is a complex submanifold, then a Hermitian structure on M restricts to a Hermitian structure on N, which is Kähler if the one on M is.

Example 23. $\mathbb{C}P^n$ admits a ($U(n+1)$-invariant) Kähler metric, known as the **Fubini-Study metric**.

Let $z_1, \cdots, z_{n+1}$ be the standard coordinates on $\mathbb{C}^{n+1}$ and $\rho = \|z\|^2 = \sum_j z_j^2$. Set

$$\begin{aligned} \tilde{\omega} &= \frac{i}{2\pi} \partial \overline{\partial} \log \rho = \frac{i}{2\pi} \left[\frac{\partial \overline{\partial} \rho}{\rho} - \frac{\partial \rho \wedge \overline{\partial} \rho}{\rho^2} \right] \\ &= \frac{i}{2\pi} \left[\frac{\|z\|^2 \sum dz_j \wedge d\overline{z_j} - (\sum \overline{z_j} dz_j) \wedge (\sum z_j d\overline{z_j})}{\|z\|^4} \right]. \end{aligned}$$

This is $U(n+1)$-invariant since it only depends on ρ, and $\mathbb{C}^\times$-invariant since the numerator and denominator are homogeneous of degree 4. Hence $\tilde{\omega}$ pushes forward to a 2-form ω on $\mathbb{C}P^n$. To see that it is positive definite (i.e. $\omega(J(\cdot), \cdot) > 0$), evaluate it at the point $(1 : 0 : 0 : \cdots : 0)$, where it is clearly positive. Then appeal to $U(n+1)$-invariance to see that it is positive definite at all points.

Let (M, h) be a complex manifold with a Hermitian metric. We can always find a local frame for h, i.e. smooth sections of $T^{1,0}M$ whose values give a unitary basis at each point. For such a frame $s_1, \cdots, s_n$, we would have

$$h(s_j, s_k) = \delta_{jk}.$$

Indeed, let's start with any frame and then apply Gram-Schmidt. If $s_1, \cdots, s_n$ is a local unitary frame and $s_1', \cdots, s_n'$ is a local dual coframe, $s_j'(s_k) = \delta_{jk}$, then we have

$$\omega = \frac{i}{2} \sum s_j' \wedge \overline{s_j'},$$

and so

$$\omega^n = \omega \wedge \cdots \wedge \omega = n! \frac{i^n}{2^n} (s_1' \wedge \overline{s_1'} \wedge \cdots \wedge s_n' \wedge \overline{s_n'}) = n! \mathrm{Vol}(g).$$

In particular, if $d\omega = 0$ and M is compact, then ω is not exact and neither is ω^k for $k \leq n$. Indeed, if ω were exact, then ω^n is exact (since if α is closed and β is exact, then $\alpha \wedge \beta$ is exact, Check this!), but if ω^n were exact, then Stokes Theorem tells us $\int_M \omega^n = 0$. However,

$$\int_M \omega^n = n! \mathrm{Vol}(M) \neq 0.$$

Contradiction! Thus $[\omega^k] \in H^{2k}_{\mathrm{dR}}(M)$ are not zero for all $1 \leq k \leq n$.

Proposition 13. (M, g, J, ω) is a Hermitian manifold, it is Kähler iff for each $\xi \in M$, there are local holomorphic coordinates $z_1, \cdots, z_n$ centered at ξ such that the Hermitian metric satisfies

$$h = \mathrm{id}_n + O\left(\sum |z_i|^2\right).$$

Proof. We prove it from two directions.

$(\Longrightarrow)$:

Start with any holomorphic coordinates $z_1, \cdots, z_n$ centered at ξ. By a constant linear change of coordinates, we can assume that $h(0) = \mathrm{id}_n$. Write

$$\omega = \frac{i}{2} \sum (\delta_{jk} + \omega_{jk}) dz_j \wedge d\overline{z_k} + O(|z|^2),$$

where ω_{jk} is a Hermitian matrix whose entries are linear functions of $\{z_j\}$ and $\{\overline{z_j}\}$. We can decompose ω_{jk} into its $\mathbb{C}$-linear and $\mathbb{C}$-anti-linear parts:

$$\omega_{jk} = \omega_{jk}^{\mathrm{hol}} + \omega_{jk}^{\mathrm{ant}},$$

and since $\omega = \omega^*$, we have $\overline{\omega_{jk}^{\mathrm{hol}}} = \omega_{kj}^{\mathrm{ant}}$. Being Kähler implies that $d\omega = 0$, giving

$$\partial \left(\sum \omega_{jk}^{\mathrm{hol}} dz_j \wedge d\overline{z_k} \right) = 0$$

at the origin. Since this is a linear function, this holds on all of the chart. Hence $\partial_{z_\ell} \omega_{jk}^{\mathrm{hol}} = \partial_{z_j} \omega_{\ell k}^{\mathrm{hol}}$. After possible shrinking of the neighborhood, We can apply the Poincaré's Lemma to find holomorphic functions ϕ_j (assume $\phi_j(0) = 0$ for simplicity) satisfying $\omega_{jk}^{\mathrm{hol}} = \partial_{z_k} \phi_j$. Set $z_j' = z_j + \phi_j(z)$. Since $\phi_j(0) = 0$, again after possible shrinking of the neighborhood, these are still holomorphic coordinates. Moreover, note

$$dz_j' = dz_j + \sum \partial_{z_k} \phi_j dz_k = dz_j + \sum \omega_{jk}^{\mathrm{hol}} dz_k,$$

so

$$\begin{aligned} \sum dz_j' \wedge d\overline{z_j'} &= \sum dz_j \wedge d\overline{z_j} + \sum \omega_{jk} dz_j \wedge d\overline{z_k} + O(|z|^2) \\ &= \frac{2}{i} \omega + O(|z|^2). \end{aligned}$$

Thus,

$$\omega = \frac{i}{2} \sum dz_j' \wedge d\overline{z_j'} + O(|z'|^2)$$

as required.

$(\Longleftarrow)$:

Given these coordinates, saying any identity involving the metric and its first derivatives is valid on M is equivalent to say it is valid on $\mathbb{C}^n$. In particular, since $d\omega = 0$ on $\mathbb{C}^n$, it is valid on M.

$\square$

We now want to compare the Hermitian geometry with the Riemannian geometry. We know that $T^{1,0}$ is holomorphic, so the Hermitian metric induces a Chern connection. However, $T^{1,0}M \simeq TM$, and the Riemannian metric induces a connection on TM. How are these related?

A special feature of connections on TM is that vector fields show up both as the directions in which we differentiate and the sections that are setting differentiated.

A covariant derivative ∇ on sections of $TM \to M$ has **torsion**

$$\begin{aligned} T : C^\infty(M, TM) &\to C^\infty(M, TM) \\ (V, W) &\mapsto \nabla_V W - \nabla_W V - [V, W]. \end{aligned}$$

This can be interpreted as follows: $\mathrm{id} : TM \to TM$ is a section of $T^*M \otimes TM$, i.e. $\mathrm{id} \in \Omega^1(M, TM)$, and

$$d^\nabla(\mathrm{id}) \in \Omega^2(M, TM), \quad d^\nabla(\mathrm{id})(V, W) = T(V, W).$$

Theorem 26. For each Riemannian metric g on a smooth manifold, there is a unique connection ∇^{LC}, called the **Levi-Civita connection**, that is a metric and is torsion-free.

Proof. Let U, V, W be vector fields on M. Starting with

$$U_g(V, W) + V_g(W, U) - W_g(U, V),$$

and using that ∇^{LC} is metric and torsion-free, we find the **Koszul form**:

$$2g(\nabla^{LC}_U V, W) = U_g(V, W) + V_g(W, U) - W_g(U, V) + g(U, [V, W]) + g(V, [W, U]) - g(W, [U, V]).$$

On the other hand, this formula defines a connection that is metric and torsion-free. □

Lemma 8. If $\alpha \in \Omega^1(M)$ is parallel with respect to a connection ∇^{T^*M} on T^*M which is torsion-free, then $d\alpha = 0$.

Proof. Through direct computation:

$$\begin{aligned} d\alpha(V, W) &= \mathscr{L}_V \alpha(W) - \mathscr{L}_W \alpha(V) - \alpha([V, W]) \\ &= (\nabla^{T^*M}_V \alpha)(W) + \alpha(\nabla^{TM}_V W) - (\nabla^{T^*M}_W \alpha)(V) - \alpha(\nabla^{TM}_W V) - \alpha([V, W]) \\ &= (\nabla^{T^*M}_V \alpha)(W) - (\nabla^{T^*M}_W \alpha)(V) + \alpha(\nabla^{TM}_V W - \nabla^{TM}_W V - [V, W]) \\ &= (\nabla^{T^*M}_V \alpha)(W) - (\nabla^{T^*M}_W \alpha)(V) + \alpha(T(V, W)) \\ &= (\nabla^{T^*M}_V \alpha)(W) - (\nabla^{T^*M}_W \alpha)(V). \qquad \text{(torsion-free)} \end{aligned}$$

□

Corollary 6. If $\nabla^{LC}\omega = 0$, then $d\omega = 0$.

Let (M, g, J, ω) be a Hermitian manifold. The map

$$\begin{aligned} \xi : TM &\to T^{1,0}M \\ v &\mapsto \frac{1}{2}(v - iJ(v)) \end{aligned}$$

is a $\mathbb{C}$-bundle isomorphism. A connection ∇ on $T^{1,0}M$ induces a connection $\xi^*\nabla$ on TM. So is makes sense to compare $\xi^*\nabla^C$ (Chern connection) with ∇^{LC} (Levi-Civita connection). Notice that on $\mathbb{C}^n$ with the standard metric, these coincide since they can both be identified with d. In a coordinate chart, we have $d + A^C$ and $d + A^{LC}$. The value of A^C and A^{LC} at a point only depends on the first derivatives of the metric at that point. So they coincide on Kähler manifold. In general, we have the following theorem:

Theorem 27. Let (M, g, J, ω) be a Hermitian manifold. The following are equivalent:

1. It is Kähler.
2. For each $p \in M$, there exists local holomorphic coordinates such that $H = \text{id} + O(|z|^2)$.
3. $\xi^*\nabla^C = \nabla^{LC}$.
4. $\xi^*\nabla^C$ is torsion-free.
5. $\nabla^{LC} J = 0$.
6. $\nabla^{LC}\omega = 0$.
7. For each $p \in M$, there exists a neighborhood of p and $f : U \to \mathbb{R}$ smooth, such that

$$\omega = i\partial\overline{\partial}f \qquad \text{on } U.$$

Here f is called a **local Kähler potential**.

Proof. By Proposition 13, 1 and 2 are equivalent. By Corollary 6, 6 implies 1. Also, from previous discussion, 3 and 4 are equivalent, and 1 implies them.

$(3 \implies 5)$:

Since ∇^C is $\mathbb{C}$-linear, we know

$$\nabla^{LC}_V(J(\omega)) = (\nabla^{LC}_V J)(\omega) + J(\nabla_V\omega),$$

and

$$(\nabla^{LC}_V J)(\omega) = \nabla_V(J(\omega)) = \nabla_V(J(\omega)) - J(\nabla_V\omega).$$

Combining these, we get $\nabla^{LC} J = 0$.

$(5 \implies 6)$:

Since $\omega(V, W) = g(J(V), W)$, by definition we get the result.

$(7 \implies 1)$:

Note that $d\omega = (\partial + \overline{\partial})(i\partial\overline{\partial}f) = 0$.

$(1 \implies 7)$:

Let U be a coordinate chart identified with polydisc. By Poincaré's Lemma, we know that $d\omega = 0$, and $\omega = d\eta$ for some η on U. Extending ω and η to $\mathbb{C}$-vector fields and letting η be real, i.e. $\eta^{1,0} = \overline{\eta^{0,1}}$. ω is of type $(1,1)$, yielding

$$d\eta = \overline{\partial}\eta^{1,0} + \partial\eta^{0,1}.$$

So $\partial\eta^{1,0} = 0 = \overline{\partial}\eta^{0,1}$. Applying the $\overline{\partial}$-Poincaré's Lemma, we know that there exists φ on U with $\partial\varphi = \eta^{1,0}$ and $\overline{\partial}\overline{\varphi} = \eta^{0,1}$. Let $f = 2\Im\varphi = i(\varphi - \overline{\varphi})$, then

$$i\partial\overline{\partial}f = -\partial\overline{\partial}\varphi + \partial\overline{\partial}\overline{\varphi} = \overline{\partial}\eta^{1,0} + \partial\eta^{0,1} = d\eta = \omega.$$

□

6 Hodge Theory

We will introduce the Hodge Theory in this chapter.

6.1 Elliptic Operators

We first set up the adjoint of a differential operator. If M is a smooth manifold, a linear differential operator of order k: $L \in \mathrm{Diff}^k(M)$ is a $\mathbb{F}$-linear map $L : C^\infty(M, \mathbb{F}) \to C^\infty(M, \mathbb{F})$ that for any choice of local coordinates takes the form

$$Lf = \sum_{|\alpha|\le k} a_\alpha(\xi) D^\alpha f = \sum_{\alpha_1+\alpha_2+\cdots+\alpha_n\le k} a_{\alpha_1,\cdots,\alpha_n}(\xi)\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n} f,$$

so L is a polynomial in vector fields.

In fancier language, L is an element of the enveloping algebra of vector fields, or equivalently, by Peetre's Theorem L is a linear map that does not increase support, i.e. $\operatorname{supp} Lf \subset \operatorname{supp} f$. Another approach by Grothendieck is to define $\mathrm{Diff}^k(M)$ inductively with respect to k. When $k = 0$, $\mathrm{Diff}^k(M)$ is just the multiplication by a smooth function. When $k > 0$, $L \in \mathrm{Diff}^k(M)$ iff $[L, f] \in \mathrm{Diff}^{k-1}(M)$ for any $f \in C^\infty(M)$.

If $E \to M$ and $F \to M$ are vector bundles over M, then $L \in \mathrm{Diff}^k(M; E, F)$ is a linear map $L : C^\infty(M, E) \to C^\infty(M, F)$ that in local coordinates has the same form as above, with $a_\alpha(\xi) \in \mathrm{Hom}(E_\xi, F_\xi)$. The explicit expression for L in local coordinates depends strongly on the choice of coordinates, but the highest order part can be defined invariantly. This is called the **principal symbol** of L.

Example 24. If $L \in \mathrm{Diff}^k(M; E, F)$, its principal symbol $\sigma_k(L)$ is the map

$$\begin{aligned} T^*_\xi M &\to \mathrm{Hom}(E_\xi, F_\xi) \\ \xi &\mapsto \sum_{|\alpha|=k} a_\alpha(\xi)(i\xi)^\alpha \end{aligned}$$

obtained from the highest order derivatives by replacing $\partial_{x_j}^{\alpha_j}$ with $i\xi_j^{\alpha_j}$.

Remark 14. The motivation comes from Fourier transform. If $M = \mathbb{R}^n$ and for $f \in C_c^\infty(\mathbb{R}^n)$

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx,$$

then

$$\mathcal{F}(\partial_{x_j} f)(\xi) = i\xi_j \mathcal{F}(f)(\xi).$$

So for any constant coefficient differential operator L on $\mathbb{R}^n$, we have

$$\mathcal{F}(Lf)(\xi) = \varphi_L(\xi)\mathcal{F}(f)(\xi)$$

for some polynomial φ_L (called the **full symbol** of L). Its homogeneous part of degree k is the principal symbol of L.

Example 25. Set $\Delta = -\sum \partial_{x_j}^2$. It satisfies

$$\mathcal{F}(\Delta f)(\xi) = -\sum (i\xi_j)^2 \mathcal{F}(f)(\xi) = |\xi|^2 \mathcal{F}(f)(\xi).$$

So

$$\sigma_2(\Delta)(\xi) = |\xi|^2.$$

To check that the notion "principal symbol" is well-defined, first set $L \in \mathrm{Diff}^k(M; E, F)$, which is defined as before. Let $\xi \in T_p^*M$. Pick $f \in C^\infty(M)$ such that $df(p) = \xi$. Note that

$$\sigma_k(L)(\xi) = \lim_{t\to\infty} \frac{e^{-itf} L(e^{itf})}{t^k}.$$

Indeed,

$$\begin{aligned} \partial_{x_j} e^{itf} &= (it\partial_{x_j} f) e^{itf}, \\ \partial_{x_j}^{\alpha_j} e^{itf} &= (it\partial_{x_j} f)^{\alpha_j} e^{itf} + \psi(t) e^{itf}, \end{aligned}$$

where $\psi(t)$ is some lower order terms in t. So

$$e^{-itf} L(e^{itf}) = t^k \sigma_\ell(L) + \psi(t).$$

Remark 15. If $k = 1$, then $\sigma_1(L)(\xi) = i[L, f](p)$ for any smooth function f such that $df(p) = \xi$. If $k = 2$, then $\sigma_2(L)(\xi) = -\frac{1}{2}[[L, f], f](p)$ for any smooth function f such that $df(p) = \xi$.

The principal symbol is local in that, if $\varphi \in C^\infty(M)$, then for any $\xi \in T_p^*M$,

$$\sigma_k(\varphi L)(\xi) = \varphi(p) \sigma_k(L)(\xi).$$

Hence

$$\sigma_k(L) \in C^\infty(T^*M, \pi^* \mathrm{Hom}(E, F)),$$

with $\pi : T^*M \to M$.

Definition 42. An operator L is called **elliptic** if $\sigma_k(L)(\xi)$ is invertible for any $\xi \in T_pM \backslash \{0\}$, where $p \in M$.

Example 26. Δ on $\mathbb{R}^n$, and $\sigma_2(\Delta)(\xi) = |\xi|^2$. It is the Example 25.

Proposition 14. The (formal) adjoint of an elliptic differential operator is again elliptic.

In order to prove the proposition, we need some setting-ups. On a Riemannian manifold (M, g), there is an L^2-pairing on $C^\infty(M)$:

$$\begin{aligned} C_c^\infty(M) \times C_c^\infty(M) &\xrightarrow{(\cdot,\cdot)_M} \mathbb{F} \\ (f_1, f_2) &\mapsto \int_M f_1 \cdot \overline{f_2} dV_g, \end{aligned}$$

and the norm

$$\|f\|_{L^2}^2 = (f, f)_{L^2(M)} = \int_M |f|^2 dV_g.$$

$L^2(M)$ is the completion of $C_c^\infty(M)$ with respect to $\|\cdot\|_{L^2}$. If E is an $\mathbb{F}$-vector bundle over M equipped with an $\mathbb{F}$-bundle metric h^E, then there is an L^2-pairing on $C_c^\infty(M,E)$:

$$\begin{aligned} C_c^\infty(M,E)\times C_c^\infty(M,E) &\xrightarrow{(\cdot,\cdot)_E} \mathbb{F} \\ (s_1,s_2) &\mapsto \int_M h^E(s_1,s_2)dV_g, \end{aligned}$$

which yields the completion $L^2(M,E)$.

Definition 43. Let $L\in \mathrm{Diff}^k(M;E,F)$. Equip M with a Riemannian metric g, and E,F with bundle metrics h^E,h^F, then the **(formal) adjoint** of L is the operator $L^*\in \mathrm{Diff}^k(M;F,E)$ determined by

$$(Ls,\tilde{s})_F=(s,L^*\tilde{s})_E,$$

where $s\in C_c^\infty(M^0,E)$ and $s\in C_c^\infty(M^0,F)$, and if M is closed then $C_c^\infty(M^0,E)=C^\infty(M,E)$ (same for F).

Proof of Proposition 14. The principal symbol of L^* satisfies

$$h^F(\sigma(L)(\xi)u,v)=h^E(u,\sigma(L^*)(\xi)v)$$

for any $\xi\in T_p^*M$, $u\in E_p$ and $v\in F_p$. That is,

$$\sigma(L^*)=\sigma(L)^*.$$

In particular, L is elliptic iff L^* is elliptic. □

Remark 16. The principal symbol is a homomorphism:

$$\sigma(L\circ L')=\sigma(L)\circ\sigma(L').$$

Theorem 28. Let M be a closed smooth manifold, $E\to M$ and $F\to M$ be two vector bundles, and $L\in \mathrm{Diff}^k(M;E,F)$. If L is elliptic, then

1. $\ker L=\ker_{C^\infty} L=\{u\in C^\infty(M,E): Lu=0\}$ is finite dimensional.
2. $\mathrm{im}\, L=L(C^\infty(M,E))$ is a closed subspace of $C^\infty(M,F)$.
3. Cokernel of L, which is $C^\infty(M,F)\big/L(C^\infty(M,E))\cong\ker L^*$ is finite dimensional.

Corollary 7.

$$\begin{aligned} C^\infty(M,E)&\cong\ker L\oplus \mathrm{im}\, L^*,\\ C^\infty(M,F)&\cong\ker L^*\oplus\mathrm{im}\, L. \end{aligned}$$

This is also true if we replace all instances of C^∞ with L^2-spaces. Moreover,

$$\ker_{C^\infty} L=\ker_{L^2} L.$$

This is called the **elliptic regularity**. In this case, $L: C^\infty(M,E)\to C^\infty(M,F)$ and $L: L^2(M,E)\to L^2(M,F)$ are **Fredholm operators**.

6.2 Hodge Cohomology

Start with the de Rham complex:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0,$$

where $d \in \mathrm{Diff}^1(M; \bigwedge^k T^*M, \bigwedge^{k+1} T^*M)$. A natural question is: What is $\sigma_1(d)$?

If $\xi \in T_p^*M$, then

$$\sigma_1(d)(\xi) = i[d, f](p)$$

for any $f \in C^\infty(M)$ such that $f(p) = \xi$. We have $\sigma_1(d)(\xi) : \bigwedge^k T_p^*M \to \bigwedge^{k+1} T_p^*M$, and

$$\begin{aligned}\sigma_1(d)(\xi)(\omega) &= i[d, f](p)(\omega) = i(d(f\omega) - fd(\omega)) = idf \wedge \omega \\ &= i\xi \wedge \omega.\end{aligned}$$

That is,

$$\sigma_1(d)(\xi) = i\xi \wedge - =: i\mathrm{ext}(\xi).$$

Consider the (formal) adjoint of d, denoted by $d^* = \delta$. Notice that an inner product on V induces one on $\bigotimes^k V$ and $\bigwedge^k V$ by demanding, on $\bigotimes^k V$,

$$\langle v_1 \otimes v_2 \otimes \cdots \otimes v_k, v_1' \otimes v_2' \otimes \cdots \otimes v_k' \rangle = \prod \langle v_j, v_j' \rangle ;$$

or on $\bigwedge^k V$,

$$\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k, v_1' \wedge v_2' \wedge \cdots \wedge v_k' \rangle = \det \left(\langle v_j, v_j' \rangle \right),$$

and extending linearly according to each factor. Another way of saying this is to declare that, if $v_1, \cdots, v_n$ is an orthonormal basis of V, then $\{v_{j_1} \wedge \cdots \wedge v_{j_k} : j_1 < j_2 < \cdots < j_k\}$ is an orthonormal basis of $\bigwedge^k V$. In particular, any $\omega \in \bigwedge^k V$ can be written uniquely as $\omega = v_1 \wedge \omega' + \omega''$, or, if $\eta \in V$ is nonzero, can be written uniquely as $\omega = \eta \wedge v' + \omega''$, where ω' and ω'' have nothing to do with η.

Given a Riemannian metric on M, we obtain bundle metrics on $\bigwedge^k T^*M$ for all k, and it makes sense to discuss $d^* = \delta : \Omega^k(M) \to \Omega^{k-1}(M)$.

Definition 44. If V is a vector field and $\omega \in \Omega^k(M)$, then the **interior product** of V and ω is

$$\mathrm{int}(V)(\omega) = V \lrcorner\, \omega \in \Omega^{k-1}(M),$$

where

$$(V \lrcorner\, \omega)(V_1, \cdots, V_{k-1}) = \omega(V, V_1, \cdots, V_{k-1}).$$

Recall that a Riemannian metric is equivalent to a bundle isomorphism $g^\flat : TM \to T^*M$. We denote the inverse by $g^\sharp : T^*M \to TM$. If $\partial_{x_1}, \cdots, \partial_{x_n}$ is a g-orthonormal basis of T_p^*M, then $dx_j = g^\flat(\partial_{x_j})$ is a g-orthonormal basis of T_pM, and is the dual basis since

$$dx_j(\partial_{x_k}) = g^\flat(\partial_{x_j})(x\partial_{x_k}) = g(\partial_{x_k}, \partial_{x_j}) = \delta_{jk}.$$

Given a nonzero $\eta \in T_p^*M$, any $\omega \in \bigwedge^k T_p^*M$ can be written uniquely as $\omega = \eta \wedge \omega' + \omega''$. Notice that

$$\mathrm{int}(g^\sharp \eta)(\omega) = |\eta|^2 \omega',$$

so

$$\mathrm{ext}(\eta)\mathrm{int}(g^\sharp \eta)(\omega) = |\eta|^2 \eta \wedge \omega',$$

and

$$\begin{aligned} &\mathrm{ext}(\eta)(\omega) = \eta \wedge \omega'', \\ &\mathrm{int}(g^\sharp \eta)\mathrm{ext}(\eta)(\omega) = |\eta|^2 \omega''. \end{aligned}$$

Hence

$$(\mathrm{ext}(\eta)\mathrm{int}(g^\sharp \eta) + \mathrm{int}(g^\sharp \eta)\mathrm{ext}(\eta))(\omega) = |\eta|^2 \omega.$$

Similarly, for any vector field V and 1-form η,

$$(\mathrm{ext}(\eta)\mathrm{int}(V) + \mathrm{int}(V)\mathrm{ext}(\eta))(\omega) = \eta(V)\omega.$$

In particular,

$$\begin{aligned} g(\eta \wedge \alpha, \beta) &= g(\eta \wedge \alpha, \eta \wedge \beta' + \beta'') \\ &= g(\eta \wedge \alpha, \eta \wedge \beta') = g(\eta, \eta) g(\alpha, \beta') \\ &= g(\alpha, \mathrm{int}(g^\sharp \eta)\beta), \end{aligned}$$

i.e. the adjoint of $\mathrm{ext}(\eta)$ is $\mathrm{int}(g^\sharp \eta)$. So

$$\sigma_1(\delta)(\xi) = \frac{1}{i}\mathrm{int}(g^\sharp \eta).$$

Definition 45. The **Hodge Laplacian**, also known as the **Laplace–de Rham operator**, on k-forms on a Riemannian manifold is the differential operator

$$\Delta_k : \Omega^k(M) \to \Omega^k(M),$$

defined as

$$\Delta_k = d\delta + \delta d = (d + \delta)^2 \mid_{\Omega^k} .$$

Since the principal symbol is a homomorphism,

$$\begin{aligned} \sigma_2(\Delta)(\xi) &= (\sigma_1(d)\sigma_1(\delta) + \sigma_1(\delta)\sigma_1(d))(\xi) \\ &= \mathrm{ext}(\xi)\mathrm{int}(g^\sharp \xi) + \mathrm{int}(g^\sharp \xi)\mathrm{ext}(\xi) = |\xi|^2, \end{aligned}$$

yielding Δ_k is elliptic. Formally, Δ_k is self-adjoint:

$$(d\delta)^* = \delta^* d^* = d\delta, \quad (\delta d)^* = d^* \delta^* = \delta d.$$

Also,

$$d\Delta_k = d\delta d = \Delta_{k+1} d, \quad \delta \Delta_k = \delta d \delta = \Delta_{k-1}\delta.$$

Notice that $\ker \Delta = \ker d \cap \ker \delta$ on C^∞-forms. Indeed, we have $\ker d \cap \ker \delta \subset \ker \Delta$. To prove the converse, let $u \in \ker \Delta$, then

$$0 = \langle \Delta u, u \rangle = \langle d\delta u, u \rangle + \langle \delta d u, u \rangle = \langle \delta u, \delta u \rangle + \langle du, du \rangle = \|\delta u\|^2 + \|du\|^2,$$

which reveals the result.

Theorem 29 (Maximum Principle)**.** The only functions f satisfying $\Delta f = 0$ (called the **harmonic functions**) on a closed, connect and oriented Riemannian manifold are the constant functions.

Theorem 30 (Hodge's Theorem for the de Rham Complex)**.** Let M be a closed Riemannian manifold. For each k, we have

$$\Omega^k(M) = \ker \Delta_k \oplus \operatorname{im} \Delta_k = \ker \Delta_k \oplus \operatorname{im} d \oplus \operatorname{im} \delta.$$

In particular,

$$H^k_{\mathrm{Hod}}(M) = \ker \Delta_k \cong H^k_{\mathrm{dR}}(M) = \frac{\ker d}{\operatorname{im} d} = \frac{\ker \Delta_k \oplus \operatorname{im} d}{\operatorname{im} d}$$

is finite dimensional. Here H^k_{Hod} is called the **Hodge cohomology**.

Remark 17. One way of saying this is that for each choice of Riemannian metric, a de Rham cohomology class has a unique representation $\omega \in \Omega^k(M)$ satisfying both $d\omega = 0$ and $\delta\omega = 0$.

We need to justify $\operatorname{im} \Delta_k = \operatorname{im} d \oplus \operatorname{im} \delta$. Indeed, note $\langle du, \delta v \rangle = 0$ for any u, v, since $d^2 = 0$. Clearly $\operatorname{im} \Delta_k \subset \operatorname{im} d \oplus \operatorname{im} \delta$. On the other hand, using $\Omega^k(M) = \ker \Delta_k \oplus \operatorname{im} \Delta_k$, we see that

$$du = d(\pi_k u + \Delta_k u') = d\Delta_k u' = d\delta d u'.$$

Hence $\operatorname{im} d \subset \operatorname{im}(d\delta)$ and $\operatorname{im} \delta \subset \operatorname{im}(\delta d)$. This implies

$$\operatorname{im} \Delta_k = \operatorname{im}(d\delta) \oplus \operatorname{im}(\delta d) = \operatorname{im} d \oplus \operatorname{im} \delta.$$

Similarly, we have the L^2 version of Theorem 30 as follows.

Theorem 31 (Hodge's Theorem for the de Rham Complex, L^2-Version)**.** Let M be a closed Riemannian manifold. For each k, denote

$$\Delta_k : \mathcal{D}_k \subset L^2(M, \bigwedge^k T^*M) \to L^2(M, \bigwedge^k T^*M),$$

and says

$$L^2(M, \bigwedge^k T^*M) = \ker_{L^2} \Delta_k \oplus \Delta_k(\mathcal{D}_k) = \ker_{L^2} \Delta_k \oplus d(\mathcal{D}_{k-1}) \oplus \delta(\mathcal{D}_{k+1}).$$

Moreover,

$$\ker_{L^2} \Delta_k = \ker_{C^\infty} \Delta_k = \ker_{C^\infty} d \cap \ker_{C^\infty} \delta,$$

so

$$H^k_{L^2}(M) = \frac{\ker_{L^2} \Delta_k}{\operatorname{im}_{L^2} d} \cong H^k_{\mathrm{Hod},L^2}(M) = H^k_{\mathrm{Hod},C^\infty}(M) \cong H^k_{\mathrm{dR}}(M).$$

6.3 Hodge Star Operator

We first give the statement of the Poincaré Duality.

Theorem 32 (Poincaré Duality)**.** Let M be a closed and orientable manifold and $\dim M = m$. Then its de Rham cohomology satisfies

$$H^k_{\mathrm{dR}}(M) \cong H^{m-k}_{\mathrm{dR}}(M),$$

for any $0 \leq k \leq m$.

In order to prove this theorem, we need some setting-ups.

Definition 46. Let (M, g) be a closed and orientable Riemannian manifold with $\dim M = m$. For any $\alpha, \beta \in \Omega^k(M)$, $0 \leq k \leq m$, we define the **Hodge star** $\star : \Omega^k(M) \xrightarrow{\simeq} \Omega^{m-k}(M)$ by

$$\alpha \wedge \star\beta = g(\alpha, \beta) dV_g,$$

where V_g is the volume form. In general coordinate chart,

$$dV_g = \sqrt{|\det g|} dx_1 \wedge \cdots \wedge dx_n.$$

Example 27. Consider $(\mathbb{R}^3, g_{\mathbb{R}^3})$, where $g_{\mathbb{R}^3}$ is the standard metric, with the volume form $dV_g = dx_1 \wedge dx_2 \wedge dx_3$. Then

$$\begin{aligned}
&\star 1 = dx_1 \wedge dx_2 \wedge dx_3, \\
&\star dx_1 = dx_2 \wedge dx_3, \quad \star dx_2 = -dx_1 \wedge dx_3, \quad \star dx_3 = dx_1 \wedge dx_2, \\
&\star (dx_1 \wedge dx_2) = dx_3, \quad \star(dx_2 \wedge dx_3) = dx_1, \quad \star(dx_1 \wedge dx_3) = -dx_2, \\
&\star (dx_1 \wedge dx_2 \wedge dx_3) = 1.
\end{aligned}$$

Example 28. Let $p \in M$. For any positively oriented basis $dx_1, dx_2, \cdots, dx_m$ of $T_p^* M$, we can define the Hodge star as

$$\star(dx_{\alpha_1} \wedge dx_{\alpha_2} \wedge \cdots \wedge dx_{\alpha_k}) = dx_{\beta_1} \wedge dx_{\beta_2} \wedge \cdots \wedge dx_{\beta_{m-k}},$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ and $dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_k} \wedge dx_{\beta_1} \wedge \cdots \wedge dx_{\beta_{m-k}} = dx_1 \wedge \cdots \wedge dx_m$.

Exercise 10. $\star^2 |_{\Omega^k} = \pm\, \mathrm{id}$. In fact,

$$\star^2 |_{\Omega^k} = (-1)^{k(m-k)}.$$

Remark 18. The L^2-pairing on $\Omega^k(M)$ is

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta,$$

which is an inner product. On can use this to express $\delta = d^*$ through $\star$ and d. Let $\alpha \in \Omega^k(M)$ and

$\beta \in \Omega^{k+1}(M)$, and M be a closed manifold, then

$$\begin{aligned}\langle d\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle &= \int_M d\alpha \wedge \star\beta \\ &= \int_M d(\alpha\wedge\star\beta) - (-1)^k \int_M \alpha \wedge d(\star\beta) \\ &= (-1)^{k+1} \int_M \alpha\wedge d(\star\beta) && \text{(Stokes' Theorem)} \\ &= (-1)^{k+1}\cdot(-1)^{k(m-k)} \int_M \alpha\wedge\star\star\, d(\star\beta) && \text{(Exercise 10)} \\ &= (-1)^{km+1+k(1-k)} \int_M \alpha\wedge\star(\star d\star\beta) \\ &= (-1)^{km+1}\langle \alpha, \star d\star\beta\rangle .\end{aligned}$$

Thus,

$$\delta\beta = d^*\beta = (-1)^{km+1}(\star d\star)\beta.$$

Corollary 8. We have the relationship:

$$\star\Delta_k = \Delta_{m-k}\star .$$

Exercise 11. Prove this Corollary.

Remark 19. The Hodge star operator commutes with the Hodge Laplacian (Exercise 11), thus defines an isomorphism between $H^k_{\mathrm{dR}}(M)$ and $H^{m-k}_{\mathrm{dR}}(M)$. So we prove the Poincaré Duality (Theorem 32).

Suppose M is a complex manifold with Hermitian metric h, with $\dim M = m$ being its complex dimension and $n = \frac{1}{2}m$ being its real dimension. Extend J to differential forms, i.e. $\omega \in \Omega^k(M)$ giving $J\omega \in \Omega^k(M)$,

$$(J\omega)(V_1, \cdots, V_k) = \omega(JV_1, \cdots, JV_k).$$

In other words, this means that after complexifying, J acts on $\Omega^{p,q}(M)$ by i^{p-q}. Denote another first order operator by

$$d^c = J^{-1}\circ d\circ J : \Omega^k(M) \to \Omega^{k+1}(M).$$

In polar coordinates on $\mathbb{C}$, $z = re^{i\theta}$, we have

$$\begin{aligned} d &= \partial_r \mathrm{ext}(dr) + \partial_\theta \mathrm{ext}(d\theta), \\ d^c &= r\partial_r \mathrm{ext}(d\theta) + \frac{1}{r}\partial_\theta \mathrm{ext}(d\theta), \\ dd^c &= (\partial_x^2 + \partial_y^2)\mathrm{ext}(dx\wedge dy).\end{aligned}$$

After complexifying, if $\omega \in \Omega^{p,q}(M)$, then

$$\begin{aligned} d^c\omega &= J^{-1}(\partial+\overline{\partial})J\omega = J^{-1}(\partial+\overline{\partial})i^{p-q}\omega \\ &= i^{p-q}(J^{-1}\partial\omega + J^{-1}\overline{\partial}\omega) = i^{p-q}\left(\frac{1}{i^{p+1-q}\partial\omega} + \frac{1}{i^{p-(q+1)}}\overline{\partial}\omega\right) \\ &= \frac{1}{i}\partial\omega + i\overline{\partial}\omega = i(\overline{\partial}-\partial)\omega.\end{aligned}$$

In particular,

$$dd^c = (\partial + \overline{\partial})i(\overline{\partial} - \partial) = 2i\partial\overline{\partial} = -d^c d.$$

On the other hand, $(d^c)^2 = J^{-1}dJJ^{-1}dJ = 0$. So we have a complex

$$0 \to \Omega^0(M) \xrightarrow{d^c} \Omega^1(M) \xrightarrow{d^c} \cdots \xrightarrow{d^c} \Omega^m(M) \to 0$$

and associated cohomology

$$H^k_{d^c} = \frac{\ker d^c \mid_{\Omega^k}}{\operatorname{im} d^c \mid_{\Omega^{k-1}}}.$$

Exercise 12. Check that the (formal) adjoint of d^c is $- \star d^c \star$.

Next, we extend $\star$ operator to complexified differential forms by requiring

$$\omega \wedge \star\overline{\eta} = h(\omega, \eta) dV_g.$$

If $u = \sum u_{\alpha,\beta} dz^\alpha \wedge d\overline{z}^\beta$ and $v = \sum v_{\alpha,\beta} dz^\alpha \wedge d\overline{z}^\beta$ both have type (p,q), then

$$h(u,v) = \sum u_{\alpha,\beta}\overline{v_{\alpha,\beta}},$$

and so (here we choose an orthogonal frame)

$$u \wedge \star\overline{v} = h(u,v) dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z_1} \wedge \cdots \wedge d\overline{z_n}.$$

In particular, since dV_g has type (n,n), $\star$ is a $\mathbb{C}$-linear isometry $\Omega^{p,q}(M) \to \Omega^{n-q,n-p}(M)$.

Note that the decomposition $\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$ is orthogonal with respect to the L^2-product. The (formal) adjoints of ∂ and $\overline{\partial}$ are

$$\partial^* : \Omega^{p,q}(M) \to \Omega^{p-1,q}(M), \quad \overline{\partial}^* : \Omega^{p,q}(M) \to \Omega^{p,q-1}(M).$$

Lemma 9. $\partial^* = - \star \overline{\partial} \star$, and $\overline{\partial}^* = - \star \partial \star$.

Proof. For any $\omega \in \Omega^{p-1,q}(M)$ and $\eta \in \Omega^{p,q}(M)$,

$$\begin{aligned}
\langle \partial\omega, \eta \rangle_{\Omega^{p,q}} &= \langle \omega, \partial^*\eta \rangle_{\Omega^{p-1,q}} \\
&= \int \partial\omega \wedge \star\overline{\eta} = \int \partial(\omega \wedge \star\overline{\eta}) - (-1)^{p-1} \int \omega \wedge \partial(\star\overline{\eta}) \\
&= \int d(\omega \wedge \star\overline{\eta}) - (-1)^{p-1} \int \omega \wedge \partial(\star\overline{\eta}) \\
&= (-1)^p \int \omega \wedge \partial(\star\overline{\eta}) && \text{(Stokes' Theorem)} \\
&= (-1)^p \cdot (-1)^{(p-1)(m-p+1)} \int \omega \wedge \star\star\, \partial(\star\overline{\eta}) && \text{(Exercise 10)} \\
&= (-1)^{p(1+m-p)-m-1} \int \omega \wedge \star\overline{\left(\star\overline{\partial}(\star\eta)\right)}.
\end{aligned}$$

Note that m is even, so if p is even, $p(1+m-p)$ is even, $p(1+m-p)-m-1$ is odd; if p is odd, $p(1+m-p)$ is even, $p(1+m-p)-m-1$ is also odd. This implies the sign to be -1. So $\partial^* = - \star \overline{\partial} \star$ follows. The other part is the same (omitted). $\square$

Now for operators $p \in \{d, d^c, \partial, \overline{\partial}\}$, define a new operator $\Delta_p = pp^* + p^*p$. We want to find the principal symbols of these operators. If $\xi \in T_p^*M$ and $f \in C^\infty(M, \mathbb{R})$ satisfies $df(p) = \xi$, then $\sigma_1(\partial)(\xi) : \bigwedge^{p,q} T_p^*M \to \bigwedge^{p+1,q} T_p^*M$ is given by

$$\sigma_1(\partial)(\xi)(\alpha) = i[\partial, f](\alpha) = i[\partial(f\alpha) - f\partial(\alpha)] = i\partial f \wedge \alpha = i\pi_{1,0}\xi \wedge \alpha.$$

Hence

$$\begin{aligned}\sigma_2(\Delta_\partial)(\xi)(\alpha) = \|\pi_{1,0}\xi\|^2\alpha &= \left\|\frac{1}{2}(\xi + iJ(\xi))\right\|^2 \alpha \\ &= \frac{1}{4}\left(\|\xi\|^2 + \|J(\xi)\|^2\right)\alpha = \frac{1}{2}\|\xi\|^2\alpha.\end{aligned}$$

Similarly,

$$\sigma_2(\Delta_{\overline{\partial}})(\xi)(\alpha) = \|\pi_{0,1}\xi\|^2 = \frac{1}{2}\|\xi\|^2\alpha.$$

Finally, note that

$$\sigma_1(d^c)(\xi)(\alpha) = \sigma_1(i(\overline{\partial} - \partial))(\xi)(\alpha) = i\left(\text{ext}(\pi_{0,1}\xi) - \text{ext}(\pi_{1,0}\xi)\right)\alpha,$$

this implies

$$\begin{aligned}\sigma_2(\Delta_{d^c})(\xi) &= \left(i(\text{ext}(\pi_{0,1}\xi) - \text{ext}(\pi_{1,0}\xi))\right)\left(-i(\text{int}(\pi_{0,1}\xi) - \text{int}(\pi_{1,0}\xi))\right) + \left(-i(\text{int}(\pi_{0,1}\xi) - \text{int}(\pi_{1,0}\xi))\right)\left(-i(\text{ext}(\pi_{0,1}\xi) - \text{ex} \right. \\ &= \text{ext}(\pi_{0,1}\xi)\text{int}(\pi_{0,1}\xi) + \text{int}(\pi_{0,1}\xi)\text{ext}(\pi_{0,1}\xi) + \text{ext}(\pi_{1,0}\xi)\text{int}(\pi_{1,0}\xi) + \text{int}(\pi_{1,0}\xi)\text{ext}(\pi_{1,0}\xi) - \text{ext}(\pi_{0,1}\xi)\text{int}(\pi_{1,0} \\ &= \left(g(\pi_{0,1}\xi, \pi_{0,1}\xi) + g(\pi_{1,0}\xi, \pi_{1,0}\xi) - g(\pi_{0,1}\xi, \pi_{1,0}\xi) - g(\pi_{1,0}\xi, \pi_{0,1}\xi)\right) \\ &= \left(\|\pi_{0,1}\xi\|^2 + \|\pi_{1,0}\xi\|^2\right) \\ &= \|\xi\|^2.\end{aligned}$$

Similarly, we find $\sigma_2(\Delta_d) = \sigma_2(\Delta_{d^c}) = 2\sigma_2(\Delta_\partial) = 2\sigma_2(\Delta_{\overline{\partial}})$. Thus, all of these are elliptic. Therefore, we have Hodge's Theorem (on closed manifolds) for them, i.e.

Theorem 33 (Hodge's Theorem for Elliptic Operators)**.** Let M be a closed Riemannian manifold, whose $\mathbb{C}$-dimension is $m = 2n$, where n is its corresponding $\mathbb{R}$-dimension. For each k, the cohomology

1. $H^{p,q}_{\text{Hod},\partial}(M) = \ker \Delta_\partial \mid_{\Omega^{p,q}} \cong H^{p,q}_{\text{dR},\partial}(M)$
2. $H^{p,q}_{\text{Hod},\overline{\partial}}(M) = \ker \Delta_{\overline{\partial}} \mid_{\Omega^{p,q}} \cong H^{p,q}_{\text{dR},\overline{\partial}}(M)$
3. $H^{k}_{\text{Hod},d^c}(M) = \ker \Delta_{d^c} \mid_{\Omega^{k}} \cong H^{k}_{\text{dR},d^c}(M)$

are finite dimensional. Moreover,

$$\Omega^{p,q}(M) = \ker \overline{\partial} \cap \ker \overline{\partial}^* \oplus \overline{\partial}\left(\Omega^{p,q-1}(M)\right) \oplus \overline{\partial}^*\left(\Omega^{p,q+1}(M)\right),$$

and the Hodge star operator $\star$ induces Poincaré Duality isomorphisms:

$$\star : H^{p,q}_{\text{Hod},\partial}(M) \xrightarrow{\cong} H^{n-p,n-q}_{\text{Hod},\partial}(M).$$

Similar results hold for other operators $\overline{\partial}, d^c$ (omitted).

6.4 Lefschetz Operator

On Kähler manifolds (M, g, J, ω), there is an important result revealing the properties of differential operators called the Kähler identities. To start with, we first give the definition of the Lefschetz operator.

Definition 47. The **Lefschetz operator** L on a Kähler manifolds (M, g, J, ω) is defined as

$$L : \Omega^k(M) \to \Omega^{k+2}(M)$$
$$\alpha \mapsto \omega \wedge \alpha$$

After complexifying, it restricts to

$$L : \Omega^{p,q}(M) \to \Omega^{p+1,q+1}(M).$$

Lemma 10. The (formal) adjoint of L, denoted by $\Lambda : \Omega^k(M) \to \Omega^{k-2}(M)$, is given by $\Lambda = (-1)^k \star L \star$.

Proof. Indeed, for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+2}(M)$,

$$\begin{aligned}\langle L\alpha, \beta\rangle &= \langle \alpha, \Lambda\beta\rangle \\ &= \int L\alpha \wedge \star\beta = \int \omega \wedge \alpha \wedge \star\beta \\ &= (-1)^{2k} \int \alpha \wedge \omega \wedge \star\beta \\ &= (-1)^{(k+2)(m-k-2)} \int \alpha \wedge \star\star(\omega \wedge \star\beta).\end{aligned}$$

Note $(k+2)(m-k-2) = k(m-k) + 2(m-2k-2)$. If k is odd, then $k(m-k)$ is odd; if k is even, then $k(m-k)$ is even. Hence $(-1)^{(k+2)(m-k-2)} = (-1)^k$, which implies the lemma. □

Lemma 11. If the open set $U \subset \mathbb{C}^n$ is endowed with the standard metric h with

$$\omega = \frac{i}{2} \sum dz_j \wedge d\overline{z_j},$$

then $[\overline{\partial}^*, L] = i\partial$.

Proof. First note the fact: since $d\overline{z_j} = dx_j - udy_j$, and $\partial_{\overline{z_j}} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$, we have

$$\mathrm{ext}(d\overline{z_j})^* = 2\mathrm{int}(\partial_{\overline{z_j}}).$$

Now, for a form $u = u_{\alpha,\beta}\, dz^\alpha \wedge d\overline{z}^\beta$,

$$\overline{\partial} u = \sum \mathrm{ext}(d\overline{z_j})\partial_{\overline{z_j}} u_{\alpha,\beta}\, dz^\alpha \wedge d\overline{z}^\beta.$$

So

$$\overline{\partial}^* u = -2 \sum \mathrm{int}(\partial_{\overline{z_j}})\partial_{z_j} u_{\alpha,\beta}\, dz^\alpha \wedge d\overline{z}^\beta.$$

Therefore,

$$\begin{aligned}
\overline{\partial}^*(Lu) &= -2\sum_j \operatorname{int}(\partial_{\overline{z_j}})\partial_{z_j}\left(\frac{i}{2}\sum_k u_{\alpha,\beta}\, dz_k \wedge d\overline{z_k} \wedge dz^\alpha \wedge d\overline{z}^\beta\right) \\
&= -i\sum_{j,k} \partial_{z_j} u_{\alpha,\beta} \operatorname{int}(\partial_{\overline{z_j}})\left(dz_k \wedge d\overline{z_k} \wedge dz^\alpha \wedge d\overline{z}^\beta\right) \\
&= -i\sum_{j,k} \partial_{z_j} u_{\alpha,\beta}\left[\operatorname{int}(\partial_{\overline{z_j}})\left(dz_k \wedge d\overline{z_k}\right) \wedge dz^\alpha \wedge d\overline{z}^\beta + dz_k \wedge d\overline{z_k} \wedge \operatorname{int}(\partial_{\overline{z_j}})\left(dz^\alpha \wedge d\overline{z}^\beta\right)\right] \\
&= -i\sum_{j} \partial_{z_j} u_{\alpha,\beta}\left[-dz_j \wedge dz^\alpha \wedge d\overline{z}^\beta\right] - \sum_j \partial_{z_j} u_{\alpha,\beta}\, \omega \wedge \operatorname{int}(\partial_{\overline{z_j}})\left(dz^\alpha \wedge d\overline{z}^\beta\right) \\
&= i\partial u + \omega \wedge \overline{\partial}^* u.
\end{aligned}$$

Hence

$$\overline{\partial}^* Lu - L\overline{\partial}^* u = i\partial u.$$

□

We give the statement of the Kähler identities.

Theorem 34 (Kähler Identities)**.** Let (M, g, J, ω) be a Kähler manifold, then

1. $[\overline{\partial}^*, L] = i\partial$.
2. $[\partial, L] = -i\overline{\partial}$, $[\Lambda, \overline{\partial}] = -i\partial^*$, $[\Lambda, \partial] = i\overline{\partial}^*$.
3. $\partial\overline{\partial}^* = -\overline{\partial}^*\partial$ and $\overline{\partial}\partial^* = -\partial^*\overline{\partial}$.
4. $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d = \frac{1}{2}\Delta_{d^c}$.
5. $d^c d^* = -d^* d^c$, $d(d^c)^* = -d^{c*}d$, $[\Lambda, d] = -(d^c)^*$.

Proof. 1. The relation at each point of M only involves the metric h and its first derivatives at this point. Hence it follows from Lemma 11.

2. Derived directly from 1 by taking conjugates or adjoints.

3. From 2, $\overline{\partial}^* = -i[\Lambda, \partial]$, thus

$$\begin{aligned}
\partial\overline{\partial}^* &= -i\partial[\Lambda, \partial] = -i\partial\left(\lambda\partial - \partial\Lambda\right) \\
&= -i\partial\Lambda\partial,
\end{aligned}$$

and

$$\begin{aligned}
-\overline{\partial}^*\partial &= i[\Lambda, \partial]\partial = i\left(\lambda\partial - \partial\Lambda\right)\partial \\
&= -i\partial\Lambda\partial.
\end{aligned}$$

Thus $\partial\overline{\partial}^* = -\overline{\partial}^*\partial$. The other part can be proved in the same way.

4. Note $\Delta_\partial = \partial\partial^* + \partial^*\partial$. From 2, $\partial^* = i[\Lambda, \overline{\partial}] = i\left(\Lambda\overline{\partial} - \overline{\partial}\Lambda\right)$. So
$$\begin{aligned}\Delta_\partial &= \partial \cdot i\left(\Lambda\overline{\partial} - \overline{\partial}\Lambda\right) + i\left(\Lambda\overline{\partial} - \overline{\partial}\Lambda\right)\partial \\ &= i\left(\partial\Lambda\overline{\partial} - \partial\overline{\partial}\Lambda + \Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial\right).\end{aligned}$$
On the other hand, one can easily find
$$\begin{aligned}\Delta_{\overline{\partial}} &= \overline{\partial} \cdot (-i)\left(\Lambda\partial - \partial\Lambda\right) - i\left(\Lambda\partial - \partial\Lambda\right)\overline{\partial} \\ &= -i\left(\overline{\partial}\Lambda\partial - \overline{\partial}\partial\Lambda + \Lambda\partial\overline{\partial} - \partial\Lambda\overline{\partial}\right).\end{aligned}$$
So
$$\begin{aligned}\Delta_\partial - \Delta_{\overline{\partial}} &= i\left(\partial\Lambda\overline{\partial} - \partial\overline{\partial}\Lambda + \Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial + \overline{\partial}\Lambda\partial - \overline{\partial}\partial\Lambda + \Lambda\partial\overline{\partial} - \partial\Lambda\overline{\partial}\right) \\ &= i\left[-\left(\overline{\partial}\partial + \partial\overline{\partial}\right)\Lambda + \Lambda\left(\overline{\partial}\partial + \partial\overline{\partial}\right)\right] \\ &= 0. && \text{(By } \partial\overline{\partial} = -\overline{\partial}\partial\text{)}\end{aligned}$$
For Δ_d, we have
$$\begin{aligned}\Delta_d = dd^* + d^*d &= \left(\partial + \overline{\partial}\right)\left(\partial^* + \overline{\partial}^*\right) + \left(\partial^* + \overline{\partial}^*\right)\left(\partial + \overline{\partial}\right) \\ &= \partial\partial^* + \partial^*\partial + \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} && \text{(Use 3)} \\ &= \Delta_\partial + \Delta_{\overline{\partial}} = 2\Delta_\partial = 2\Delta_{\overline{\partial}}.\end{aligned}$$
Since Δ_{d^c} preserves (p,q)-type, we have $\Delta_{d^c} = J^{-1}\Delta_d J = \Delta_{d^c}$.

5. To prove the first equation, note that
$$\begin{aligned}d^c d^* + d^* d^c &= i\left(\overline{\partial} - \partial\right)\left(\overline{\partial}^* + \partial^*\right) + i\left(\overline{\partial}^* + \partial^*\right)\left(\overline{\partial} - \partial\right) \\ &= i\left(\overline{\partial}\overline{\partial}^* + \overline{\partial}\partial^* - \partial\overline{\partial}^* - \partial\partial^* + \overline{\partial}^*\overline{\partial} - \overline{\partial}^*\partial + \partial^*\overline{\partial} - \partial^*\partial\right) \\ &= i\left(\overline{\partial}\overline{\partial}^* - \partial\partial^* + \overline{\partial}^*\overline{\partial} - \partial^*\partial\right) && \text{(Use 3)} \\ &= i\left(\Delta_{\overline{\partial}} - \Delta_\partial\right) \\ &= 0.\end{aligned}$$
To prove the second equation, note that
$$\left(d^c\right)^* d = -i\left(\overline{\partial}^* - \partial^*\right)\left(\partial + \overline{\partial}\right) = -i\left(\overline{\partial}^*\partial + \overline{\partial}^*\overline{\partial} - \partial^*\partial - \partial^*\overline{\partial}\right),$$
also
$$d\left(d^c\right)^* = -i\left(\partial + \overline{\partial}\right)\left(\overline{\partial}^* - \partial^*\right) = -i\left(\partial\overline{\partial}^* + \overline{\partial}\overline{\partial}^* - \partial\partial^* - \overline{\partial}\partial^*\right).$$
These imply
$$d\left(d^c\right)^* + \left(d^c\right)^* d = -i\left(\Delta_{\overline{\partial}} - \Delta_\partial\right) = 0.$$
To prove the third equation, note that
$$[\Lambda, d] = [\Lambda, \partial + \overline{\partial}] = [\Lambda, \partial] + [\Lambda, \overline{\partial}] = i\overline{\partial}^* - i\partial^* = i\left(\overline{\partial}^* - \partial^*\right) = -i\left(d^c\right)^*.$$

□

Exercise 13. Prove $[\Delta_\partial, L] = 0$.

Corollary 9. If (M, g, J, ω) is a Kähler manifold, then

$$d\omega = \partial\omega = \overline{\partial}\omega = d^c\omega = \delta\omega = \partial^*\omega = \overline{\partial}^*\omega = (d^c)^*\,\omega = 0.$$

Proof. $d\omega = 0$ is the definition. Since ω has type $(1,1)$, from $d\omega = 0$ we know $\overline{\partial}\omega = 0 = d^c\omega = \partial\omega$. Now from Theorem 34, $\partial^*\omega = i[\Lambda, \overline{\partial}]\omega = i\Lambda\overline{\partial}\omega - i\overline{\partial}\Lambda\omega$. We claim $\Lambda\omega = 1$. Indeed, for any $f \in C_c^\infty(M)$,

$$\langle \Lambda\omega, f\rangle_M = \langle \omega, Lf\rangle = \langle \omega, f\omega\rangle = \int_M \overline{f}\omega \wedge \star\omega = \int_M \overline{f} dV_g = \langle 1, f\rangle_M\,.$$

This implies $\partial^*\omega = 0$. Similarly one can check $(d^c)^* = 0$ (Check). □

Definition 48. For any operator $\mathfrak{p} \in \{d, d^c, \partial, \overline{\partial}\}$, a form α is called **$\mathfrak{p}$-harmonic** if $\mathfrak{p}\alpha = 0$.

Corollary 10. If (M, g, J, ω) is a Kähler manifold, then

$$H^k_{\mathrm{dR}}(M) \cong H^k_{d^c}(M) \cong \bigoplus_{p+q=k} H^{p,q}_\partial(M) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M).$$

Moreover, conjugation induces an isomorphism $\overline{H^{p,q}_\partial(M)} \cong H^{q,p}_\partial(M)$, and $\star$ operator induces an isomorphism

$$\star : H^{p,q}_{\overline{\partial}}(M) \to H^{n-q,n-p}_{\overline{\partial}}(M).$$

Proof. Since $\Delta_d = \Delta_{d^c} = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$, we can deduce

$$\ker \Delta_d \mid_{\Omega^k} = \ker \Delta_{d^c} \mid_{\Omega^k} = \ker \Delta_\partial \mid_{\bigoplus_{p+q=k} \Omega^{p,q}} = \ker \Delta_{\overline{\partial}} \mid_{\bigoplus_{p+q=k} \Omega^{p,q}}\,.$$

Also, for any harmonic form α of type (p,q), $\overline{\alpha}$ has type (q,p), and $\overline{\Delta_\partial\overline{\alpha}} = \Delta_{\overline{\partial}}\alpha = 0$. So $\overline{\alpha}$ is also harmonic. □

Corollary 11 (dd^c-Lemma)**.** If α is a differential form such that $d\alpha = 0$, $d^c\alpha = 0$ and $\alpha = d\gamma$ for some γ, then $\alpha = dd^c\beta$ for some β.

Proof. Write $\gamma = \gamma_0 + d^c\gamma_1 + (d^c)^*\,\gamma_2$ using $\Omega^{k-1}(M) = \ker \Delta_{d^c} \oplus \mathrm{im}\ (d^c) \oplus \mathrm{im}\ (d^c)^*$ (Theorem 33). So

$$\alpha = d\gamma = d\gamma_0 + dd^c\gamma_1 + d\,(d^c)^*\,\gamma_2.$$

Since $\ker \Delta_{d^c} = \ker \Delta_d = \ker d \cap \ker d^*$, $d\gamma_0 = 0$. On the other hand,

$$0 = d^c\alpha = d^c dd^c\gamma_1 + d^c d\,(d^c)^*\,\gamma_2 = -(d^c)^2 d\gamma_1 - d^c\,(d^c)^*\,d\gamma_2 = -d^c\,(d^c)^*\,d\gamma_2.$$

Thus, $-\,(d^c)^*\,d\gamma_2 = d\,(d^c)^*\,\gamma_2 \in \ker d^c \cap \mathrm{im}\ (d^c)^* = \{0\}$. So $\alpha = dd^c\gamma_1$. Write $\beta = \gamma_1$ and we are done. □

A linear statement like

$$\ker d \cap \ker d^c \cap \mathrm{im}\, d = \mathrm{im}\, dd^c$$

holds over $\mathbb{R}$, then it continues to hold over $\mathbb{C}$. Over $\mathbb{C}$, $dd^c = 2i\partial\overline{\partial}$. This is equivalent to the following lemma:

Lemma 12 ($\partial\overline{\partial}$-Lemma)**.** If (M, g, J, ω) is a Kähler manifold, and if $\alpha \in \Omega^{p,q}(M)$ is d-closed and either ∂ or $\overline{\partial}$-exact, then there exists $\beta \in \Omega^{p-1,q-1}(M)$ such that $\alpha = \partial\overline{\partial}\beta$.

Exercise 14. Prove the $\partial\overline{\partial}$-Lemma. This is much similar to the proof of dd^c-Lemma (Corollary 11). From the Exercise, one can convince himself or herself that the Hodge-Dolbeault decomposition

$$H^k_{\text{dR}}(M) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M)$$

does not depend on the specific Kähler structure of M.

Example 29. Consider $M = \mathbb{C}P^n$. Through induction and the Mayer-Victoris Theorem, one can show

$$H^k_{\text{dR}}(\mathbb{C}P^n) = \begin{cases} \mathbb{C} & \text{, if } 0 \le k \le 2n \text{ for even } k \\ 0 & \text{, otherwise} \end{cases}$$

Let (M, g, J, ω) be the Fubini-Study Kähler structure. We know that $[\omega^\ell] \in H^{2\ell}_{\text{dR}}(M)$ is a nonzero cohomology class for any $0 \le 2\ell \le 2n$. Thus it is a basis of harmonic forms. After complexifying, $\omega^\ell \in \Omega^{p,q}(M)$, we see that

$$H^{p,q}_{\overline{\partial}}(\mathbb{C}P^n) = \begin{cases} \mathbb{C} & \text{, if } p = q \le n \\ 0 & \text{, otherwise} \end{cases}$$

Corollary 12. Let M be a closed Kähler manifold and $m = \dim M$ be its $\mathbb{C}$-dimension. Let $b_k = \dim H^k_{\text{dR}}(M)$, $h^{p,q} = \dim H^{p,q}_{\overline{\partial}}(M)$. We have the following consequences:

1. $b_k = \sum_{p+q=k} h^{p,q}$.
2. $h^{p,q} = h^{q,p} = h^{m-q,m-p} = h^{m-p,m-q}$.
3. $h^{p,p} \neq 0$ for any $p \in \{1, \cdots, n\}$.
4. b_k is even if k is odd.

Proof. $1 \sim 3$ are trivial. 4 follows from 1 and 2. □

With this corollary in hand, we can show that a manifold with a complex structure, and possibly with a symplectic structure, need **NOT** to be Kähler.

Example 30. Take $\mathbb{C}^2\backslash\{0\}/\sim$, where $(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$ for any $\lambda \in \mathbb{Z}\backslash\{0\}$. It is homeomorphic to $S^1 \times S^3$. It inherits a complex structure from $\mathbb{C}^2$, but it is not Kähler. This is because $b_1 = 1$, contradicting to consequence 4 in Corollary 12. Also $b_2 = 0$, so it is not even symplectic!

Exercise 15 (Iwasawa Manifold)**.** Let $G \subset \text{GL}_3(\mathbb{C})$ be the subgroup of matrices of the form

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\Gamma \subset G$ be the subgroup with $z_1, z_2, z_3 \in (\mathbb{Z} + i\mathbb{Z})$. Show that:

1. G/Γ is a $\mathbb{C}$-manifold.
2. The forms $dz_1, dz_2, dz_3 - z_1 dz_2$ on G are invariant under the left action of Γ. So they descend to holomorphic $(1,0)$-forms $\alpha_1, \alpha_2, \alpha_3$ on G/Γ such that $d\alpha_3 = -\alpha_1 \wedge \alpha_2$.

Since $\overline{\partial}^* : \Omega^{p,q}(M) \to \Omega^{p,q-1}(M)$, $\overline{\partial}^* |_{\Omega^{1,0}} \equiv 0$, a $(1,0)$-form being holomorphic is equivalent to it being $\Delta_{\overline{\partial}}$-harmonic. If G/Γ were Kähler, then $\Delta_{\overline{\partial}}\alpha_3 = 0$, yielding $\Delta_d \alpha_2 = 0$, which implies $d\alpha_3 = 0$. Hence G/Γ with the induced complex structure is not Kähler. However, its Betti numbers **DO** satisfy the Corollary 12.

The following identity is important:

Lemma 13 (Lefschetz Identity)**.** $[L, \Lambda] : \Omega^k(M) \to \Omega^k(M)$ satisfies

$$[L, \Lambda]\alpha = (k - m)\alpha,$$

for any $\alpha \in \Omega^k(M)$, where M has $\mathbb{C}$-dimension m.

Proof. Since the identity only involves the value of the metric at a given point, so it suffices to prove it for $\mathbb{C}^m$ with the standard Hermitian metric.

Induction on m. For the base case $m = 1$ with the standard coordinate $z = x + iy$, we have that L acts as $1 \mapsto \omega$ on $\bigwedge^0(\mathbb{C}^m)^*$ and coincides with the zero map otherwise. Likewise, Λ acts as $\omega \mapsto 1$ on $\bigwedge^2(\mathbb{C}^m)^*$ and coincides with the zero map otherwise. Thus, $[L, \Lambda]$ acts as $-\Lambda L = -1$ on $\bigwedge^0(\mathbb{C}^m)^*$ and as $L\Lambda = 1$ on $\bigwedge^2(\mathbb{C}^m)^*$, as needed (zero maps on $\bigwedge^1(\mathbb{C}^m)^*$).

Inductively, suppose $\mathbb{C}^m = W_1 \oplus W_2$ is an orthogonal splitting into two complex subspaces, compatible with the Kähler structure. Then $\bigwedge^\bullet(\mathbb{C}^m)^* = \bigwedge^\bullet W_1^* \otimes \bigwedge^\bullet W_2^*$ with $W_1 \cong \mathbb{C}^{m_1}$, $W_2 \cong \mathbb{C}^{m_2}$, and the symplectic form ω on C^m decomposes into $\omega_1 \oplus \omega_2$, where ω_1 and ω_2 are symplectic forms on W_1, W_2, respectively. Correspondingly, $L = L_1 + L_2$ as operators on $\bigwedge^\bullet(\mathbb{C}^m)^*$, where L_1 acts on $\bigwedge^\bullet W_1^*$ by $L_1 \otimes \mathrm{id}$ and L_2 acts on $\bigwedge^\bullet W_2^*$ by $\mathrm{id} \otimes L_2$. Similarly we can proceed the same with $\Lambda = \Lambda_1 + \Lambda_2$. Let $\alpha \in \bigwedge^\bullet(\mathbb{C}^m)^*$, then we can WLOG suppose that it is split, i.e. $\alpha = \alpha_1 \otimes \alpha_2$ for $\alpha_j \in \Omega^{k_j}(M)$. Thus,

$$\begin{aligned}
[L, \Lambda]\alpha &= [L, \Lambda](\alpha_1 \otimes \alpha_2) \\
&= (L_1 + L_2)(\Lambda_1\alpha_1 \otimes \alpha_2 + \alpha_1 \otimes \Lambda_2\alpha_2) - (\Lambda_1 + \Lambda_2)(L_1\alpha_1 \otimes \alpha_2 + \alpha_1 \otimes L_2\alpha_2) \\
&= [L_1, \Lambda_1]\alpha_1 \otimes \alpha_2 + \alpha_1 \otimes [L_2, \Lambda_2]\alpha_2 \\
&= (k_1 - m_1)\alpha_1 \times \alpha_2 + (k_2 - m_2)\alpha_1 \otimes \alpha_2 \\
&= (k_1 + k_2 - (m_1 + m_2))\alpha_1 \otimes \alpha_2.
\end{aligned}$$

Hence by induction, we conclude our proof. □

Remark 20. Let $\jmath : \Omega^\bullet(M) \to \Omega^\bullet(M)$ satisfy $\jmath\alpha = k\alpha$ for any $\alpha \in \Omega^k(M)$. Then

$$[L, \Lambda] = (\jmath - m)\,\mathrm{id}.$$

We call $\jmath$ a **number operator**.

We can generalize the Lefschetz Identity as follows:

Proposition 15 (Generalized Lefschetz Identity)**.** For $0 \leq k \leq m$ and $0 \leq r \leq m-k$, we have

$$[L^r, \Lambda]\alpha = (r(k-m) + r(r-1))\, L^{r-1}\alpha,$$

for any $\alpha \in \Omega^k(M)$, where M has $\mathbb{C}$-dimension m.

Proof. Inductively, from the fact $[L^r, \Lambda] = L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1}$, we know

$$\begin{aligned}
[L^r, \Lambda]\alpha &= L[L^{r-1}, \Lambda]\alpha + [L, \Lambda]L^{r-1}\alpha \\
&= L\left((r-1)(k-m) + (r-2)(r-1)\right) L^{r-2}\alpha + (2r-2+k-m)L^{r-1}\alpha \\
&= \big((r-1)(k-m) + r^2 - 3r + 2 + (k-m) + 2r - 2\big)\, L^{r-1}\alpha \\
&= (r(k-m) + r(r-1))\, L^{r-1}\alpha.
\end{aligned}$$

The base case was proved in Lemma 13. Hence we conclude the proof. □

Remark 21. Let (M, g, J, ω) be a Kähler manifold of $\mathbb{C}$-dimension m. With generalized Lefschetz Identity in hand, one can show that for $k \leq m$, the map $L^{m-k} : \Omega^k(M) \to \Omega^{2m-k}(M)$ is an isomorphism. (Check!)

6.5 Lefschetz Decomposition

Our current theme is to understand how the three compatible structures of a Kähler manifold (M, g, J, ω) interact with cohomology. We have observed how g will act in Hodge Theorem (Theorem 30, 31 and 33) and how J will act in Hodge-Dolbeault decomposition (Exercise 14). We will see how ω acts (called the Lefschetz decomposition) in this section.

Definition 49. A form α is called **primitive** if it is not in the image of L, i.e. $\alpha \neq \omega \wedge \tilde{\alpha}$ for any $\tilde{\alpha}$. We are interested in those who is not primitive because they come from a lower-degree form.

One can write $\Omega^\bullet = \operatorname{im} L \oplus \ker L^* = \operatorname{im} L \oplus \ker \Lambda$. Thus, a form α is primitive if $\Lambda\alpha = 0$.

Lemma 14. For $k \leq m$, $m = \dim M$, $\alpha \in \Omega^k(M)$ is primitive iff $L^{m-k+1}\alpha = 0$.

Proof. We know $[L^{m-k+1}, \Lambda]\alpha = 0$ by Proposition 15. This implies $L^{m-k+1}\Lambda\alpha = \Lambda L^{m-k+1}\alpha$. Now $\Lambda\alpha \in \Omega^{k-2}(M)$, and $L^{m-(k-2)}$ is an isomorphism on $\Omega^{k-2}(M)$. So $L^{m-(k-1)}$ is injective on $\Omega^{k-2}(M)$, yielding that $L^{m-(k-1)}\Lambda\alpha = 0$ is equivalent to $\Lambda\alpha = 0$. Also, $L^{m-k+1}\alpha \in \Omega^{2m-k+2}(M)$ and $\Lambda^{m-k+2} : \Omega^{2m-k+2}(M) \to \Omega^{k-2}(M)$ is an isomorphism. So Λ is injective on $\Omega^{2m-k+2}(M)$, yielding that $\Lambda L^{m-(k-1)}\alpha = 0$ is equivalent to $L^{m-(k-1)}\alpha = 0$. Hence, α is primitive iff $\Lambda\alpha = 0$, iff $L^{m-k+1}\alpha = 0$. □

Theorem 35 (Lefschetz Decomposition of Differential Forms)**.** Every $\alpha \in \Omega^k(M)$ admits a unique decomposition of the form $\alpha = \sum L^r\alpha_r$ with α_r being of degree $k - 2r \leq \min(2m-k, k)$ and primitive.

Proof. WLOG, we assume $k \leq m$. Start with uniqueness. Suppose $\sum_{r\geq 0} L^r \alpha_r = 0$. We want to show $\alpha_r = 0$. If $\alpha_0 = 0$, then $L\left(\sum L^{r-1}\alpha_r\right) = 0$ implies $\sum L^{r-1}\alpha_r = 0$ and we are done by induction. Now suppose $\alpha_0 \neq 0$. Since $\alpha_0 \in \Omega^k(M)$ and it is primitive, we know $L^{m-k+1}\alpha_0 = 0$. From

$$L^{m-k+1}\left(\sum L^r \alpha_r\right) = 0 = L^{m-k+2}\underbrace{\left(\sum_{r>0} L^{r-1}\alpha_r\right)}_{\text{degree } k-2},$$

and the fact that L^{m-k+2} is an isomorphism on Ω^{k-2}, we know $\sum_{r>0} L^{r-1}\alpha_r = 0$. Induction on k, we get $\alpha_r = 0$ for all $r > 0$, implying $\alpha_0 = 0$. Combining the previous result, we are done.

To prove the existence, first note

$$L^{m-k+1}\alpha \in \Omega^{2m-k+2}(M) = L^{m-k+2}\left(\Omega^{k-2}(M)\right).$$

Thus, there exists $\beta \in \Omega^{k-2}(M)$ such that $L^{m-k+1}\alpha = L^{m-k+2}\beta$. So $\alpha_0 = \alpha - L\beta$ is primitive and $\alpha = \alpha_0 + L\beta$. Induction on degrees, we can assume that β has a Lefschetz decomposition and so does α. □

Remark 22. If $\alpha = \sum L^r \alpha_r$, then we may complexify it into

$$\pi_{p,q}\alpha = \sum L^r \pi_{p-r,q-r}\alpha_r.$$

Lemma 15. On a Kähler manifold (M, g, J, ω), we have $[\Delta_d, L] = 0$.

Proof. We know that $[\partial, L]\alpha = \partial(\omega \wedge \alpha) - \omega \wedge \partial\alpha = \partial\omega \wedge \alpha = 0$ and $[\partial^*, L] = -i\overline{\partial}$. Hence

$$\begin{aligned}[\Delta_d, L] &= 2[\Delta_\partial, L] = 2\left([\partial\partial^*, L] + [\partial^*\partial, L]\right)\\ &= 2\left(\partial[\partial^*, L] + [\partial^*, L]\partial\right) = -2i(\partial\overline{\partial} + \overline{\partial}\partial) = 0.\end{aligned}$$

□

The Lefschetz decomposition (Theorem 35) descends to cohomology. We have the following theorem to describe the consequence:

Theorem 36 (Hard Lefschetz Theorem)**.** Let (M, g, J, ω) be a Kähler manifold of complex dimension m. For all $k \leq m$, L^{m-k} induces an isomorphism $H^k_{\mathrm{dR}}(M) \to H^{2m-k}_{\mathrm{dR}}(M)$.

Proof. Denote $\mathcal{H}^k(M) = \ker \Delta_d \mid_{\Omega^k(M)} \cong H^k_{\mathrm{dR}}(M)$. From Lemma 15, $[\Delta_d, L] = 0$, $L^{m-k} : \mathcal{H}^k(M) \to \mathcal{H}^{2m-k}(M)$ is injective. On the other hand, $\dim \mathcal{H}^k(M) = \dim \mathcal{H}^{2m-k}(M)$ since $\star$ is an isomorphism. Thus, L^{m-k} is also surjective. □

Corollary 13 (Lefschetz Decomposition of Cohomology)**.** If $H^k(M)_{\mathrm{prim}} = \ker L^{m-k+1} \subset H^k(M)$ for $k \leq m$, then for any k, $H^k(M) = \bigoplus_r L^r H^{k-2r}(M)_{\mathrm{prim}}$.

Exercise 16. Prove this corollary.

Remark 23. If $k \leq m$, $b_k \leq b_{k+2}$ ($h^{p,q} \leq h^{p+1,q+1}$, respectively); and if $k \geq m$, $b_k \geq b_{k+2}$ ($h^{p,q} \geq h^{p+1,q+1}$, respectively). Thus $\dim H^k(M)_{\mathrm{prim}} = b_k - b_{k-2}$.

We see that L^{m-k} and $\star$ play similar roles in decomposition and duality. Naturally, we would ask if there is any relationship between these two operators. To state the relation, recall that J extends to a map on differential forms by pulling back along $TM \to T^*M$. After complexifying, $J\mid_{\Omega^{p,q}}$ is multiplication by i^{p-q}. We have the following proposition.

Proposition 16. If $\alpha \in \Omega^k(M)$ is primitive, then

$$\star\frac{L^j\alpha}{j!} = (-1)^{\frac{k(k+1)}{2}}\frac{L^{m-k-j}J(\alpha)}{(m-k-j)!}.$$

Proof. It suffices to see that this holds for $\mathbb{C}^m$ with standard metric.

Induction on m. For $m = 1$, pick an orthonormal basis x, y such that $J(dx) = -dy$. One can check that

$$\begin{aligned} k = 0, j = 0: &\quad \star\alpha = \alpha dx \wedge dy = LJ(\alpha),\\ k = 0, j = 1: &\quad \star L\alpha = \star(\alpha dx \wedge dy) = \alpha = LJ(\alpha),\\ k = 1, j = 0: &\quad \star dx = dy = -J(dx), \quad \star dy = -dx = -J(dy). \end{aligned}$$

Assume inductively that the proposition holds on $W = \mathbb{C}^{m-1}$. Let $V = \mathbb{C}^m = W \oplus \mathbb{C}$. We can write $\bigwedge^\bullet V^* = \bigwedge^\bullet W^* \otimes \bigwedge^\bullet \mathbb{C}^*$, and correspondingly $L = L_1 + L_2$, $\Lambda = \Lambda_1 + \Lambda_2$, where $L_1 = L_W \otimes \mathrm{id}$, $L_2 = \mathrm{id} \otimes L_{\mathbb{C}}$, $\Lambda_1 = \Lambda_W \otimes \mathrm{id}$ and $\Lambda_2 = \mathrm{id} \otimes \Lambda_{\mathbb{C}}$. One can WLOG suppose α is split, i.e. $\alpha = \alpha_1 \otimes \alpha_2$ for $\alpha_1 \in \bigwedge^{k_1} W^*$ and $\alpha_2 \in \bigwedge^{k_2} \mathbb{C}^*$. So

$$\star\alpha = \star(\alpha_1 \otimes \alpha_2) = (-1)^{k_1k_2}(\star_W\alpha_1 \otimes \star_{\mathbb{C}}\alpha_2),$$

and $J(\alpha_1 \otimes \alpha_2) = J(\alpha_1) \otimes J(\alpha_2)$. Any $\alpha \in \bigwedge^k V^*$ decomposes as $\alpha = \beta + \beta' \otimes dx + \beta'' dy + \beta''' \otimes dx \wedge dy$. So

$$\Lambda\alpha = \Lambda_W\beta + \Lambda_W\beta' \otimes dx + \Lambda_W\beta'' dy + \Lambda_W\beta''' \otimes dx \wedge dy + \beta'''.$$

Since α is primitive, $\Lambda\alpha = 0$. This yields $\Lambda_W\beta + \beta''' = 0$ and $\Lambda_W\beta' = \Lambda_W\beta'' = \Lambda_W\beta''' = 0$, i.e. $\beta', \beta'', \beta'''$ are primitive. Since $\Lambda_W^2\beta = 0$, the Lefschetz decomposition of β is $\beta = \gamma + L_W\gamma'$ with γ, γ' primitive, one have

$$\Lambda_W\beta = \Lambda_W L_W\gamma' = [\Lambda_W, L_W]\gamma' = (\dim_C W - (k-2))\,\gamma'.$$

Hence $\beta''' = (k-m-1)\gamma'$. Thus α being primitive implies $\alpha = \gamma + L_W\gamma' + \beta' \otimes dx + \beta'' \otimes dy + (k-m-1)\gamma' \otimes dx \wedge dy$, with $\gamma, \gamma', \beta', \beta''$ primitive. Since $\mathbb{C}$ is 1-dimensional, $L^j = (L_1 + L_2)^j = L_W^j \otimes \mathrm{id} + jL_W^{j-1} \otimes L_{\mathbb{C}}$. So

$$\begin{aligned} L^j\alpha =& L_W^j\gamma + jL_W^{j-1}\gamma \otimes dx \wedge dy + L_W^{j+1}\gamma' + jL_W^j\gamma' \otimes dx \wedge dy\\ &+ L_W^j\beta' \otimes dx + L_W^j\beta'' \otimes dy + (k-m-1)L_W^j\gamma' \otimes dx \wedge dy\\ =& L_W^j\gamma + jL_W^{j-1}\gamma \otimes dx \wedge dy + L_W^{j+1}\gamma' - (m+1-k-j)L_W^j\gamma' \otimes dx \wedge dy\\ &+ L_W^j\beta' \otimes dx + L_W^j\beta'' \otimes dy. \end{aligned}$$

From the inductive hypothesis,

$$(-1)^{\frac{k(k+1)}{2}}(m-k-j)!\,\frac{\star L^j\alpha}{j!} = (m-k-j)L_W^{m-1-k-j}J_W(\gamma)\otimes dx\wedge dy - (j+1)L_W^{m-k-j}J_W(\gamma')\otimes dx\wedge dy$$
$$+ L_W^{m-k-j}J_W(\gamma) + L_W^{m+1-k-j}J_W(\gamma') - L_W^{m-k-j}J_W(\beta')\otimes\star_{\mathbb{C}}dx - L_W^{m-k-j}J_W(\beta'')\otimes\star_{\mathbb{C}}dy \qquad (\#)$$

On the other hand, $L^{m-k-j}J(\alpha)$ can be computed similarly (Exercise!), which coincides (#). By induction, we are done. □

6.6 Hodge-Riemann Bilinear Relations

Definition 50. Define a bilinear form Q on $\Omega^*(M)$ by

1. $Q(\alpha,\beta)=0$ if the degree of α does not equal to the degree of β.
2. If $\alpha,\beta\in\Omega^k(M)$, then

$$Q(\alpha,\beta) = (-1)^{\frac{k(k+1)}{2}}\int_M L^{m-k}(\alpha\wedge\beta) = (-1)^{\frac{k(k+1)}{2}}\int_M \omega^{m-k}\wedge\alpha\wedge\beta.$$

We call Q an **intersection form** on $\Omega^*(M)$.

Lemma 16. 1. Q is symmetric for k even and anti-symmetric for k odd.

2. $Q(L\alpha, L\beta) = -Q(\alpha,\beta)$ for all $\alpha,\beta\in\Omega^k(M)$.
3. The Lefschetz decomposition $H^k(M)=\bigoplus L^rH^{k-2r}(M)_{\text{prim}}$ is orthogonal for Q.

Proof. 1. Follows from $\alpha\wedge\beta=(-1)^k\beta\wedge\alpha$ for any $\alpha,\beta\in\Omega^k(M)$.

2. Note

$$Q(L\alpha,L\beta) = (-1)^{\frac{(k+2)(k+3)}{2}}\int_M L^{m-k-2}(L\alpha\wedge L\beta) = (-1)^{\frac{(k+2)(k+3)}{2}}\int_M \omega^{m-k}\wedge\alpha\wedge\beta.$$

So it suffice to check the sign. Since

$$\frac{(k+2)(k+3)}{2}-\frac{k(k+1)}{2}=\frac{4k+6}{2}=2k+3$$

is odd, $(-1)^{\frac{(k+2)(k+3)}{2}} = -(-1)^{\frac{k(k+1)}{2}}$.

3. Suppose $\alpha=L^r\alpha_0$, $\beta=L^s\beta_0$ with α_0,β_0 primitive and $r<s$. Since $\alpha_0\in\Omega^{k-2s}(M)_{\text{prim}}$, $L^{m-k+2r+1}\alpha_0=0$. We see

$$Q(\alpha,\beta)=Q(L^r\alpha_0,L^s\beta_0)=(-1)^rQ(\alpha_0,L^{s-r}\beta_0)=\pm\int L^{m-k+2r}(\alpha_0\wedge L^{s-r}\beta_0)$$
$$=\pm\int L^{m-k+2r+1}\alpha_0\wedge L^{s-r-1}\beta_0=0.$$

□

Theorem 37 (Hodge-Riemann Bilinear Relations)**.** Let (M, g, J, ω) be a closed Kähler manifold with complex dimension m. We have the following consequences:

1. $H^{p,q}(M)_{\text{prim}}$ and $H^{r,s}(M)_{\text{prim}}$ are orthogonal with respect to Q, unless $(p,q) = (r,s)$.

2. If $\alpha \in H^{p,q}(M)_{\text{prim}}$ is nonzero, then

$$i^{p-q}Q(\alpha, \overline{\alpha}) > 0.$$

   In particular, Q is non-degenerate.

Proof. Note that Q descends to cohomology since, by Stokes Theorem, if α and β are closed and either of them is exact, then

$$\int_M L^{m-k}(\alpha \wedge \beta) = 0.$$

1. If $\alpha \in \Omega^{p,q}(M)$ and $\beta \in \Omega^{r,s}(M)$, then $L^{m-k}(\alpha\wedge\beta)$ has type $(m-k+p+r, m-k+q+s)$, where the volume form has type (m,m). So the integral vanishes, unless $-k+p+r = 0 = -k+q+s$, i.e. $p+r = k = q+s$. But $k = p+q$, yielding $r = q$, $s = p$.

2. If $\alpha \in \Omega^{p,q}(M)$, then $\Lambda\overline{\alpha} = \overline{\Lambda\alpha} = 0$. Then $\overline{\alpha} \in \Omega^{q,p}(M)_{\text{prim}}$. So, $\star\overline{\alpha} = (-1)^{\frac{k(k+1)}{2}} i^{p-q} \frac{L^{m-k}\overline{\alpha}}{(m-k)!}$ by Proposition 16. Observe

$$\begin{aligned} i^{p-q}Q(\alpha,\overline{\alpha}) &= (-1)^{\frac{k(k+1)}{2}} i^{p-q} \int L^{m-k}(\alpha \wedge \overline{\alpha}) \\ &= (-1)^{\frac{k(k+1)}{2}} i^{p-q} \int \alpha \wedge L^{m-k}\overline{\alpha} \\ &= (m-k)!\, \|\alpha\|^2 \geq 0. \end{aligned}$$

   This reveals the desired result and shows that Q is non-degenerate.

□

In other word, the second result of Theorem 37 is to say that the form $i^{p-q}Q$ is positive definite on the complex subspace $H^{p,q}(M)_{\text{prim}}$. The Hodge index theorem is an immediate of Theorem 37. This theorem describes the index (or the signature) of the intersection form Q on $H^m(M,\mathbb{R})$, where $m = \dim_{\mathbb{C}} M$ is assumed to be even and M is closed Kähler. In order to give the statement of the theorem, we first define the index (or the signature) of Q.

Definition 51. Diagonalize Q gives a set of positive and negative eigenvalues. The **signature** of Q is then defined to be the number of positive eigenvalues minus number of negative eigenvalues. That is, one can find a basis $\{\alpha_j\}_{j\leq N}$ such that

$$(Q(a_j, a_\ell))_{j,\ell\leq N} = \begin{pmatrix} \mathrm{id}_r & 0 \\ 0 & -\mathrm{id}_s \end{pmatrix}$$

for some r, s. The signature of Q is then defined to be $r - s$.

Note 5. Note that on $H^m(M)$, Q doesn't use ω or J, and $Q(\alpha,\beta) = \pm\int_M \alpha\wedge\beta$. Define

$$\widetilde{Q}(\alpha,\beta) = \int_M \alpha\wedge\beta.$$

If M is orientable, $\widetilde{Q}$ is non-degenerate since $\widetilde{Q}(\alpha,\star\alpha) = \int_M \alpha\wedge\star\alpha = \|\alpha\|^2 \geq 0$. The **signature** of M is then defined to be the signature of $\widetilde{Q}$, denoted by $\sigma(M)$. It vanishes unless the real dimension of M is a multiple of 4.

Theorem 38 (Hodge Index Theorem)**.** Let (M,g,J,ω) be a closed Kähler manifold with complex dimension m, which is assumed to be even, then

$$\sigma(M) = \sum_{p,q}(-1)^p h^{p,q}.$$

(We defined $h^{p,q}$ in Corollary 12.)

Proof. Extend $\widetilde{Q}$ to a Hermitian form on $H^m(M,\mathbb{C})$, $\widetilde{Q}(\alpha,\beta) = \int\alpha\wedge\overline{\beta}$. We have an orthogonal decomposition $H^m(M,\mathbb{C}) = \bigoplus L^r H^{p,q}(M)_{\text{prim}}$. From the Hodge-Riemann bilinear relations, $(-1)^p\widetilde{Q}$ is positive definite on $L^r H^{p,q}(M)_{\text{prim}}$ (note m is even). Thus

$$\begin{aligned}
\sigma(M) &= \sum_{p+q=m-2r}(-1)^p \dim H^{p,q}(M)_{\text{prim}}\\
&= \sum_{p+q=m-2r}(-1)^p\left(h^{p,q}-h^{p-1,q-1}\right)\\
&= \sum_{p+q=m-2r}(-1)^p h^{p,q} + (-1)^{p-1}h^{p-1,q-1}\\
&= \sum_{p+q=m}(-1)^p h^{p,q} + 2\sum_{\substack{p+q=m-2r\\ r\neq 0}}(-1)^p h^{p,q}\\
&= \sum_{p+q=m}(-1)^p h^{p,q} + \sum_{\substack{p+q \text{ even}\\ p+q\neq m}}(-1)^p h^{p,q}\\
&= \sum_{p+q \text{ even}}(-1)^p h^{p,q}.
\end{aligned}$$

On the other hand, by applying complex conjugation,

$$\sum_{p+q \text{ odd}}(-1)^p h^{p,q} = \sum_{p+q \text{ odd}}(-1)^p h^{q,p} = -\sum_{p+q \text{ odd}}(-1)^q h^{q,p} = 0.$$

Hence,

$$\sigma(M) = \sum_{p,q}(-1)^p h^{p,q}.$$

□

6.7 Cohomology with Holomorphic Coefficients

In this section, we will allow the cohomology to have coefficients in holomorphic vector bundle, instead of $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$.

Definition 52. If $E \to M$ is a $\mathbb{C}$-vector bundle over a Hermitian manifold (M, g, J, ω) of complex dimension m, $p \in M$, and h_E is a Hermitian bundle metric on E inducing $h_E^\flat : E \to E^*$, which is a $\mathbb{C}$-antilinear bundle isomorphism, then we define

$$\bar{\star}_E : \bigwedge^{p,q} T_p^* M \otimes E_p \to \bigwedge^{m-p,m-q} T_p^* M \otimes E_p^*$$

by requiring $\bar{\star}_E(\alpha \otimes s) = \overline{\star \alpha} \otimes h_E^\flat(s)$.

Remark 24. $\bar{\star}_E$ is a $\mathbb{C}$-antilinear isomorphism such that for any $\alpha, \beta \in \Omega^{p,q}(M, E)$

$$\alpha \wedge \bar{\star}_E \beta = h_E(\alpha, \beta) dV_g.$$

Indeed, $\bar{\star}_E \star \bar{\star}_E = (-1)^{p+q}$ on $\bigwedge^{p,q} T_p^* M \otimes E$.

If $(E, h_E) \to M$ is also holomorphic, then we have

$$\begin{aligned} \overline{\partial}_E &: \Omega^{p,q}(M, E) \to \Omega^{p,q+1}(M, E), \\ \overline{\partial}_E^* &: \Omega^{p,q}(M, E) \to \Omega^{p,q-1}(M, E). \end{aligned}$$

Exercise 17. Prove that the Laplacian $\Delta_{\overline{\partial}_E} = \overline{\partial}_E \overline{\partial}_E^* + \overline{\partial}_E^* \overline{\partial}_E$ is elliptic. From this deduce that there is a Hodge decomposition:

$$\Omega^{p,q}(M, E) = \underbrace{\ker \overline{\partial}_E \cap \ker \overline{\partial}_E^*}_{\ker \Delta_{\overline{\partial}_E}} \oplus \operatorname{im} \overline{\partial}_E \oplus \operatorname{im} \overline{\partial}_E^*,$$

and

$$H^{p,q}(M, E) \cong \ker \Delta_{\overline{\partial}_E} |_{\Omega^{p,q}(M,E)} .$$

We can express $\overline{\partial}_E^*$ as $-\bar{\star}_{E^*} \overline{\partial}_{E^*} \bar{\star}_E$. This generalizes $\overline{\partial}^* = - \star \partial \star$, since

$$-\bar{\star} \left(\overline{\partial} \bar{\star} \alpha \right) = -\bar{\star} \left(\overline{\partial} (\overline{\star \alpha}) \right) = -\bar{\star} \left(\overline{\partial \star \alpha} \right) = - \star \partial \star \alpha.$$

Exercise 18. Show that $\bar{\star}_E \Delta_{\overline{\partial}_E} = \Delta_{\overline{\partial}_{E^*}} \bar{\star}_{E^*}$. So we have a $\mathbb{C}$-antilinear isomorphism

$$H^{p,q}(M, E) \xrightarrow{\bar{\star}_E} H^{m-p,m-q}(M, E^*).$$

A better way of thinking about this is that the natural pairing

$$\begin{aligned} H^{p,q}(M, E) \otimes H^{m-p,m-q}(M, E^*) &\to \mathbb{C} \\ (\alpha, \beta) &\longmapsto \int_M \alpha \wedge_E \beta \end{aligned}$$

is non-degenerate since $(\alpha, \bar{\star}_E \alpha) \mapsto \int_M h_E(\alpha, \alpha) dV_g = \|\alpha\|_{h_E}^2$. This puts these vector spaces into duality, i.e. we have a $\mathbb{C}$-linear isomorphism

$$\left(H^{m-p,m-q}(M, E^*) \right)^* \cong H^{p,q}(M, E).$$

This is the Poincaré duality. In this context, it is known as the **Serre duality**.

Remark 25. From the Dolbeault Theorem (Exercise 14), the duality becomes

$$H^q(M, \Omega^p \otimes E) \cong \left(H^{m-q}(M, \Omega^{m-p} \otimes E^*)\right)^*.$$

The sheaf Ω_M^m is known as the **structure sheaf** of M, denoted by K_M. It satisfies

$$H^q(M, E) \cong \left(H^{m-q}(M, K_M \otimes E^*)\right)^*.$$

Recall that, if $E \to M$ is a holomorphic vector bundle, then for each Hermitian metric h_E, there is a unique Chern connection ∇_E on E such that $\nabla_E^{0,1} = \overline{\partial}_E$. We can define

$$d^\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E).$$

There is a form $R^\nabla \in \Omega^2(M, \mathrm{End}(E))$ with $\left(d^\nabla\right)^2 \alpha = R^\nabla \wedge_{\mathrm{End}} \alpha$. Therefore, it is valid to talk about $H^k(M, E)$ only when ∇_E is flat. In particular, if we want to generalize the Dolbeault decomposition, the Lefschetz decomposition and so on, we need to restrict attention to holomorphic vector bundle with a flat Chern connection.

For any holomorphic bundle $E \to M$ with a Hermitian structure over a Kähler manifold M, we can generalize the idea of Lefschetz operator:

$$\begin{aligned} L : \Omega^k(M, E) &\to \Omega^{k+2}(M, E) \\ (\alpha \otimes s) &\mapsto (\omega \wedge \alpha) \otimes s \end{aligned}$$

In the same way, we can generalize the adjoint Λ to $\Omega^k(M, E)$.

Exercise 19. Check that the generalized Lefschetz operator L and its adjoint Λ satisfy

1. L commutes with d^∇, ∂^∇ and $\overline{\partial}_E$.
2. Λ commutes with $\left(d^\nabla\right)^*$, $\left(\partial^\nabla\right)^*$ and $\left(\overline{\partial}_E\right)^*$.
3. $[L, \left(\partial^\nabla\right)^*] = i\overline{\partial}_E$, $[L, \overline{\partial}_E^*] = -i\partial^\nabla$

These properties again imply that

$$[L, \Delta_{\overline{\partial}_E}] = -[L, \Delta_{\partial^\nabla}] = -i\left(\partial^\nabla \overline{\partial}_E + \overline{\partial}_E \partial^\nabla\right).$$

However, this doesn't vanish. Instead, one can check it equals to $R^\nabla \wedge_{\mathrm{End}} -$.

Corollary 14.

$$\Delta_{d^\nabla} = \Delta_{\partial^\nabla} + \Delta_{\overline{\partial}_E}, \quad \Delta_{\overline{\partial}_E} - 2\Delta_{\partial^\nabla} = \left[iR^\nabla \wedge_{\mathrm{End}} -, \Lambda\right].$$

If the Chern connection is flat, then

$$\Delta_{d^\nabla} = 2\Delta_{\partial^\nabla} = 2\Delta_{\overline{\partial}_E}.$$

With this corollary, we can establish the generalized Dolbeault decomposition, Lefschetz decomposition, $\partial^\nabla \overline{\partial}_E$-Lemma and so on, in a similar manner as before. (Exercise!)

In order to obtain a full Hodge theory on holomorphic bundles $E \to M$ over a closed Kähler manifold M, it suffices to find a Hermitian metric on E whose Chern connection is flat. However, there is a topological obstruction here. It comes from the **Bianchi Identity**: $d^\nabla R^\nabla = 0$, where d^∇ uses the induced connection on $\mathrm{End}(E)$. Indeed,

$$\begin{aligned} d^\nabla R^\nabla \wedge \alpha &= d^\nabla \left(R^\nabla \wedge \alpha\right) - R^\nabla \wedge d^\nabla \alpha \\ &= d^\nabla \left((d^\nabla)^2 \alpha\right) - \left(d^\nabla\right)^2 \left(d^\nabla \alpha\right) = \left(d^\nabla\right)^3 \alpha - \left(d^\nabla\right)^3 \alpha = 0. \end{aligned}$$

For a holomorphic bundle $E \to M$, this implies that $\overline{\partial}_{\mathrm{End}(E)} R^\nabla = 0$, where ∇ is a Chern connection. Hence R^∇ defines a cohomology class $[R^\nabla] \in H^{1,1}(M, \mathrm{End}(E))$. If $\widetilde{h}_E$ is a Hermitian metric with Chern connection $\widetilde{\nabla}$, then $\widetilde{\nabla} - \nabla = a \in \Omega^1(M, \mathrm{End}(E))$. Since $\widetilde{\nabla}^{0,1} - \nabla^{0,1} = \overline{\partial}_E - \overline{\partial}_E = 0$, a has type $(1,0)$. Also, $R^{\widetilde{\nabla}} = R^\nabla + \nabla a + a \wedge_{\mathrm{End}} a$ are both $(1,1)$-forms. Thus,

$$\nabla a + a \wedge_{\mathrm{End}} a = \left(\nabla a + a \wedge_{\mathrm{End}} a\right)^{1,1} = \left(\nabla a\right)^{1,1} = \overline{\partial}_{\mathrm{End}(E)} a,$$

yielding $[R^{\widetilde{\nabla}}] = [R^\nabla]$ in $H^{1,1}(M, \mathrm{End}(E))$. This cohomology class is known as the **Atiyah class** of E, denoted by $a(E)$.

A simpler version of the same argument works for line bundles. If $E \to M$ is a $\mathbb{C}$-vector bundle of rank 1, then $\mathrm{End}(E)$ is canonically the trivial $\mathbb{C}$-bundle $M \times \mathbb{C}$. So $R^\nabla \in \Omega^2(M)$ and $d^\nabla R^\nabla = dR^\nabla = 0$ for any ∇. The cohomology class $c_1(E) \coloneqq [R^\nabla] \in H^2(M, \mathbb{C})$ is known as the **first Chern class** of $E \to M$, and it is independent if the choice of ∇. Often, we take coefficients in $\mathbb{Z}$ instead of $\mathbb{C}$; that is, we prefer to define $\frac{1}{2\pi i}[R^\nabla] \in H^2(M, \mathbb{Z})$ as the first Chern class.

Remark 26. Every $\mathbb{C}$-vector bundle $E \to M$ of rank 1 is classified by a map $M \xrightarrow{\phi} \mathbb{C}P^\infty$. This is well-defined up to homotopy. From the knowledge of algebraic topology, we know $H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[x]$, where $x \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$. The first Chern class can be identified as

$$c_1(E) = \frac{1}{2\pi i} \left[R^\nabla\right] = \phi^*[x].$$

In general, for a $\mathbb{C}$-vector bundle $E \to M$ of rank r, $d^\nabla R^\nabla = 0$, $[R^\nabla] \in H^2(M, \mathrm{End}(E))$. If $f : \mathrm{Gr}_r(\mathbb{C}) \to \mathbb{C}$ is a polynomial function such that $f(STS^{-1}) = f(T)$, then

$$f(R^\nabla) \in \Omega^{\mathrm{even}}(M, \mathbb{C}) = \bigoplus_k \Omega^{2k}(M, \mathbb{C}).$$

One can show that $df(R^\nabla) = 0$. So we can show that $[f(R^\nabla)] \in H^{\mathrm{even}}(M, \mathbb{C})$ is independent of the choice of the connection.

Definition 53. These classes are called the **characteristic classes** of E. Define $c_k(x)$ by

$$\sum c_k(x) t^n = \det\left(I_r - t \cdot \frac{x}{2\pi i}\right).$$

Then $c_k(E) = \left[c_k\left(R^\nabla\right)\right] \in H^{2k}(M, \mathbb{C})$ is called the k^{th} **Chern class** of E.

Recall the pullback

$$\begin{array}{ccc} E & \longrightarrow & \gamma^r \\ \big\downarrow & \ulcorner & \big\downarrow \\ M & \xrightarrow{\phi} & \mathrm{Gr}_r \end{array}$$

i.e. $E = \phi^*\gamma^r$. Hence, in terms of ϕ, $c_*(E) = \phi^*\left(c_k(\gamma^r)\right)$. Consequently, $\{c_k(\gamma^r)\}$ generate $H^*(Gr_r)$.

Definition 54. We call a connection ∇_E on E is **compatible with the holomorphic structure** if $\nabla_E^{0,1} = \overline{\partial}_E$. If furthermore, ∇_E sends holomorphic sections to holomorphic sections (possibly locally), then ∇_E is said to be a **holomorphic connection**.

Corollary 15 (Atiyah)**.** $a(E) = 0$ iff $E \to M$ admits a holomorphic connection.

7 Kodaira Embedding Theorem

The aim of this chapter is to prove the following theorem:

Theorem 39 (Kodaira Embedding Theorem)**.** A closed (or compact) Kähler manifold endowed with a positive line bundle admits a projective embedding.

The main technique used is recasting local problems in global ones with the help of "blowing up", namely replacing a point of a complex manifold with a hypersurface. In order to tackle the problem, we must introduce the important notion of "positivity".

7.1 Kodaira Vanishing

Definition 55. We say that a (real) differential form α of type $(1,1)$ on a complex manifold M is **positive**, if for all nonzero v in the real tangent space of M,

$$\alpha\left(v, J(v)\right) > 0,$$

where J is the complex structure of M.

Corollary 16. If α is closed and positive, then by setting $g(\omega_1, \omega_2) = \alpha(\omega_1, J(\omega_2))$, the manifold (M, g, J, α) is Kähler.

Definition 56. A cohomology class $c \in H^{1,1}(M, \mathbb{C})$ is **positive** if it has a positive representation.

Definition 57. A holomorphic line bundle $E \to M$ is **positive** if its first Chern class is positive.

For the start, consider the holomorphic line bundle over $\mathbb{C}P^n$. First note that the tautological bundles have natural Hermitian metrics. Indeed, if $E \to M$ is $\gamma^r(\mathbb{C}^N) \to \mathrm{Gr}_r(\mathbb{C}^N)$, then by definition $\gamma^r(\mathbb{C}^N) = \{(W, v) \in \mathrm{Gr}_r(\mathbb{C}^N) \times \mathbb{C}^N : v \in W\}$ is a subbundle of the trivial bundle $\underline{\mathbb{C}^N} \to \mathrm{Gr}_r(\mathbb{C}^N)$. The standard metric on $\mathbb{C}^N$ induces a bundle metric on $\underline{\mathbb{C}^N}$, which restricts to a bundle metric on $\gamma^r(\mathbb{C}^N) \to \mathrm{Gr}_r(\mathbb{C}^N)$. The Chern connection on $\underline{\mathbb{C}^N}$ is given by d, and the one on γ^r is the projection of d onto γ^r. That is, if $s : U \to \gamma^r$ is a local section, then we can regard it as a map $s : U \to \mathbb{C}^N$ such that $s(w) \in W$, and $\nabla_V^{\gamma^r} s = \pi^{\gamma^r}(ds(V))$.

For the tautological bundle $\mathcal{O}_{\mathbb{C}P^n}(-1) \to \mathbb{C}P^n$, this construction gives a metric $\frac{|dz|^2}{dz_\alpha^2}$ on U_α. Working on the details, let's equip $\mathbb{C}P^n$ with a standard atlas $\{U_\alpha = \{(z_0 : \cdots : z_n) \in \mathbb{C}P^n : z_\alpha \neq 0\}\}$. The tautological line bundle $L = \mathcal{O}(-1)$ has transition functions $g_{\alpha\beta} = \left(\frac{z_\beta}{z_\alpha}\right)^{-1} = \frac{z_\alpha}{z_\beta}$. Let $s : \mathbb{C}P^n \to L$ be a section of L. It can be decomposed into maps $s_\alpha : U_\alpha \to \mathbb{C}$ such that $s_{beta} = g_{\alpha\beta} s_\alpha$. A Hermitian metric on L can similarly be decomposed into maps $h_\alpha : U_\alpha \to \mathbb{R}^+$ such that $h_\beta = |g_{\alpha\beta}|^{-2} h_\alpha$ for which if $p \in U_\alpha$,

$$|s|_h^2(p) = |s_\alpha(p)|^2 h_\alpha(p).$$

It is well-defined. Given $p \in U_\alpha \cap U_\beta$, we see

$$|s_\beta(p)|^2 h_\beta(p) = |g_{\alpha\beta} s_\alpha|^2 |g_{\alpha\beta}|^{-2} h_\alpha = |s_\alpha(p)|^2 h_\alpha(p).$$

Now

$$h_\alpha(z_0 : \cdots : z_n) = |z_\alpha|^{-2}\left(|z_0|^2 + \cdots + |z_n|^2\right)$$

satisfies

$$h_\beta([z]) = |z_\beta|^2|z_\alpha|^{-2}h_\alpha([z]) = |g_{\alpha\beta}|^{-2}h_\alpha([z]).$$

So this defines a Hermitian metric on L. In particular, in the chart $z \mapsto (z_0 : \cdots : \underbrace{1}_{\alpha^{\text{th}}\text{ position}} : \cdots : z_n)$, we have $h_\alpha = 1 + |z|^2 = 1 + z\overline{z}$.

Recall that, locally, the Chern connection associated to h is $\nabla = h^{-1}\partial h$ and the corresponding curvature is $R^\nabla = \overline{\partial}\nabla$. Here we have

$$\nabla = h^{-1}\partial h = \frac{\overline{z}dz}{1 + |z|^2},$$

and the curvature is locally

$$R^\nabla = \overline{\partial}\nabla = \frac{d\overline{z} \wedge dz}{\left(1 + |z|^2\right)^2} = -\frac{dz \wedge d\overline{z}}{\left(1 + |z|^2\right)^2}.$$

This shows that the tautological line bundle $L \to \mathbb{C}P^n$ is not positive.

Remark 27. A metric on $L = \mathcal{O}(-1)$ induces metrics on $\mathcal{O}(-k) = \mathcal{O}(-1)^{\otimes k}$ and $\mathcal{O}(k) = \mathcal{O}(-k)^*$ for all $k > 0$. The corresponding Chern connection has a curvature on $\mathcal{O}(k)$:

$$R_k^\nabla = k \cdot \frac{dz \wedge d\overline{z}}{\left(1 + |z|^2\right)^2},$$

for all $k \in \mathbb{Z}$. Thus $\mathcal{O}(k)$ is positive iff k is positive.

Note 6. For $\mathcal{O}(1)$, $\omega_{FS} = \frac{i}{2\pi}R_1^\nabla$ is the "fundamental form" of the Fubini-Study metric.

Example 31. In general, let $E \to M$ be a Hermitian holomorphic line bundle with the Hermitian metric h. For $p \in M$, let s be a nonzero local frame on a neighborhood U of p in M, then the corresponding curvature is $R^\nabla = \partial\overline{\partial}\log\widetilde{h}$, where $\widetilde{h} = \langle s, s\rangle_h$. This is called the **Chern curvature** of E. Indeed, in a holomorphic trivialization, connections compatible with the holomorphic structure have the form $\nabla^E = d + A$ by Proposition 12, where A is a $(1,0)$-form. The Hermitian structure h is given in the trivialization by a smooth real-valued positive function, which we continue to denote h. Being compatible with the Hermitian metric h amounts to $Ah + h\overline{A} = dh$ which, when combined with the fact that A is of type $(1,0)$, gives $A = \partial\log h$. So $R^\nabla = \overline{\partial}A = \overline{\partial}\partial\log h$.

Lemma 17. If M is a closed Kähler manifold and $c \in H^{1,1}(M, \mathbb{C})$ has a positive representation, then it does not have a negative representation.

Proof. Suppose c has both a positive representation α_1 and a negative representation α_2. By definition, $\omega = \alpha_1 - \alpha_2$ is positive and closed, $d\omega = 0$. ω is then a Kähler form. We have shown in Lecture 5.4 that $\omega^n = n!\text{Vol}(g)$, where g is the Riemannian metric of M. This implies that $\int_M \omega^n > 0$. However, since ω is exact, ω^n is exact. It follows from Stokes' Theorem that $\int_M \omega^n = 0$. Contradiction! $\square$

Lemma 18. If M is Kähler and $E \to M$ is a holomorphic line bundle, then E is positive iff there is a Hermitian metric on E, denoted by h_E, whose curvature of the corresponding Chern connection ∇ satisfies $iR^{\nabla} > 0$.

Proof. Since $c_1 = \left[\frac{i}{2\pi}R^{\nabla}\right]$, it is clear that $iR^{\nabla} > 0$ implies the positivity of E. Conversely, suppose E is positive, then $c_1(E)$ is positive. We can endow M with a Kähler structure (M, g, J, ω) such that $[\omega] \in 2\pi c_1(E)$. Let $\widetilde{h}_E$ be any Hermitian metric on E and $\widetilde{R^{\nabla}}$ be the curvature of its Chern connection. So $\widetilde{R^{\nabla}} = \overline{\partial}\partial \log \widetilde{h}_E(s, s)$, where s is any nonzero local section of E. In particular, $\overline{\partial}\widetilde{R^{\nabla}} = 0$. Now $[\widetilde{iR^{\nabla}}] = 2\pi c_1(E) = [\omega]$, so by $\partial\overline{\partial}$-lemma (Lemma 12), $\omega - \widetilde{iR^{\nabla}} = i\partial\overline{\partial}\Phi$ for some Φ. Let $h = \widetilde{h}e^{\Phi}$, then the curvature of the Chern connection of h satisfies

$$iR^{\nabla} = i\partial\overline{\partial}\log h = \widetilde{iR^{\nabla}} + \left(\omega - \widetilde{iR^{\nabla}}\right) = \omega > 0.$$

□

Proposition 17. Let M be Kähler, $E \to M$ be a positive holomorphic line bundle endowed with h_E such that $i\lambda R^{\nabla} > 0$ for some $\lambda \in \mathbb{R}$. If $(M, g, J, \omega = i\lambda R^{\nabla})$ is the resulting Kähler structure on M, then we have, for $m = \dim_{\mathbb{C}} M$,

$$\left(\Delta_{\overline{\partial}_E} - \Delta_{\partial^{\nabla}}\right)|_{\Omega^{p,q}} = \frac{1}{\lambda}(p + q - m).$$

Exercise 20. Prove this Proposition. Hint: note that

$$\Delta_{\overline{\partial}_E} - \Delta_{\partial^{\nabla}} = [iR^{\nabla} \wedge_{\mathrm{End}} -, \Lambda].$$

The main result for this section is the following:

Theorem 40 (Kodaira Vanishing Theorem)**.** Let M be a closed Kähler manifold and $E \to M$ be a holomorphic line bundle. Denote $m = \dim_{\mathbb{C}} M$, then

1. If E is positive, then $H^{p,q}(M, E) = 0$ for $p + q > m$.

2. If E is negative (i.e. not positive), then $H^{p,q}(M, E) = 0$ for $p + q < m$.

Proof. Serre duality (see Lecture 6.7) implies that case 1 and 2 are equivalent, since E being positive is equivalent to say E^* is negative. It suffices to prove case 2.

Pick a Hermitian metric h_E on E such that the curvature of its Chern connection satisfies $iR^{\nabla} < 0$. Pick the corresponding Kähler structure on M, i.e. $(M, g, J, \omega = -iR^{\nabla})$. If α is a $\Delta_{\overline{\partial}_E}$-harmonic form with coefficients in E of type (p, q), then

$$\left(\Delta_{\overline{\partial}_E} - \Delta_{\partial^{\nabla}}\right)\alpha = -\Delta_{\partial^{\nabla}}\alpha = -(p + q - m)\alpha.$$

Thus

$$\begin{aligned}
(m - p - q)(\alpha, \alpha) &= -\left(\Delta_{\partial^{\nabla}}\alpha, \alpha\right) \\
&= -\left((\partial^{\nabla}(\partial^{\nabla})^* + (\partial^{\nabla})^*\partial^{\nabla})\alpha, \alpha\right) \\
&= -\left[\left(\partial^{\nabla}\alpha, \partial^{\nabla}\alpha\right) + \left((\partial^{\nabla})^*\alpha, (\partial^{\nabla})^*\alpha\right)\right] \\
&= -\left[\|\partial^{\nabla}\alpha\|^2 + \|(\partial^{\nabla})^*\alpha\|^2\right] \le 0.
\end{aligned}$$

Since $(\alpha, \alpha) = \|\alpha\|^2 \geq 0$, either $m - p - q \leq 0$ or $\alpha = 0$. However, $H^{p,q}(M, E) \cong \ker \Delta_{\overline{\partial}_E} |_{\Omega^{p,q}(M,E)}$, the result follows. □

Note 7. A differential form of type (p, q) with coefficients in E is the same as a differential form of type $(0, q)$ with coefficients in $\bigwedge^{p,0} T^*M \otimes E$. Now for $m = \dim_{\mathbb{C}} M$, we define

$$K := \bigwedge^{m,0} T^*M.$$

Corollary 17. If M is closed and $E \to M$ is a holomorphic line bundle with $K^* \otimes E$ being positive, then $H^{0,q}(M, E) = 0$ for any $q > 1$.

Proof. Observe that

$$H^{0,q}(M, E) = H^{0,q}(M, K \otimes K^* \otimes E) = H^{m,q}(M, K^* \otimes E).$$

By Kodaira Vanishing Theorem, this cohomology vanishes when $m + q > m$, which is straightforward since $q > 1$. □

The Kodaira Vanishing Theorem is strong, but sometimes one might find a weaker version is more practicable for use. Let (M, g, J, ω) be a Kähler manifold and (E, h_E) be a Hermitian holomorphic line bundle with Chern connection ∇^E. For each $p \in M$, we can pick a local orthonormal frame of TM, say $(x_1, y_1, \cdots, x_m.y_m)$, with $J(x_j) = y_j$. Define $z_j = \frac{1}{2}(x_j + iy_j)$. So $(z_1, \overline{z_1}, \cdots, z_m.\overline{z_m})$ is a local orthonormal frame of $TM \otimes \mathbb{C}$.

Denote the dual frame of the chosen orthonormal frame by $(\theta_1, \overline{\theta_1}, \cdots, \theta_m, \overline{\theta_m})$. Write

$$R^\nabla = \sum_{j=1}^m a_j \theta_j \wedge \overline{\theta_j}.$$

For a local section σ of E and a differential form $\alpha = \theta_J \wedge \overline{\theta_{J'}} \otimes \sigma$, we have

$$\left[\theta_j \wedge \overline{\theta_j} \wedge -, \Lambda\right] \alpha = \begin{cases} \alpha & \text{, if } j \in J \cap J' \\ -\alpha & \text{, if } j \notin J \cup J' \\ 0 & \text{, otherwise} \end{cases}$$

It follows that

$$\left[iR^\nabla \wedge_{\text{End}} -, \Lambda\right] = \left[\sum_{j \in J \cap J'} a_j - \sum_{j \notin J \cup J'} a_j\right] \alpha.$$

This refines the case $[L, \Lambda]\alpha = p + q - m$, which corresponds to $a_j = 1$ for every j since

$$\text{card}(J \cap J') - (m - \text{card}(J \cup J')) = \text{card}(J \cap J') + \text{card}(J \cup J') - m = \text{card}(J) + \text{card}(J') - m.$$

Theorem 41 (Kodaira Vanishing Theorem, Weak Version)**.** Let (M, g, J, ω) be a closed Kähler manifold and $E \to M$ be a holomorphic line bundle with a Hermitian metric h_E and its corresponding Chern connection R^∇. For every $\varepsilon > 0$ and vector fields V, W, define $\kappa : M \to \mathbb{R}$ by

$$\left|R^\nabla(V, W) + i\kappa\omega(V, W)\right| < \varepsilon |V||W|,$$

then $H^{p,q}(M,E)=0$ for all p,q satisfying

$$\varepsilon\,(m-|p-q|)<\kappa(p+q-m).$$

Proof. Exercise. □

7.2 Kodaira Embedding Theorem

First we restate our main theorem to a form we can get a hand on:

Theorem 42 (Kodaira Embedding Theorem)**.** If M is a closed (or compact) Kähler manifold and $L\to M$ is a holomorphic line bundle over M with Chern connection ∇ whose curvature equals $-i\omega$, then M is projective.

In fact, there is an embedding $\psi : M\to\mathbb{C}P^N$ for some N such that $\psi^*\omega_{FS}=\ell\omega$ for some ℓ. Before we start to tackle the problem, we first introduce an useful tool.

Theorem 43 (Chow)**.** Every closed analytic subset of $\mathbb{C}P^n$ is an algebraic set.

We will not prove the theorem in this course.

Remark 28. The hypothesis is equivalent to saying that M is a Kähler manifold and $[\omega]\in H^{1,1}(M,\mathbb{C})$ is in the image of $H^2(M,\mathbb{Z})$ (after multiplying by $\frac{1}{2\pi}$). This comes from the long exact sequence in the sheaf cohomology

$$\cdots\to H^1(M,\mathcal{O}^\times)\to H^2(M,\mathbb{Z})\to \underbrace{H^2(M,\mathcal{O})}_{\text{equal to } H^{0,2}(M)}\to\cdots.$$

This tells us that $\left[\frac{\omega}{2\pi}\right]$ is in the image of $H^1(M,\mathcal{O}^\times)$. This classifies holomorphic line bundles.

Definition 58. We say that a Kähler manifold M is **Hodge** if $\left[\frac{\omega}{2\pi}\right]$ is integral; that is, in the image of $H^2(M,\mathbb{Z})$.

So Kodaira embedding is equivalent to that M is Hodge iff it is a projective algebraic variety.

Remark 29. We know that $V_N=H^{0,0}(M,L^{\otimes N})$ is a finite-dimensional complex vector space. The proof of Kodaira embedding theorem proceed as follows:

Step 1. For $N\gg 0$, $L^{\otimes N}$ is basepoint-free. That is, for any $p\in M$, there is a global holomorphic section of $L^{\otimes N}$ that does not vanish at p. Given this, we can define a map $M\xrightarrow{e_N}\mathbb{P}(V_N)$, where $\mathbb{P}(V_N)$ is the projective space of V_N. Namely, choose a basis $s_1,\cdots,s_\ell$ of V_N and set $e_N(p)=[s_1(p):\cdots:s_\ell(p)]\in\mathbb{P}(V_N)$. Since $L^{\otimes N}$ has rank 1, this map is well-defined and independent of the choice of basis.

Step 2. Show that e_N is injective for $N\gg 0$ by showing that for any $x,y\in M$, there is a global holomorphic section of $L^{\otimes N}$, denoted by s, such that $s(x)\neq s(y)$.

Step 3. Show that e_N has injective derivative for $N\gg 0$, i.e. global holomorphic sections of $L^{\otimes N}$ separate tangent vectors.

The classical proof establishes these steps by using the Kodaira vanishing theorem. However, we will discuss an approach of Donaldson who proved something similar for symplectic manifolds.

Let (M, g, J, ω) be a symplectic manifold, where J is a compatible almost complex structure so that $\omega(J(u), J(v)) = \omega(u, v)$, and $g(u, v) = \omega(u, J(v))$ is a Riemannian metric. Let $L \to M$ be a complex line bundle with covariant derivative ∇ whose curvature is $-i\omega$. First, we will consider the trivial line bundle $\xi \to \mathbb{C}^n$. Here the standard symplectic form is $\omega_0 = \frac{i}{2}\sum dz_j \wedge d\overline{z_j}$, which is exact, i.e. $\omega_0 = idA_0$ for $A_0 = \frac{1}{4}\sum z_j d\overline{z_j} - \overline{z_j}dz_j$. Let ∇^ξ be the connection $d + A_0$. The curvature of ∇^ξ is $R^\nabla = dA_0 + A_0 \wedge A_0 = dA_0 = -i\omega_0$. Since this has type $(1,1)$, we see that $-\nabla^\xi$ induced a holomorphic structure on ξ. The $\overline{\partial}^\xi$ operator is

$$\overline{\partial}^\xi(f) = \overline{\partial} f + A_0^{0,1} f.$$

Consider the section $\sigma(z) = \exp\left(-\frac{1}{4}|z|^2\right) = \exp\left(-\frac{1}{4}z \cdot \overline{z}\right)$. This satisfies

$$\overline{\partial}\sigma = \sum \left(-\frac{1}{4}z_j \exp\left(-\frac{1}{4}|z|^2\right)\right) d\overline{z_j} = -A_0^{0,1}\sigma,$$
$$\partial\sigma = \sum \left(-\frac{1}{4}\overline{z_j} \exp\left(-\frac{1}{4}|z|^2\right)\right) dz_j = A_0^{1,0}\sigma.$$

Hence $\overline{\partial}^\xi\sigma = 0$ and $\partial^\nabla\sigma = 2A_0^{1,0}\sigma$. In particular we have a holomorphic section of ξ with exponential decay.

For the line bundle $\xi^{\otimes k} \to \mathbb{C}^n$, the connection ∇^ξ induced a connection $\nabla^{\otimes k} = d + A_0^{(k)} = d + kA_0$ with the curvature $-ik\omega_0$, and the section $\sigma_k(z) = \exp\left(-\frac{1}{4}|z|^2\right)$ is holomorphic. We want to think of this as $\exp\left(-\frac{1}{4}|k^{1/2}z|^2\right)$, and as "taking the k^{th} tensor power of ξ locally has the same effect as scaling the coordinates by $k^{-1/2}$", or "working with $\xi \to B_{k^{-1/2}}(0)$ is like working with $\xi^{\otimes k} \to B_1(0)$".

On an arbitrary symplectic manifold, it is possible to choose local coordinates around any point $p \in U$ such that $\omega |_U = \omega_0$ (known as the Darboux coordinates). However, in these coordinates $J = J_0 + O(|z|)$. Intuitively, the closer we are to the origin of the coordinate chart, the more (M, g, J, ω) looks like $(\mathbb{C}^n, g_0, J_0, \omega_0)$.

Example 32. Closed oriented surfaces are Hodge manifolds. To see this, first note that these surfaces are Kähler manifold. $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ is a lattice inside $H^2(M, \mathbb{R}) \cong \mathbb{R}$. So by scaling g if needed, we can rearrange for $\left[\frac{\omega}{2\pi}\right]$ to be in the image of $H^2(M, \mathbb{Z})$. Obviously, the same argument works for any Kähler manifold with second Betti number $b_2 = 1$.

Definition 59. Let (M, g, J, ω) be a symplectic manifold, and $L \to M$ be a complex line bundle with connection ∇. A sequence of sections $s_k \in C^\infty(M, L^{\otimes k})$ is:

1. **uniformly bounded**, if for all $r \in \mathbb{N}$, there exists a C_r such that $\sup_{p\in M} |\nabla^r s_k|_g \leq C_r k^{r/2}$;
2. **approximately holomorphic**, if for all $r \in \mathbb{N}$, there exists a C_r such that $\sup_{p\in M} |\nabla^{r-1}\overline{\partial} s_k|_g \leq C_r k^{(r-1)/2}$;

3. **uniformly concentrated at** $p \in M$, if for all $r \in \mathbb{N}$, there exists a polynomial P and a constant $\lambda > 0$ such that for every $q \in M$,

$$\sup_{0 \le t \le r} |k^{-t/2} \nabla^t s_k(q)| \le P\left(\sqrt{k} d(p,q)\right) \exp\left(-\lambda k d(p,q)^2\right).$$

Proposition 18. Let (M, g, J, ω) be a closed symplectic manifold and $L \to M$ be a $\mathbb{C}$-line bundle with connection ∇ whose curvature is $R^\nabla = -i\omega$.

1. For each $p \in M$, there exists a sequence of sections $s_{k,p} \in C^\infty(M, L^{\otimes k})$ which is uniformly bounded, approximately holomorphic and uniformly concentrated at p, satisfying $|s_{k,p}| \ge c > 0$ over $B_{k^{-1/2}}(p)$.
2. If in addition, ω is the symplectic form of a Kähler structure on M, then there exists a sequence $\widetilde{s_{k,p}}$ of holomorphic sections such that

$$\sup\left(k^{r/2} |\nabla^r s_{k,p} - \nabla \widetilde{s_{k,p}}|\right) \le C e^{-\lambda k/3}.$$

Proof. Assuming that (M, g, J, ω) is a Kähler manifold. Fix $p \in M$ and choose holomorphic coordinates centered at p such that $g_{jk} = \delta_{jk} + O(|z|)$.

1. Let u be a local holomorphic section of L with $u(p) = 1$. We have $-i\omega = R^\nabla = \overline{\partial}\partial \log |u|^2$ near p. Near p,

$$\log |u|^2 = \sum_j a_j z_j + \overline{a_j z_j} + \sum b_{jk} z_j \overline{z_k} + c_{jk} z_j z_k + \overline{c_{jk} z_j z_k} + O\left(|z|^3\right).$$

   Replacing u with $\exp\left(-\sum a_j z_j - \sum c_{jk} z_j z_k\right) u$ preserves holomorphicity and satisfies $\log |u|^2 = \sum b_{jk} z_k \overline{z_k} + O(|z|^3)$. Then $\overline{\partial}\partial \log |u|^2 = -i\omega$, implying $b_{jk} = -\frac{1}{2}$ for any j, k (metric tensor on T_pM). So $\log |u|^2 = -\frac{1}{2}|z|_g^2 + O(|z|^3)$. Using this section, we obtain local holomorphic sections of $L^{\otimes k}$ satisfying $|u^{(k)}| = \exp\left(-\frac{k}{4}|z|^2 + kO(|z|^3)\right)$. Estimating the growth of derivatives of $\log |u|^2$ gives us uniform concentratedness as long as $|z| \ll 1$, which is sufficient since we care about $u^{(k)}$ in a ball of radius $k^{-1/3}$ and 0 outside the ball of radius $2k^{-1/3}$.

   Since the cut-off occurs in the region where $|z| \sim k^{-1/3}$, we have $|u^{(k)}| \sim \exp\left(-k\frac{|z|^2}{4}\right) \sim \exp\left(-k^{1/3}\right)$. Thus we get

$$\sup |\overline{\partial} s_{k,p}| = \sup |u^{(k)} \overline{\partial} x_k| \le O\left(\exp(-\lambda k^{1/3})\right),$$

   since $\overline{\partial} x_k \equiv 0$ except for $|z| \sim k^{-1/3}$ and $|\overline{\partial} x_k| \le O(k^{1/3})$.

2. Before we prove the second part, we will need the following lemma:

   Lemma 19. Given $\sigma \in C^\infty(M, L^{\otimes k})$, there exists $\xi \in C^\infty(M, L^{\otimes k})$ such that $\|\xi\|_{L^2} \le \frac{c}{\sqrt{k}} \|\overline{\partial}\sigma\|_{L^2}$, and $\sigma + \xi$ is holomorphic.

Apply the lemma to $s_{k,p}$. Noticing that these are each supported in a ball of volume tending to $k^{-2n/3}$, we can find $\xi_{k,p}$ satisfying

$$\|\xi_{k,p}\|_{L^2} \le \frac{c}{\sqrt{k}}\|\overline{\partial} s_{k,p}\|_{L^2} \le O\left(k^{-2n/3-1/2}\exp\left(-\lambda k^{-1/3}\right)\right).$$

Through Cauchy's formula for holomorphic functions ans these L^2-estimates, one can get pointwise estimates. At each point inside $B_{k^{-1/3}}(p)$, $x \neq 1$, so $s_{k,p}$ is holomorphic there, so is $\xi_{k,p}$. $\|\xi_{k,p}\|_{C^r}$ is controlled by $\|\xi_{k,p}\| \sim \exp(-\lambda k^{1/3})$. Finally, outside this ball we can use the same argument to see that $\|s_{k,p}+\xi_{k,p}\|$ is controlled by $\exp(-\lambda k^{1/3})$.

□

It remains to prove Lemma 19.

Proof. Start by recalling that for any holomorphic line bundle $E \to M$ with Chern connection ∇, we have $\Delta_{\overline{\partial}_E} - 2\Delta_{\partial^\nabla} = \left[iR^\nabla \wedge_{\text{End}} -, \Lambda\right]$ on $\Omega^{0,q}(M,E)$ (Corollary 14). When working with $\Omega^{0,q}(M,E)$, we can use $E \cong \underline{K} \otimes \underline{K^*} \otimes E$, where $\underline{K} = \Omega^{n,0}(M)$, to get an isomorphism

$$\varphi : \Omega^{0,q}(M,E) \xrightarrow{\cong} \Omega^{n,q}(M, \underline{K}^* \otimes E).$$

Since $\tilde{E} \coloneqq \underline{K}^* \otimes E$ is also a holomorphic line bundle, we have

$$\Delta_{\overline{\partial}_{\tilde{E}}} - 2\Delta_{\partial^{\tilde{\nabla}}} = \left[iR^{\tilde{\nabla}} \wedge_{\text{End}} -, \Lambda\right]. \qquad (*)$$

But if we only look at the image of φ, then we only care about this identity on forms of type $(n,*)$. One can show that

$$\begin{aligned}
&\varphi^{-1}\Delta_{\overline{\partial}_{\tilde{E}}}\varphi = \Delta_{\overline{\partial}_E},\\
&\varphi^{-1}\Delta_{\partial^{\tilde{\nabla}}}\varphi \ge 0,\\
&\varphi^{-1}\left[iR^{\tilde{\nabla}} \wedge_{\text{End}} -, \Lambda\right]\varphi = \sum R^{\tilde{\nabla}}\left\langle e_j, \overline{e_k}\right\rangle \text{ext}(e^k)\text{int}(\overline{e_j}) =: \mathfrak{H}^{\tilde{\nabla}},
\end{aligned}$$

where $\{e_j\}$ is a local orthonormal frame of $T^{1,0}M$ and $\{e^j\}$ is the dual frame. Conjugating $(*)$ by φ, we conclude that for any $\alpha \in \Omega^{0,q}(M,E)$, we have

$$\left\langle \Delta_{\overline{\partial}_E}\alpha, \alpha\right\rangle = 2\left\langle \varphi^{-1}\Delta_{\partial^{\tilde{\nabla}}}\varphi\alpha, \alpha\right\rangle + \left\langle \mathfrak{H}^{\tilde{\nabla}}\alpha, \alpha\right\rangle \ge \left\langle \mathfrak{H}^{\tilde{\nabla}}\alpha, \alpha\right\rangle.$$

Note that id $E = L^{\otimes k}$, then this says that

$$\left\langle \Delta_{\overline{\partial}_E}\alpha, \alpha\right\rangle \ge \left\langle \mathfrak{H}^{\nabla_L}\alpha, \alpha\right\rangle + \left\langle \mathfrak{H}^{\nabla_{\underline{K}^*}}\alpha, \alpha\right\rangle.$$

So if M is closed (or compact) and L is positive, we conclude that there exists some $C_0, C_1 > 0$, such that

$$\left\langle \Delta_{\overline{\partial}_k}\alpha, \alpha\right\rangle \ge (kC_0 - C_1)\|\alpha\|_{L^2}^2,$$

for any $\alpha \in \Omega^{0,q}(M, L^{\otimes k})$ and $q > 0$. For $k \gg 0$, we see that $\Delta_{\overline{\partial}_k}$ is invertible on its domain in L^2, and its inverse G_k has L^2-norm $O(1/k)$. Finally, given $s \in C^\infty(M, L^{\otimes k})$, set $\xi = -\overline{\partial}_k^* G_k \overline{\partial}_k s$. Then

1. $s+\xi$ is holomorphic, since

$$\begin{aligned}\overline{\partial}_k(s+\xi) &= \overline{\partial}_k s - \overline{\partial}_k\overline{\partial}_k^* G_k\overline{\partial}_k s\\ &= \overline{\partial}_k s - \left[\left(\Delta_{\overline{\partial}_k} - \overline{\partial}_k^*\overline{\partial}_k\right) G_k\overline{\partial}_k s\right]\\ &= \overline{\partial}_k^*\overline{\partial}_k G_k\overline{\partial}_k s \in \operatorname{im}\overline{\partial}_k \cap \operatorname{im}\overline{\partial}_k^* = \{0\}.\end{aligned}$$

2. The L^2-norm of ξ is given by

$$\begin{aligned}\|\xi\|_{L^2}^2 &= \left\langle \overline{\partial}_k^* G_k\overline{\partial}_k s, \overline{\partial}_k^* G_k\overline{\partial}_k s\right\rangle\\ &= \left\langle \overline{\partial}_k\overline{\partial}_k^* G_k\overline{\partial}_k s, G_k\overline{\partial}_k s\right\rangle\\ &= \left\langle \overline{\partial} s, G_k\overline{\partial}_k s\right\rangle\\ &\leq \|G_k\|\cdot\|\overline{\partial}_k s\| \leq \frac{C}{k}\|\overline{\partial}_k s\|_{L^2}^2.\end{aligned}$$

Now we finish the proof of the lemma. □

Remark 30. Using the sections $\widetilde{s_{k,p}}$, one can prove Kodaire embedding following the steps in Remark 29.

8 Overview on L^2-Hodge Theory

This chapter is subject to introduce the Hodge decomposition for the L^2-differential forms. Note that differential operators are not a priori definition on L^2, so the first thing is to describe how to make a such definition. This leads us to the conception of Sobolev spaces.

8.1 Sobolev Spaces

The key of Sobolev spaces is the "weak derivatives" of a function. We present the formal definition to this terminology.

Definition 60. If $f \in C^1(\mathbb{R}, \mathbb{C})$, then for any $\varphi \in C_c^\infty(\mathbb{R})$, we have

$$\int \varphi(\xi)\overline{f'(\xi)}d\xi = -\int \varphi'(\xi)\overline{f(\xi)}d\xi.$$

So we can identify $f'(\xi)$ with the functional $\Lambda : C_c^\infty(\mathbb{R}) \to \mathbb{C}$, sending φ to $-\int \varphi'(\xi)\overline{f(\xi)}d\xi$. We call this functional Λ the **weak derivative** of f.

The advantage of using this functional expression is that it makes sense for f which is not differentiable in the common sense.

Remark 31. If the functional Λ extends to all $\varphi \in L^2$, then by Riesz Representation, there exists $h \in L^2$ such that

$$\Lambda(\varphi) = \langle \varphi, h\rangle = \int \varphi(\xi)\overline{h(\xi)}d\xi.$$

That is, for all $\varphi \in C_c^\infty$, there exists $h \in L^2$ such that

$$-\int \varphi'(\xi)\overline{h(\xi)}d\xi = \int \varphi(\xi)\overline{h(\xi)}d\xi.$$

In this case, we identify Λ and h, and call h the **weak derivative** of f.

Example 33. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$. If $\varphi \in C_c^\infty(\mathbb{R})$, then the weak derivative of f applied to φ is

$$\begin{aligned} -\int \varphi'(x)f(x)dx &= -\int_{-\infty}^0 \varphi'(x)\cdot(-x)dx - \int_0^\infty \varphi'(x)\cdot xdx \\ &= -\int_\infty^0 \varphi(x)dx + \int_0^\infty \varphi(x)dx \\ &= \int_{-\infty}^\infty \varphi(x)\cdot \operatorname{sign}(x)dx, \end{aligned}$$

where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{, if } x > 0 \\ -1 & \text{, if } x < 0 \end{cases}$$

We can keep going and find the weak derivative of $\operatorname{sign}(x)$ is the functional $\Lambda : C_c^\infty(\mathbb{R}) \to \mathbb{C}$, sending φ to $-\int \varphi'(x)\cdot \operatorname{sign}(x)dx$. Now

$$\Lambda(\varphi) = \int_{-\infty}^0 \varphi'(x)dx - \int_0^\infty \varphi'(x)dx = 2\varphi(0) =: 2\int_\infty^\infty \varphi(x)\delta_0(x)dx,$$

where $\delta_0(x)$ is a formal definition, called the **Dirac-δ function**. We need to point out that this definition is not an accurate description - δ_0 is a functional rather than a function. For reasons why it got a name "function", readers are encouraged to look in Wikipedia to find some historical anecdotes.

Definition 61. The functionals on $C_c^\infty(\mathbb{R})$ are said to be **distributions**.

Definition 62. If $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, i.e. u is a locally integrable function, and α be a multi-index, we say that $v \in L^1_{\text{loc}}(\mathbb{R}^n)$ is the **weak** (or **distributive**) $\alpha^{\mathbf{th}}$ **partial derivative** of u if, for all $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} D^\alpha \varphi \overline{u} = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \varphi \overline{v}.$$

Definition 63. The $k^{\mathbf{th}}$ **Sobolev space**, for $k \in \mathbb{N}$, is

$$H^k(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : D^\alpha f \in L^2(\mathbb{R}^n) \text{ for all multi-index } \alpha, |\alpha| \leq k\}. \qquad (*)$$

Using the Fourier transform

$$\mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx,$$

an equivalent definition to $(*)$ is

$$H^k(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : p(\xi)\mathcal{F}(f)(\xi) \in L^2(\mathbb{R}^n) \text{ for all polynomial } p, \deg p \leq k\}. \qquad (\#)$$

Proposition 19. Sobolev spaces are Hilbert spaces with respect to

$$\langle f, g \rangle_{H^k} = \sum_{|\alpha| \leq k} (D^\alpha f, D^\alpha g),$$

for all $f, g \in H^k(\mathbb{R}^n)$. An equivalent Hilbert space structure is given by

$$\langle f, g \rangle_{H^k} = \int_{\mathbb{R}^n} \mathcal{F}(f)(\xi) \overline{\mathcal{F}(g)(\xi)} \left(1 + |\xi|^2\right)^{k/2}.$$

Proof. Exercise. □

Remark 32. The inner product via integrals (second Hilbert space structure in the previous Proposition) makes sense for $k \in \mathbb{R}$. One can prove that the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the resulting norm is $H^k(\mathbb{R}^n)$ for $k \in \mathbb{R}$.

If M is a compact manifold and $E \to M$ is a vector bundle, we choose a Riemannian metric on M and a Hermitian metric on E. One can use these to define an L^2-inner product on sections of E. The topological space $L^2(M, E)$ is independent of these choices. We can now define the **Sobolev spaces** on (M, E), denoted by $H^s(M, E)$, in the two equivalent ways:

1. Pick a finite cover of charts trivializing E with subordinate partition of unity $\{x_j\}$ and declare $u \in H^{k(M,E)}$ if $x_j u \in H^k(\mathbb{R}^n, C^r)$ for any j. The norm is defined by $\|u\|^2_{H^k} = \sum \|x_j u\|^2_{H^k}$.

2. Pick a metric connection ∇_E and take $\|u\|^2_{H^k} = \sum_j \int \|\nabla^j_E u\|^2 dV_g$ for $k \in \mathbb{N}$.

So it is clear that $H^k(M, E) \subset H^{k'}(M, E)$ for $k \geq k'$.

Theorem 44 (Rellich–Kondrachov)**.** If M is compact, then the inclusion $H^k(M,E) \hookrightarrow H^{k'}(M,E)$ is a compact operator whenever $k > k'$.

Recall the **Fourier inversion** says that if

$$\mathcal{F}^*(u)(x) = \frac{1}{(2\pi)^{n/2}} \int u(\xi) e^{ix\xi} d\xi,$$

then $\mathcal{F}^*$ is the inverse of $\mathcal{F}$ (and its adjoint, since $\mathcal{F}$ is self-adjoint) as maps between $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Here $\mathcal{S}(\mathbb{R}^n)$ is the is the **Schwartz space**, that is,

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty, \alpha, \beta \in \mathbb{N}^n\},$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $D^\beta = \partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}$, for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Let $\mathcal{S}'(\mathbb{R}^n)$ be the dual space of $\mathcal{S}(\mathbb{R}^n)$. Elements in $\mathcal{S}'(\mathbb{R}^n)$ are known as the **tempered distributions**.

For any two functions $f, g \in \mathcal{S}(\mathbb{R}^n)$, the L^2-pairing satisfies $\langle \mathcal{F}(f), h\rangle = \langle f, \mathcal{F}^*(h)\rangle$. So if Λ is a tempered distribution, we define $\mathcal{F}(\Lambda)$ to be the tempered distribution $\mathcal{F}(\Lambda)(f) = \Lambda(\mathcal{F}^*(f))$. Thus if Λ is given by $\Lambda(h) = \langle h, f\rangle$, then

$$\mathcal{F}(\Lambda)(h) = \Lambda(\mathcal{F}^*(h)) = \langle \mathcal{F}^*(h), f\rangle = \langle h, \mathcal{F}(f)\rangle .$$

Similarly extending $\mathcal{F}^*$ to tempered distributions, we see that $\mathcal{F}^*\mathcal{F} = \mathrm{id} = \mathcal{F}\mathcal{F}^*$ holds on distributions.

Remark 33. If $s > \frac{n}{2}$, then $H^s(\mathbb{R}^n) \subset C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. This is because for any $f \in H^s(\mathbb{R}^n)$,

$$\begin{aligned} |f(x)| &= \left| \frac{1}{(2\pi)^{n/2}} \int \mathcal{F}(f)(\xi) e^{ix\xi} d\xi \right| \\ &\le \frac{1}{(2\pi)^{n/2}} \int |\mathcal{F}(f)(\xi)|\, d\xi \\ &\le \frac{1}{(2\pi)^{n/2}} \int |\mathcal{F}(f)(\xi)|^2 \left(1 + |\xi|^2\right)^s d\xi \int \left(1 + |\xi|^2\right)^{-s} d\xi, \end{aligned}$$

where $\int \left(1 + |\xi|^2\right)^{-s} d\xi$ is finite precisely when $s > \frac{n}{2}$. Similarly, if $s > \frac{n}{2} + k$ for $k \in \mathbb{N}$, then $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$. In particular,

$$\bigcap_s H^s(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n).$$

8.2 Pseudo-differential Operators

Suppose $P \in \mathrm{Diff}^\ell(\mathbb{R}^n)$, with

$$Pf(\xi) = \sum_{|\alpha|\le\ell} a_\alpha(\xi) D^\ell f.$$

We use the convenient notation $D_x = \frac{1}{2}\partial_x$ (similar for others) as we end up with fewer signs when we integrate by parts or intertwine with Fourier transform. Let $p(x,y) = \sum_{|\alpha|\le\ell} a_\alpha(x) y^\alpha$ and $\sigma(P) = \sum_{|\alpha|=\ell} a_\alpha(x) y^\alpha$. We can write P as an iterated integral

$$\begin{aligned} Pf(x) &= \frac{1}{(2\pi)^{n/2}} \int e^{ixy} p(x,y) \mathcal{F}(f)(y) dy \\ &= \frac{1}{(2\pi)^n} \int e^{i(x-z)y} p(x,y) f(z) dz dy. \end{aligned}$$

From which it is clear that P defines a continuous map $H^k(\mathbb{R}^n) \to H^{k-\ell}(\mathbb{R}^n)$ for any k. Similarly on a manifold, $P \in \mathrm{Diff}^\ell(M; E, F)$ defines map $H^k(M, E) \to H^{k-\ell}(M, F)$.

Remark 34. Alternatively, on $\mathbb{R}^n$, we can express

$$\begin{aligned} Pf(x) = \mathcal{F}^*\mathcal{F}(Pf) &= \mathcal{F}^* \left[\frac{1}{(2\pi)^{n/2}} \int e^{-izy} \sum_{|\alpha| \leq \ell} a_\alpha(z) D^\alpha f(z) dz \right] \\ &= \mathcal{F}^* \left[\frac{1}{(2\pi)^{n/2}} \int e^{-izy} \sum_{|\alpha| \leq \ell} b_\alpha(z) y^\alpha f(z) dz \right] \\ &= \frac{1}{(2\pi)^n} \int e^{i(x-z)y} p'(z, y) f(z) dz dy. \end{aligned}$$

In these expression, $p(x, y), p'(z, y)$ are smooth in x, y, z, especially are polynomials of degree ℓ in y.

Definition 64. An **amplitude of order** ℓ is a function $a(x, y, z)$ which is smooth in all variables and satisfying that, for every multi-indices α, β, γ, there exists constant $C_{\alpha,\beta,\gamma}$ such that

$$\sup |D_x^\alpha D_y^\beta D_z^\gamma a(x, y, z)| \leq C_{\alpha,\beta,\gamma} \left(1 + |z|\right)^{\ell - |\gamma|}.$$

Even better, we will demand that, at the diagonal $x = y$, a has an asymptotic expression in z, namely

$$a(x, y, z) \sim \sum_{j=0}^{\infty} a_{(\ell-j)}(x, z),$$

when $|z| \to \infty$ at $x = y$, and $a_{(\ell-j)}(x, z)$ is homogeneous in z of degree $\ell - j$. The leading term $a_{(\ell)}(x, z)$ is known as the **principal symbol**.

Definition 65. An operator of the form

$$Af(x) = \frac{1}{(2\pi)^n} \int e^{i(x-z)y} a(x, z, y) f(z) dz dy$$

is called a **pseudo-differential operator of order** ℓ, where $a(x, z, y)$ is an amplitude of order ℓ.

Write $\Psi^\ell(\mathbb{R}^n)$ for the space of pseudo-differential operators of order ℓ. If $A \in \Psi^\ell(\mathbb{R}^n)$, then A defines a continuous map

$$H^k(\mathbb{R}^n) \to H^{k-\ell}(\mathbb{R}^n)$$

for all k. Denote $\Psi^{-\infty} = \bigcap_\ell \Psi^\ell$. These are precisely the **smoothing operators**, i.e. operators of the form

$$Af(x) = \int \kappa_A(x, y) f(y) dy,$$

where $\kappa_A \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. This defines a continuous map $H^k(\mathbb{R}^n) \to \bigcap_{k'} H^{k'}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$.

Remark 35. If M is a closed manifold with $E, F \to M$ being the vector bundles over M, then an operator $A : C^\infty(M, E) \to C^\infty(M, F)$ is in $\Psi^\ell(M; E, F)$, if for some atlas and subordinate partition of unity $\{x_j\}$, we have $x_j A x_j \in \Psi^\ell(\mathbb{R}^n) \otimes \mathrm{Hom}(E, F)$ for any j, with $x_j A x_k$ being a smoothing operator whenever $j \neq k$. The principal symbol of $A \in \Psi^\ell(M; E, F)$ is well-defined as a section of $\mathrm{Hom}(E, F)$ pulled back to T^*M.

Suppose $A \in \Psi^{\ell}(\mathbb{R}^n)$ and $B \in \Psi^{\ell'}(\mathbb{R}^n)$, where

$$\begin{aligned} Af(x) &= \int e^{ixy} a(x,y)\mathcal{F}(f)(y)dy, \\ \mathcal{F}(Bf)(y) &= \int e^{-izy} b(z,y) f(z) dz. \end{aligned}$$

Then the composition of A and B is defined by

$$(A \circ B)f(x) = \int e^{i(x-z)y} a(x,y)b(z,y)f(z)dzdy,$$

where $a(x,y)b(z,y)$ is an amplitude of order $\ell + \ell'$.

Proposition 20. The principal symbol is a homomorphism and gets involved in a short exact sequence

$$0 \to \Psi^{\ell-1}(M;E,F) \to \Psi^{\ell}(M;E,F) \xrightarrow{\sigma_\ell} C^\infty(T^*M; \mathrm{Hom}(E,F)) \to 0.$$

Proof. See Proposition 6.3 of [8]. □

Remark 36. Pseudo-differentials are "asymptotically complete". That is, if $A_j \in \Psi^{\ell_j}(M;E,F)$ and ℓ_j is decreasing with $\ell_j \to -\infty$, then there exists $\tilde{A} \in \Psi^{\ell_1}(M;E,F)$ with the property that

$$\tilde{A} - \sum_{j=1}^{N} A_j \in \Psi^{\ell_{N+1}}(M;E,F),$$

which we denote $\tilde{A} \sim \sum_{j\geq 1} A_j$.

Definition 66. $A \in \Psi^{\ell}(M;E,F)$ is said to be **elliptic** if $\sigma_\ell(A)(x)$ is invertible for all $x \neq 0$.

Theorem 45 (Hadamard)**.** If $A \in \Psi^{\ell}(M;E,F)$ is elliptic, then there is a pesudo-differential $B \in \Psi^{-\ell}(M;F,E)$ such that $AB - \mathrm{id} \in \Psi^{-\infty}(M,F)$ and $BA - \mathrm{id} \in \Psi^{-\infty}(M,E)$.

Proof. Let $B_0 \in \Psi^{-\ell}(M;F,E)$ such that $\sigma_{-\ell}(B) = \sigma_\ell(A)^{-1}$. Then we have $AB_0 \in \Psi^0(M,F)$, satisfying

$$\sigma_0(AB_0) = \sigma_\ell(A)\sigma_\ell(B_0) = \mathrm{id} = \sigma_0(\mathrm{id}).$$

Thus $AB_0 - \mathrm{id} \in \Psi^{-1}(M,F)$. Suppose inductively that we have found $B_0, B_1, \cdots, B_{N-1}$ with $B_j \in \Psi^{-\ell-j}(M;F,E)$ such that $R_N = A(\sum B_j) - \mathrm{id} \in \Psi^{-N}(M,F)$. Pick $B_N \in \Psi^{-\ell-N}(M;F,E)$ such that $\sigma_{\ell-N}(B_N) = -\sigma_\ell(A)^{-1}\sigma_{-N}(R_N)$ to complete the inductive step. Let B be the asymptotic sum of $\{B_j\}$, i.e. $B \sim \sum_{j\geq 0} B_j$. We claim that $AB - \mathrm{id} \in \Psi^{-\infty}(M,F)$, or equivalently $AB - \mathrm{id} \in \Psi^{-k}(M,F)$ for all $k > 0$.

Indeed, $B - \sum_{j=0}^{k-1} B_j \in \Psi^{-\ell-k}(M;F,E)$, so $A\left(B - \sum_{j=0}^{k-1} B_j\right) \in \Psi^{-k}(M,F)$. On the other hand, $A\left(\sum_{j=0}^{k-1} B_j\right) - \mathrm{id} \in \Psi^{-k}(M,F)$, so $A\left(B - \sum_{j=0}^{k-1} B_j\right) \in \Psi^{-k}(M,F) + A\left(\sum_{j=0}^{k-1} B_j\right) - \mathrm{id} \in \Psi^{-k}(M,F)$. Thus $R = AB - \mathrm{id} \in \Psi^{-\infty}(M,F)$.

Similarly, we can construct $\tilde{B} \in \Psi^{-\ell}(M;F,E)$ such that $\tilde{R} = \tilde{B}A - \mathrm{id} \in \Psi^{-\infty}(M,E)$. But then

$$\begin{aligned} \tilde{B}AB &= \tilde{B}(R + \mathrm{id}) = \tilde{B}R + \tilde{B} \\ &= (\tilde{R} + \mathrm{id})B = \tilde{R}B + B. \end{aligned}$$

This implies that

$$B - \tilde{B} = \tilde{B}R - \tilde{R}B \in \Psi^{-\infty}(M; F, E).$$

Hence

$$BA - \mathrm{id} = (B - \tilde{B})A + \tilde{B}A - \mathrm{id} \in \Psi^{-\infty}(M; E).$$

□

8.3 Elliptic Estimate

Let (M, g) be a closed Riemannian manifold, $E, F \to M$ be vector bundles with Hermitian metric, and $r > 0$.

Theorem 46 (Elliptic Estimate)**.** Let $A \in \Psi^r(M; E, F)$ be elliptic. For each $s \in \mathbb{R}$, there exists $C > 0$ such that, if $v \in H^t(M, E)$ for some t and $Av \in H^s(M, F)$, then $v \in H^{s+r}(M, E)$ and

$$\|v\|_{H^{s+r}} \leq C\left(\|Av\|_{H^s} + \|v\|_{H^t}\right)$$

Proof. Let $B \in \Psi^{-r}(M; F, E)$ be given through Hadamard's Theorem (Theorem 45), that is, $R = BA - \mathrm{id} \in \Psi^{-\infty}(M, E)$. Then

$$v = BAv - Rv \in H^{s+r}(M, E),$$

since $BAv \in H^{s+r}(M, E)$ and $Rv \in C^\infty(M, E)$. Also, $B : H^s(M, F) \to H^{s+r}(M, E)$ is continuous, so

$$\|BAv\|_{H^{s+r}} \leq C\|Av\|_{H^s},$$

and $R : H^t(M, F) \to H^{s+r}(M, E)$ is continuous, so

$$\|Rv\|_{H^{s+r}} \leq C\|v\|_{H^t}.$$

□

Remark 37. One can prove the elliptic estimate in a way without anything related to pseudo-differentials.

Namely, consider something like $\mathrm{id} + \Delta \in \mathrm{Diff}^2(M)$. At a given point $p \in M$, we can choose coordinates so that the metric, at p, is the Euclidean metric.

Step 1. Freezing coefficients at this point p, we have $\mathrm{id} + \Delta_{\mathbb{R}^n} \in \mathrm{Diff}^2(\mathbb{R}^n)$. This operator satisfies $(\mathrm{id} + \Delta_{\mathbb{R}^n})(f) = \mathcal{F}^*\left((1 + |\xi|^2)\mathcal{F}(f)\right)$. So the inverse is given by

$$(\mathrm{id} + \Delta_{\mathbb{R}^n})^{-1}(f) = \mathcal{F}^*\left((1 + |\xi|^2)^{-1}\mathcal{F}(f)\right).$$

This is obviously well-behaved as a map between Sobolev spaces. For $v \in H^2(\mathbb{R}^n)$, we have

$$\|v\|_{H^2} = \|(\mathrm{id} + \Delta)^{-1}(\mathrm{id} + \Delta)v\|_{H^2} \leq C\|(\mathrm{id} + \Delta)v\|_{L^2}.$$

Step 2. Back on M. If we localize at a coordinate chart instead of a point, then

$$(\mathrm{id} + \Delta_g)v = \left[\mathrm{id} + \Delta_{g_p} + (\Delta_g - \Delta_{g_p})\right] v.$$

So $v = (\mathrm{id} + \Delta_{g_p})^{-1} \left[\mathrm{id} + \Delta_{g_p} + (\Delta_g - \Delta_{g_p})\right] v$, and

$$\|v\|_{H^2} \leq C\| \left(\mathrm{id} + \Delta_{g_p}\right) v\|_{L^2} + C'\|v\|_{L^2}.$$

The elliptic estimate on M is obtained by patching these together. Now "gluing together" the inverse of $\mathrm{id} + \Delta_{g_p}$ for all $p \in M$, we get a pseudo-differential parametrix for $\mathrm{id} + \Delta_g$. So actually this method is not too far from the pseudo-differential approach.

Corollary 18. If M is closed and $A \in \Psi^r(M; E, F)$ is elliptic, then

1. $\ker(A : H^{s+r}(M, E) \to H^s(M, F)) \subset C^\infty(M, E)$ is finite dimensional and independent of s.
2. For any $s \in \mathbb{R}$, $A(H^{s+r}(M, E))$ is a closed subspace of $H^s(M, F)$.

Proof. 1. If $Au \in C^\infty(M, F) = \cap_t H^t(M, F)$, then by Theorem 46, $u \in \cap_k H^{t+r}(M, F) = C^\infty(M, E)$. The elliptic estimate, applied to elements of $\ker A$, says that there exists $C > 0$ such that $\|u\|_{H^{s+r}} \leq C\|u\|_{H^s}$. Thus the identity map is continuous as a map $\mathrm{id} : (\ker A, \| \cdot \|_{H^{s+r}}) \to (\ker A, \| \cdot \|_{H^s})$. Since the inclusion $H^{s+r}(M, E) \hookrightarrow H^s(M, E)$ is a compact operator, any sequence in $\ker A$ bounded in $H^{s+r}M, E$ has a subsequence that converges in H^s, hence in H^{s+r}. Let $\{e_j\}$ be a basis of $\ker A$ that is orthonormal in H^{s+r}. If $\{e_j\}$ were infinite, it would have a convergent subsequence. However, this is not the case since $\|e_j - e_k\|^2 = (e_j, e_j) + (e_k, e_k) = 2$ for any j, k.

2. Let (u_j) be a sequence in $H^{s+r}(M, E)$ such that $Au_j = v_j$ converges to v in $H^s(M, F)$. We want to show $v \in \mathrm{im}\, A$.

   Assuming that $u_j \perp \ker A$, we will show that they have a convergent subsequence. First assume that (u_j) has a subsequence (u_{j_k}) that is bounded in $H^{s+r}(M, E)$. Then (u_{j_k}) converges in $H^s(M, E)$. By the elliptic estimate $\|u_j\|_{H^{s+r}} \leq C\left(\|Au_j\|_{H^s} + \|u_j\|_{H^s}\right)$ for all j, the subsequence (u_{j_k}) is convergent in $H^{s+r}(M, E)$, say $u_{j_k} \to u_\infty$. Then the continuity of A implies that $Au_\infty = v$. Otherwise, assume that $\|u_j\|_{H^{s+r}} \to \infty$. Now let $w_j = \frac{u_j}{\|u_j\|_{H^{s+r}}}$. So $(w_j) \subset H^{s+r}(M, E)$ and $Aw_j = \frac{Au_j}{\|u_j\|_{H^{s+r}}} \to 0$ in $H^s(M, E)$. We apply the proceeding argument since $\|w_j\|_{H^{s+r}} = 1$, and conclude that (w_j) has a subsequence that converges in H^{s+r}, say $(w_{j_k}) \to w_\infty$. However, $Aw_\infty = 0$ implies $w \in (\ker A)^\perp$, so $w_\infty = 0$. On the other hand, $\|w_\infty\|_{H^{s+r}} = 1$, contradiction! Thus (v_j) must have a bounded subsequence.

□

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