Morse Theory and the Symplectic Quotient

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1 Introduction

Let (M, ω) be a compact symplectic manifold, and let K be a compact Lie group with a Hamiltonian action on M and moment map $\mu : M \to \mathfrak{k}^*$. Fix a K-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{k}$ to identify $\mathfrak{k} \cong \mathfrak{k}^*$.

Suppose K acts freely on $\mu^{-1}(0)$, then the *Marsden-Weinstein reduction* $M /\!\!/ K := \mu^{-1}(0)/K$ is a symplectic manifold that inherits the symplectic form on M.

What is the topology of this symplectic quotient? We may, for instance, try to compute the rational cohomology $H^*(M /\!\!/ K, \mathbb{Q})$, which turns out to isomorphic to the equivariant cohomology

$$H_K^*(\mu^{-1}(0), \mathbb{Q}).$$

A key observation in Kirwan's thesis [13] is that the submanifold $\mu^{-1}(0)$ can also be seen as the set of points on which the smooth function

$$f : M \to \mathbb{R}, f(x) = ||\mu(x)||^2$$

takes the minimum value 0. Therefore, f can be seen as a generalized Morse function, which, just as in classical Morse theory, provides a version of "smooth stratification" of the manifold M. Utilizing the stratification and adopting techniques from Morse theory, [13] arrives at the following result:

The map

$$H_K^*(M) \to H_K^*(\mu^{-1}(0)) \cong H^*(M /\!\!/ K),$$

yielded by the inclusion $\mu^{-1}(0) \to M$, is surjective.

In showing this, inductive formulas that compute the Poincaré polynomial of $M /\!\!/ K$ from the Poincaré polynomials of certain critical point sets of $f = ||\mu||^2$ are also derived.

The following aims to provide a brief summary of the theory, with an emphasis on establishing the surjectivity of the restriction map rather than computational aspects. This is followed by an overview of its infinite-dimensional analogue - the Yang-Mills theory over Riemann surfaces [1].

2 Equivariant cohomology

Explicitly describing $M /\!\!/ K$ as a topological space might not be so straightforward, and a better way is to work with M itself and use the equivariant cohomology H_K^*.

Definition 1. Let G be a group. A principal G-bundle $EG \to BG$ is called a *universal bundle* if the total space EG is contractible.

Remark 1. Universal bundles exist and satisfy the following universal property ([3], §VI.1):

(i) given any other bundle $E' \to B'$, there exists a base change $h : B' \to BG$ such that the pullback $h^*EG \cong E'$ as bundles over B';

(ii) the above base change $h : B' \to BG$ is unique up to homotopy.

Universal bundles can thus be seen as some "large" spaces that can 'absorb' the K-action on M and be utilized to define the equivariant cohomology:

Definition 2. Let K be a compact Lie group that acts on a manifold M.

(i) The *homotopy quotient* $EK \times_K M$ is defined as the quotient space of $EK \times M$ under equivalence relation $(e \cdot g^{-1}, x) \sim (e, g \cdot x)$, where $x \in M, e \in EK, g \in K$;

(ii) The *equivariant cohomology* $H^*_K(X)$ is defined as $H^*(EK \times_K X)$.

Remark 2. Other models, such as the Cartan model and Weil model, are also available and more suitable for working with equivairant differential forms. Detailed exposition on these models can be found in [9].

Proposition 1. $M /\!\!/ K$ is weakly homotopy equivalent to $EK \times_K \mu^{-1}(0)$. In particular,

$$H^*_K(\mu^{-1}(0), \mathbb{Q}) = H^*(EK \times_G \mu^{-1}(0), \mathbb{Q}) \cong H^*(M /\!\!/ K).$$

Proof. Because the K-action on M is free, $\mu^{-1}(0) \to \mu^{-1}(0)/K$ is a principal K-bundle, and $EK \times_G \mu^{-1}(0) \to \mu^{-1}(0)/K : [(e, x)] \mapsto [x])$ is fibre bundle with fibre EK.

This induces an exact sequence in homotopy groups:

$$\cdots \to \pi_q(EG) \to \pi_q(EG \times_G \mu^{-1}(0)) \to \pi_q(\mu^{-1}(0)) \to \pi_{q-1}(EG) \to \cdots$$

Because EG is contractible, there are isomorphisms

$$\pi_q(EG \times_G \mu^{-1}(0)) \cong \pi_q(\mu^{-1}(0)),$$

which induce isomorphisms in cohomology. □

Example ([1], §1) Let S^1 act on S^2 by rotating about the z-axis. A universal bundle $ES^1 \to BS^1$ may be constructed by taking the union of Hopf fibrations $S^{2l+1} \to \mathbb{CP}^l$ across $l \geq 1$ under the inclusions $S^{2l+1} \subset S^{2l+3}$ and $\mathbb{CP}^l \subset \mathbb{CP}^{l+1}$ that are induced by $\mathbb{R}^{2l+2} \to \mathbb{R}^{2l+4}$. Because $\pi_k(S^{2l+1}) = 0$ for $k < 2l+1$, the union $ES^1 = \bigcup_{l \geq 1} S^{2l+1}$ will be weakly contractible, hence contractible.

The homotopy quotient $S^2 \times_{S^1} ES^1$ is then a S^2-bundle over the classifying space $BS^1 = \bigcup_{l \geq 1} \mathbb{CP}^l$, hence a trivial bundle. By Künneth formula,

$$H^*_{S^1}(S^2) = H^*(S^2 \times_{S^1} ES^1) \cong H^*(S^2) \otimes H^*(BS^1).$$

3 Constructing the stratification

3.1 The indices

Our stratification will be indexed by elements in the Lie algebra $\mathfrak{k}$. Hence we shall first investigate the action generated by such single elements. This subsection follows §3-4 of [13].

Let $T \subset K$ be a maximal torus, let $\beta \in \mathfrak{t} \subset \mathfrak{k}$, and let $T_\beta := \overline{\exp(t\beta)} \subset K$ be the one-parameter subgroup generated by β. Both T and T_β have Hamiltonian actions on M, and the moment maps are given by the composition with restriction maps $M \xrightarrow{\mu} \mathfrak{k}^* \to \mathfrak{t}^* \to \mathrm{Lie}(T_\beta)^*$. Let μ_T denote the moment map for the T-action.

Let $\mu_\beta : M \to \mathbb{R}$ be defined as $x \mapsto \langle \mu(x), \beta \rangle$.

Lemma 1. $x \in M$ is fixed by T_β if and only if it is a critical point of μ_β.

Proof. Let X_β denote the vector field in M generated by β. By definition of the moment map, $\iota_{X_\beta}\omega = d\langle\mu(x), \beta\rangle = d\mu_\beta$. Thus, $d\mu_\beta = 0 \Leftrightarrow \iota_{X_\beta}\omega = 0 \Leftrightarrow x$ is fixed by the flow $\exp(t\beta)$, or equivalently the T_β-action. $\square$

Furthermore, critical points of f (which we denote as $\text{Crit}(f)$) can be described by their restrctions to one-parameter subgroups:

Lemma 2. Let $x \in M, \mu(x) = \beta \in \mathfrak{t}$. The following are equivalent:
(i) $X_\beta|_x = 0$; (ii) $x \in \text{Crit}(f)$; (iii) $x \in \text{Crit}(||\mu_T||^2)$; (iv) $x \in \text{Crit}(\mu_\beta)$.

Proof. (i) $\Leftrightarrow$ (ii),(iii): Throughout the following, we identify Lie algebras with their duals via the inner product. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $\mathfrak{g}$, let $\mu_i = \langle\mu, e_i\rangle$ so that $\mu(x) = \sum_{i=1}^m \mu_i(x)e_i$.

Because ω is non-degenerate, $df = d(\sum_i |\mu_i(x)|^2) = \sum \mu_i d\mu_i$ vanishes at x if and only if its ω-dual does so. Because $\iota_{X_{e_i}}\omega = d(\langle\mu, e_i\rangle) = d\mu_i$, the ω-dual of df equals $\sum_i \mu_i X_{e_i} = \sum_i \langle\mu, e_i\rangle X_{e_i} = X_{\mu(x)}$. Thus $df|_x = 0 \Leftarrow X_\beta|_x = 0$. Since X_β is the same for the T-action on M, the same holds for $||\mu_T||^2$.

(i) $\Leftrightarrow$ (iv): $X_\beta|_x = 0$ if and only if x is fixed by the flow $\exp(t\beta)$, or equivalently T_β, which is equivalent to x being a critical point of μ_β. $\square$

Now we pick the indices and critical point sets for the stratification.

Definition 3. Fix a maximal torus $T \subseteq K$. Let $\mathcal{B} = \mu(\text{Crit}(||\mu_T||^2)) \cap \mathfrak{t}_+$, where $\mathfrak{t}_+$ is a fixed positive Weyl chamber, and $\mathfrak{t} \cong \mathfrak{t}^*$ are identified via the inner product.

For $\beta \in \mathcal{B}$, define $Z_\beta = \text{Crit}(\mu_\beta) \cap \mu_\beta^{-1}(||\beta||^2)$, and $C_\beta = K \cdot (Z_\beta \cap \mu^{-1}(\beta))$.

Remark 3. From previous results, Z_β is T-invariant and fixed by T_β. Also, C_β are disjoint closed subsets of M.

The positive Weyl chamber is specified so that

Lemma 3. $\text{Crit}(f)$ is the disjoint union of $\{C_\beta : \beta \in \mathcal{B}\}$.

Proof. Let $x \in \text{Crit}(f)$. There exists $g \in K : \text{Ad}_g^*\mu(x) \in \mathfrak{t}_+$. Because f is K-equivariant, $x \in \text{Crit}(f) \Leftrightarrow g \cdot x \in \text{Crit}(f)) \Leftrightarrow g \cdot x \in Z_{\mu(g\cdot x)}$ (applying lemma 2). In this case $\beta := \mu(g \cdot x) = \text{Ad}_g^*\mu(x)$ lies in $\mathcal{B}$, so $x \in \{C_\beta : \beta \in \mathcal{B}\}$.

Suppose $x \in C_{\beta_1} \cap C_{\beta_2}$, then $\exists g_i \in K : \text{Ad}_{g_i}^*\mu(x) = \beta_i$, so that $\beta_1, \beta_2 \in \mathfrak{t}_+$ lie in the same coadjoint orbit. This implies that $\beta_1 = \beta_2$, since every coadjoint orbit intersects a positive Weyl chamber at a unique point. Therefore, C_β are disjoint. $\square$

3.2 Morse theory

After dealing with the indices, we construct the stratification by looking at the gradient flows that converge to points in C_β. To do this, we need more definitions.

Definition 4. A finite collection of locally closed submanifolds of M $\{S_\beta : \beta \in \mathcal{B}\}$ forms a smooth stratification if they partition M, and there exists a strict partial order $>$ on $\mathcal{B}$ such that $\bar{S}_\beta \subseteq \bigcup_{\gamma\geq\beta} S_\gamma$.

Definition 5. A smooth function $F : M \to \mathbb{R}$ is *minimally degenerate* if the following holds:

(i) $\mathrm{Crit}(F)$ is the finite disjoint union of closed subsets $\{M_\alpha\}$ such that F is constant on each M_α;

(ii) for each α there exists a locally closed submanifold Σ_α that contains M_α and has an orientable normal bundle in M;

(iii) on each Σ_α, F attains its minimum on M_α;

(iv) at every $x \in M_\alpha$, $T_x\Sigma_\alpha$ is maximal among subspaces of T_xM on which the Hessian $H_x(F)$ is positive-definite.

Σ_α is called the *minimising submanifold* of F along M_α.

Remark 4. Classical Morse theory requires the Hessian to be non-degenerate at every critical point, but this can fail even for reasonable moment maps. The above definition relaxes the requirement by seeking only a submanifold on which the Hessian is positive-definite.

As a simple example, consider the action of $\mathbb{T}^2$ on $\mathbb{CP}^2$ given by

$$[z_0, z_1, z_2] \mapsto [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2].$$

On the chart $\{z_0 \neq 0\}$, the squared norm of the moment map equals

$$(z_1, z_2) \mapsto \Big(\frac{|z_1|^2}{1 + |z_1|^2 + |z_2|^2}\Big)^2 + \Big(\frac{|z_2|^2}{1 + |z_1|^2 + |z_2|^2}\Big)^2.$$

The function is degenerate at $[1, 0, 0]$, and a minimising submanifold is given trivially by the singleton itself.

Examples (i) A smooth function $f : M \to \mathbb{R}$ is Morse-Bott if its critical set is a disjoint union of closed submanifolds such that the Hessian of f is non-degenerate in the normal direction. Then Morse-Bott functions are minimally degenerate, with minimising submanifolds given by tubular neighborhoods;

(ii) In [14], it is shown that if $f_1, \ldots, f_n \in C^\infty(M)$ satisfies that $\{\mathrm{grad}(f_i)\}$ commute with each other, and that any linear combination of $\{f_i\}$ is a classical Morse function, then given any strictly convex smooth function $c : \mathbb{R}^n \to \mathbb{R}$, the function $c(f_1, \ldots, f_n) : M \to \mathbb{R}$ is a minimally degenerate function.

To construct a stratification from a minimally degenerate function, we use gradient flows:

Definition 6. Let (M, g) be a Riemannian manifold, $F : M \to \mathbb{R}$ a smooth function. Let ∇F be the vector field that is g-dual to $-dF$, ie., $-dF = \iota_{\nabla F}\, g$. For a non-critical point q, denote the gradient flow of ∇F as the curve $\gamma_{F,q} : \mathbb{R} \to M$ such that $\dot\gamma_{F,q} = \nabla F, \gamma_{F,q}(0) = q$.

Define $\omega_F(q)$ as the set of points $y \in M$ such that any open neighborbood of y contains $\gamma_{F,q}(t)$ for sufficiently large t.

Given a disjoint union $\mathrm{Crit}(F) = \sqcup_{\alpha\in\mathcal{A}} C_\alpha$, define $S_{F,\alpha} = \{q \in M : \omega_F(q) \subseteq C_\alpha\}$.

Remark 5. (i) $Y_{F,\alpha}$ should be seen as the union of trajectories that converge to $Y_{F,\alpha}$. By solving the equations for points in C_α, one can check that $C_\alpha \subseteq Y_{F,\alpha}$.

(ii) By construction, the value of F decreases along the gradient flow, so $y \in \omega_F(q) \Rightarrow F(y) \leq F(q)$.

Theorem 1 ([13], Theorem 10.4). Let $F : M \to \mathbb{R}$ be a minimally degenerate function with critical point set $\sqcup_{\alpha\in\mathcal{A}} C_\alpha$ and mimimising submanifolds Σ_α. If the gradient flow of f is tangential to each Σ_α, then

(i) $S_{F,\alpha}$ forms a smooth stratification of M;

(ii) $S_{F,\alpha}$ coincides with Σ_α in some neighborhood of C_α;

(iii) The inclusion $C_\alpha \to S_{F,\alpha}$ induces isomorphism of of Cech cohomology;

(iv) If there is a G-action on M from a compact group K such that F, Σ_α and the Riemannian metric g are all K-invariant, then $C_\alpha \to S_{F,\alpha}$ also induces an equivalence of equivariant cohomology.

Proof. (Sketch) (i) In this setting, the strict partial order is given by $\alpha_1 > \alpha_2 \Leftrightarrow f(C_{\alpha_1}) > f(C_{\alpha_2})$. The definition of smooth stratification can then be verified by carefully applying remark 4(ii);

(ii) On the one hand, by shrinking the open neighborhood U_α of C_α in M, one can ensure that for any point $x \in U_\alpha \cap \Sigma_\alpha$, the gradient flow never escapes $U_\alpha \cap \Sigma_\alpha$: this follows since within Σ_α, the function F takes its minimum on S_α, while F decreases on gradient flows;

The other direction requires analysis of the gradient flow near S_C: working in local coordinates, one shows that if the initial point is away from a neighborhood within Σ_α, then the gradient flow stays away from the neighborbood: the bound is given by the assumption that the Hessian of F is positive definite on $T\Sigma_\alpha$;

(iii) & (iv) Consider the compact neighborhoods of C_α in S_α given by $N_{\alpha,\epsilon} := \{x \in S_\alpha : f(x) \leq f(C_\alpha) + \epsilon\}$. The gradient flows induce retractions from S_α to $N_{\alpha,\epsilon}$. Taking $\cap_{\epsilon>0} N_{\alpha,\epsilon} = C_\alpha$, one sees that $C_\alpha \to S_\alpha$ and $C_\alpha \times_G EG \to S_\alpha \times_K EK$ induces equivalences in (equivariant) Cech cohomology. □

We apply the above theory in our setting. For the following, fix a K-invariant Riemannian metric g on M: this can be done by picking any Riemannian metric and averaging over the K-action; alternatively, if M itself is Kähler then the real part of the Kähler form suffices.

Definition 7. Let (M, g) be a Riemannian manifold. For $\beta \in \mathfrak{k}$, let $\nabla\mu_\beta$ be the vector field that is g-dual to $-d\mu_\beta$, ie., $-d\mu_\beta = \iota_{\nabla\mu_\beta} g$. For a non-critical point q, denote the flow of $\nabla\mu_\beta$ as $\gamma_q : \mathbb{R} \to M$, so that

$$\dot{\gamma}_{\beta,q} = \nabla\mu_\beta, \gamma_{\beta,q}(0) = q.$$

Define $\omega_\beta(q)$ as the set of points $y \in M$ such that any open neighborbood of y contains $\gamma_{\beta,q}(t)$ for sufficiently large t, and define $Y_\beta = \{q \in M : \omega_\beta(q) \subseteq Z_\beta\}$.

The following lemmas check that minimising submanifolds of $C_\beta = K \cdot (Z_\beta \cap \mu^{-1}(\beta))$ can be taken as their open neighborboods in $K \cdot (Y_\beta)$.

Lemma 4. (i) Z_β is a closed submanifold of M, and the T_β-action on the normal bundle TM/TZ_β does not fix any non-zero vector;

(ii) Y_β are locally closed submanifolds of M.

Proof. (i) For any fixed point x, there exists a diffeomorphism from an open neighborhood of the zero section in $T_\beta \times_{T_\beta} T_x M$ to an open neighborhood of x in M. The fixed point set is diffeomorphic to the subbundle (hence a submanifold) $T_\beta \times_{T_\beta} V$, where V is the subspace consisting of vectors fixed under the T_β-action ([3], §I.2b).

Since $Z_\beta = \mathrm{Crit}(\mu_\beta)$ is closed, it is a closed submanifold of M. The diffeomorphism between Z_β and $T_\beta \times_{T_\beta} V$ also implies the second claim.

(ii) ([3], IV.1b) The Riemannian metric and the symplectic form gives a K-invariant almost complex structure. Let $x \in Z_\beta$. The T_β-action induces a decomposition into complex, T_β-invariant subspaces $T_xM = V_0 \oplus V_1 \oplus \cdots \oplus V_k$, where on each V_j, T_β acts by a scalar $\exp(i\lambda_j)$, and $V_0 = T_xZ_\beta$. By (i), $\lambda_j \neq 0$ if $j \neq 0$. Adopting coordinates $(v_0, \ldots, v_j)$ based on the decomposition and applying coordinates $v_j = p_j + iq_j$ for the Hamiltonian function μ_β, the Hessian may be written as the quadratic form $\sum_{i=1}^k \lambda_i |v_i|^2$, which is non-degenerate on the normal bundle T_xM/T_xZ_β.

Therefore, μ_β is a Morse function in the sense of Bott. The Hessian of μ_β is positive definite on the normal bundle TM/TZ_β. It follows that Y_β, which are the stable manifolds for μ_β over Z_β are locally closed submanifolds of M, TY_β corresponds exactly to the non-negative eigenspaces in the above decomposition ([1], §1). □

Remark 6. Stable manifolds for Morse-Bott functions have a number of further properties ([1], §1):

(i) In the decomposition $T_xM = V_0 \oplus V_1 \oplus \cdots \oplus V_k$, T_xY_β corresponds to the non-negative eigenspaces;

(ii) The Morse indices of Z_β equal to the codimensions of Y_β.

Lemma 5 ([13], Lemma 4.10)**.** Let $x \in Z_\beta \cap \mu^{-1}(\beta)$. Define $\mathrm{Stab}\ \beta = \{g \in K : \mathrm{Ad}_g\beta = \beta\}$, and $\mathrm{stab}\ \beta = \{\alpha \in \mathfrak{k} : [\alpha, \beta] = 0\}$.

(i) $g \in K$ satisfies $g \cdot x \in Y_\beta$ if and only if $g \in \mathrm{Stab}\ \beta$;

(ii) $\alpha \in \mathfrak{k}$ satisfies $X_\alpha(x) \in T_xY_\beta$ if and only if $\alpha \in \mathrm{stab}\ \beta$.

Lemma 6. $K \cdot (Y_\beta)$ is smooth in some $K-$invariant neighborbood of C_β.

Proof. (Sketch) It suffices to show that $K \cdot (Y_\beta)$ is smooth in some neighborbood of $Z_\beta \cap \mu^{-1}(\beta)$: applying K-action to the neighborbood provides a $K-$invariant neighborbood of C_β.

Consider the map $\sigma : K \times_{\mathrm{Stab}\ \beta} Y_\beta \to M$ given by $(g, y) \mapsto g \cdot y$. Clearly the image of the map lies in $K \cdot (Y_\beta)$. The level set $\{y \in Y_\beta : \mu_\beta(y) \leq ||\beta||^2 + \epsilon\}$ is a compact neighborhood of $Z_\beta \cap \mu^{-1}(\beta)$. Mapping it via the G-action σ shows that for $x \in Z_\beta \cap \mu^{-1}(\beta)$, a neighborhood around $(1, x)$ gets mapped to a neigborhood around x in $K \cdot (Y_\beta)$.

The differential of σ maps $(\alpha, v) \in T_x(K \times_{\mathrm{Stab}\ \beta} Y_\beta)$ to $X_\alpha(x) + v(x) \in T_xM$, and one can check that because

$$T_{(1,x)}(G \times_{\mathrm{Stab}\ \beta} Y_\beta) \cong \mathfrak{g} \times T_xY_\beta/\{(\alpha, v) : \alpha \in \mathrm{stab}\ \beta, v(x) = X_\alpha(x)\},$$

this map is injective, which also holds in a neighborhood of $(1, x)$. Therefore, $K \cdot (Y_\beta)$ is smooth in some $K-$invariant neighborbood of C_β. □

Proposition 2. $f : M \to \mathbb{R}$ is a minimally degenerate function, with $\mathrm{Crit}(f) = \sqcup_{\beta \in \mathcal{B}} C_\beta$ and minimising submanifolds Σ_β the G-invariant neighborhoods of C_β described in the lemma.

Proof. Σ_β are locally closed submanifolds since Y_β are; their normal bundles correspond to the negative eigenspace (in particular a complex subbundle) in the decomposition, hence are orientable. Because TY_β is the non-negative eigenspace, and $f = \mu_\beta$ on x, T_xY_β is maximal among all subspaces on which $H_x f$ is positive definite.

In Σ_β, f takes its minimum on C_β: on $G \cdot (Y_\beta)$, $||\mu||^2||\beta||^2 \geq |\langle \mu, \beta\rangle|^2 \geq ||\beta||^2$, where as f equals $||\beta||^2$ on C_β. □

By construction, the gradient flows to f are tangential to Σ_β, thus we have shown:

Theorem 2. Theorem 1 is applicable to the setting $f : M \to \mathbb{R}$ given by $f = ||\mu||^2$, $\mathrm{Crit}(f) = \sqcup_{\beta\in\mathcal{B}}(S_\beta)_{\beta\in\mathcal{B}}$. More precisely, there exists a smooth stratification S_β given by the gradient flow construction, such that S_β coincides with the minimising submanifolds in some neighborbood C_β. The inclusion $C_\beta \to S_\beta$ induces isomorphisms in both Cech and equivariant cohomology.

4 The Kirwan map

We now apply the stratification to equivariant cohomology, which yields the main results. The following discussion uses standard results on (equivariant) characteristic classes, which can be found in [2], §2.

Let

$$U_\beta = \bigcup_{\gamma\leq\beta} S_\beta, U'_\beta = \bigcup_{\gamma<\beta} S_\beta = U_\beta \setminus S_\beta,$$

and denote $(U_\beta)_K := EK \times_K U_\beta$ the homotopy quotient. The inclusion $U'_\beta \subset U_\beta$ induces an inclusion in their homotopy quotients, which gives a long exact sequence[1] in equivariant cohomology:

$$\cdots \to H^n_K(U_\beta, U'_\beta) \to H^n_K(U_\beta) \to H^n_K(U'_\beta) \to H^{n+1}_K(U_\beta, U'_\beta) \to \cdots$$

The relative cohomology can be handled by Thom isomorphism: By taking connected components if necessary, assume that μ_β have constant index $d(\beta)$ on Z_β, so that S_β also has constant codimension $d(\beta)$ in M. Applying excision to a tubular neighborbood T for $U'_\beta \subset U_\beta$,

$$H^n_K(U_\beta, U'_\beta) \cong H^n(T_K, T_K \setminus (U'_\beta)_K).$$

Applying Thom isomorphism,

$$\cdot \cup e_K(S_\beta) : H^{n-d(\beta)}_K(S_\beta) \to H^n(T_K, T_K \setminus (U'_\beta)_K) \cong H^n_K(U_\beta, U'_\beta).$$

The map is given by taking the cup product with $e_K(\nu_{S_\beta/U_\beta})$, which denotes the equivariant Euler class[2] of the normal bundle ν_{S_β/U_β} of S_β in U_β. It increases degree by $d(\beta)$ because the codimension of S_β in U_β is $d(\beta)$.

Putting the isomorphism in the long exact sequence:

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[1] It is probably easier to keep track of this sequence if we work with the Cartan model of equivariant cohomology, where the maps correspond to restriction of equivariant differential forms.

[2] As the normal bundle is a oriented vector bundle over S_β with an induced K-action, it pulls back to a K-vector bundle $EK \times_K \nu_{S_\beta/U_\beta} \to EK \times_K S_\beta$, of which the Euler class is defined as the equivariant Euler class.

$$\cdots \to H_G^{n-d(\beta)}(S_\beta) \to H_K^n(U_\beta) \to H_K^n(U'_\beta) \to H_K^{n-d(\beta)+1}(S_\beta) \to \cdots$$

The following lemma will be used to show that the long exact sequence splits into short exact sequences.

Lemma 7. (i)

$$H_K^*(C_\beta) \cong H^*_{\mathrm{Stab}\ \beta}(Z_\beta \cap \mu^{-1}(\beta));$$

(ii) The map

$$H_K^{n-d(\beta)}(S_\beta) \to H_K^n(U_\beta)$$

is injective.

Proof. (i) Consider the map

$$\sigma : K \times_{\mathrm{Stab}\ \beta} (Z_\beta \cap \mu^{-1}(\beta)) \to C_\beta, (g, x) \mapsto g \cdot x.$$

This is a continuous bijection: firstly, it is clearly surjective; suppose

$$g_i \in K, x_i \in Z_\beta \cap \mu^{-1}(\beta) : g_1 \cdot x_1 = g_2 \cdot x_2,$$

then $h = g_2^{-1} g_1$ satisfies that $hx_1 \in Z_\beta \cap \mu^{-1}(\beta)$. By Lemma 5, $h \in \mathrm{Stab}\ \beta$ so $(g_1, x_1) = (g_2, x_2)$.

Since the map is from a compact space to a Hausdorff space, it is a homeomorphism. From general results about fibre bundles,

$$(C_\beta)_K \cong ((Z_\beta \cap \mu^{-1}(\beta)) \times_{\mathrm{Stab}\ \beta} K) \times_G EK \cong (Z_\beta \cap \mu^{-1}(\beta)) \times_{\mathrm{Stab}\ \beta} EK,$$

and EK is a contractible space with a free Stab β-action, hence $EK \to EK/\mathrm{Stab}\ \beta$ is a model for the universal bundle of Stab β. Passing to cohomology, the homeomorphisms yield

$$H^*((C_\beta)_K) \cong H^*((Z_\beta \cap \mu^{-1}(\beta))_{\mathrm{Stab}\ \beta}),$$

ie., $H_K^*(C_\beta) \cong H^*_{\mathrm{Stab}\ \beta}(Z_\beta \cap \mu^{-1}(\beta))$.

(ii) The inclusion $i : S_\beta \to U_\beta$ induces $i^* : H_K^n(U_\beta) \to H_K^n(S_\beta)$. The composition of the two maps

$$H_K^{n-d(\beta)}(S_\beta) \to H_K^n(U_\beta) \to H_K^n(S_\beta)$$

is the Gysin operation, which is given by taking the cup product with the equivariant Euler class $e_K(\nu_{S_\beta/U_\beta})$ of the normal bundle for $S_\beta \subset U_\beta$ ([2], 2.19]). To show (ii), it suffices to show that $e_K(\nu_{S_\beta/U_\beta})$ is not a zero-divisor in $H_K^*(S_\beta)$.

Applying Theorem 1 (iv), the inclusion $C_\beta \to S_\beta$ induces an equivalence in equivariant cohomology

$$H_K^*(S_\beta) \cong H_K^*(C_\beta),$$

where $e_K(\nu_{S_\beta/U_\beta})$ corresponds to its restriction on C_β.

Consider the isomorphism in (i). Because it is induced by a homeomorphism, $e_K(\nu_{S_\beta/U_\beta})$ corresponds to its restriction, namely the Stab β-equivariant Euler class on $Z_\beta \cap \mu^{-1}(\beta)$, which we denote as e_N. Therefore, it suffices to show that e_N is not a zero divisor.

The subtorus $T_\beta \subseteq \mathrm{Stab}\ \beta$ fixes $Z_\beta \cap \mu^{-1}(\beta)$ pointwise. By lemma 4 (i), the T_β action does not fix any non-zero vector on e_N. Thus the following localization theorem of Atiyah-Bott guarantees that e_N is invertible after passing to a localization. Hence it cannot be a zero divisor. □

Theorem 3 ([2], §3)**.** Let T be a torus, M a smooth T-manifold, and $M^T \subset M$ the fixed point locus under the T-action.

The equivariant pushforward

$$i_* : H_T^*(M^T; \mathbb{C}) \to H_T^*(M; \mathbb{C})$$

and restriction maps

$$i^* : H_T^*(M; \mathbb{C}) \to H_T^*(M^T; \mathbb{C})$$

become isomorphisms after localizing at $\prod_P e_T(\nu_P)|_p \in H^*(\mathrm{pt}; \mathbb{C})$, where P are the connected components of M^T, and $p \in P$.

In particular, $i^* i_* 1 = e_T(\nu_{M^T})$ is invertible after the localization.

Corollary 1. (i) The stratification $\{S_\beta\}_{\beta \in \mathcal{B}}$ is *equivariantly perfect*: the long exact sequence

$$\cdots \to H_K^{n-d(\beta)}(S_\beta) \to H_K^n(U_\beta) \to H_K^n(U'_\beta) \to H_K^{n-d(\beta)+1}(S_\beta) \to \cdots$$

splits into short exact sequences

$$0 \to H_K^{n-d(\beta)}(S_\beta) \to H_K^n(U_\beta) \to H_K^n(U'_\beta) \to 0;$$

(ii) The map

$$H_K^*(M) \to H_K^*(\mu^{-1}(0)) \cong H^*(M /\!\!/ K) \tag{1}$$

induced by the inclusion $\mu^{-1}(0) \to M$ is surjective.

Proof. (i) From the above lemma, $H_K^{n-d(\beta)}(S_\beta) \to H_K^n(U_\beta)$ is injective. By exactness, the two maps

$$H_K^{n-1}(U'_\beta) \to H_K^{n-d(\beta)}(S_\beta), H_K^n(U'_\beta) \to H_K^{n-d(\beta)+1}(S_\beta)$$

equal the zero map. Hence there is the short exact sequence

$$0 \to H_K^{n-d(\beta)}(S_\beta) \to H_K^n(U_\beta) \to H_K^n(U'_\beta) \to 0;$$

(ii) In particular, each map $H_K^n(U_\beta) \to H_K^n(U'_\beta)$ induced by inclusions $U'_\beta \to U_\beta$ is surjective. Composing the inclusions

$$\mu^{-1}(0) = U_0 \to \cdots \to U_\beta \to \cdots \to \cup_{\beta \in \mathcal{B}} S_\beta = M$$

implies

$$H_K^*(M) \to H_K^*(\mu^{-1}(0)) \cong H^*(M /\!\!/ K)$$

is surjective. □

Corollary 2. Denote $h_K^n(M) := \dim H_K^n(M)$, and define $P_t^K(M) := \sum_{i=0}^\infty h_K^i(M) t^i$ as the equivariant Poincaré polynomial. Then

$$P_t^K(M) = \sum_\beta t^{d(\beta)} P_t^{(\mathrm{Stab}\ \beta)}(Z_\beta \cap \mu^{-1}(\beta)).$$

Proof. The short exact sequences in corollary 1 yield $h_K^{n-d(\beta)} S_\beta + h_K^n(U'_\beta) = h_K^n(U_\beta)$. Applying the formula for successive strata, $h_K^n(M) = \sum_\beta h_K^{n-d(\beta)}(S_\beta)$.

Hence,

$$P_t^K(M) = \sum_{i=0}^{\infty} \sum_\beta h_K^{i-d(\beta)}(S_\beta) t^i = \sum_\beta \sum_{i=0}^{\infty} t^d(\beta)(h_K^{i-d(\beta)}(S_\beta) t^{i-d(\beta)}) = \sum_\beta t^d(\beta) P_t^K(S_\beta).$$

Because $C_\beta \to S_\beta$ induces an equivalence in equivariant cohomology, $P_t^K(S_\beta) = P_t^K(C_\beta)$. Applying lemma 7(i), $P_t^K(C_\beta) = P_t^{(\mathrm{Stab}\ \beta)}(Z_\beta \cap \mu^{-1}(\beta))$ and the result follows. $\square$

5 Two simple examples

5.1 Projective space

Let us carry out the procedures in a trivial example. Consider the diagonal $U(1)$-action on $\mathbb{C}^{n+1}$. Identify $\mathfrak{u}(1) \cong \mathbb{R} \cong \langle \beta \rangle$ for some generator $\beta \in \mathfrak{u}(1)$.

The $U(1)$-action on $\mathbb{C}^{n+1}$ is Hamiltonian and a moment map is given by

$$\mu((z_1, \ldots, z_{n+1})) = \sum_{j=1}^{n+1} |z_j|^2 - 1.$$

The symplectic quotient is then

$$\mu^{-1}(0)/U(1) = S^{2n+1}/U(1) \cong \mathbb{CP}^n.$$

Clearly, the critical points of f are the origin $\{0\}$ on which $\mu = -1, f = 1$, and the unit sphere S^{2n+1} on which $\mu = f = 0$. Denote $C_1 = \{0\}, C_0 = S^{2n+1}$. Both C_1, C_0 are themselves submanifolds, so f is indeed minimally degenerate.

The stratification is given by gradient flows of f towards the critical loci. As $f = \mu^2 = (\sum_{j=1}^{n+1} |z_j|^2 - 1)^2$,

$$\nabla \mu_\beta = \mu \sum_{j=1}^{n+1} -2x_j \partial x_j - 2y_j \partial y_j.$$

By inspecting signs[3] of the gradient $\nabla \mu_\beta$, we see that a gradient flow $\gamma_{f,\underline{w}}$ initiating from $\underline{w} \in \mathbb{C}^{n+1} \setminus (C_0 \cup C_1)$ satisfies that $||\gamma_{f,w}(t)||^2 \to 1$ as $t \to \infty$. Therefore, $S_0 = \mathbb{C}^{n+1} \setminus \{0\}$, and $S_1 = \{0\}$ is the stratification.

5.2 Delzant spaces

By a theorem of Delzant, symplectic toric manifolds can be classified by convex polytopes via a certain construction that we outline below, following §28 of [4].

Definition 8. A polytope in $\mathbb{R}^n$ is the convex hull of finitely many points, with its vertices being a minimal subset of the points that has the same convex hull. A k-dimensional face is the convex hull of some $(k+1)$ vertices that lie entirely on the boundary of the polytope: in particular, a 1-dimensional face is an edge, and a $(n-1)$-dimensional face is a facet.

[3] Suppose the initial point $\underline{w}$ is inside the unit sphere, then $\mu(\underline{w}) < 0$ near $\underline{w}$, so the flow points towards the unit sphere, and similarly for $\underline{w}$ outside the unit sphere.

Let v be some vertex of a polytope, and let $e_1, \dots, e_k$ be the edges joining v.

A polytope is Delzant if:

(i) For each vertex v, the number of edges k equals n;

(ii) Each edge can be written as $v + tu_i$ for some $u_i \in \mathbb{Z}^n$;

(iii) At each vertex, the vectors $u_i, i = 1, \dots n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Fix some polytope Δ and let d be the number of facets in the polytope, necessarily $d \geq n$. Let $v_i, i = 1, \dots d$ be the outside-pointing normal vectors of the facets, which can be chosen to be integer valued vectors such that the entries do not have any common divisor.

The map $\mathbb{R}^d \to \mathbb{R}^n$ given by $e_i \mapsto v_i$ is a surjective map that induces surjection $T^d \to T^n$. Let the subtorus $N \subset T^d$ be the kernel of the homomorphism, then its Lie algebra $\mathfrak{n} \subset \mathbb{R}^d$ can be identified with the kernel of the linear map $\mathbb{R}^d \to \mathbb{R}^n$.

Let T^d act on $\mathbb{C}^d$ diagonally. This is clearly a Hamiltonian action and restricts to a Hamiltonian N-action with moment map $\mu : \mathbb{C}^d \to (\mathbb{R}^d)^* \cong \mathbb{R}^d$ given by

$$(z_1, \dots, z_d) \mapsto -\pi(|z_1|^2, \dots |z_d|^2) + \underline{c}$$

for some suitable constant[4] $\underline{c} \in \mathbb{R}^d$ such that N acts freely on $\mu^{-1}(0)$. The Delzant space X_Δ is then the symplectic reduction $\mathbb{C}^d /\!\!/ N$.

Here the ambient space $\mathbb{C}^d$ is no longer compact, and Atiyah's convexity results (explained in [13], §3.3-6) need re-interpretation. Still, local results as in lemma 2 remain valid, hence:

Lemma 8. The critical points of $f = ||\mu||^2$ are $(\xi_1, \dots, \xi_d)$ where some non-empty subset $I' \subset \{1, \dots d\}$ satisfies:

(i) $\sum_{j \in (I')^c} c_j v_j = 0$;

(ii) $\xi_{i'} = 0$ for $i' \in I'$;

(iii) $\pi|\xi_j|^2 = c_j$ for $j \in (I')^c$.

Hence the critical loci can be seen as a disjoint union of $(d - |I'|)$-dimensional tori embedded in $\mathbb{C}^{(d-|I'|)}$ subspaces cut out by $z_{i'} = 0$ for $i' \in I'$.

Proof. Let $\beta = (\beta_1, \dots, \beta_d) \in \mathfrak{n} \subset \mathbb{R}^d$ and consider elements (ξ_i) stablized by $\exp(t\beta)$. For each index i, either $\beta_i = 0$ or $\xi = 0$.

Suppose $\xi_{i'} = 0$ for some non-empty subset $i' \in I' \subset \{1, \dots d\}$, then necessarily $\beta_j = 0$ for $j \in (I')^c$. Conditions (i) and (iii) now come from imposing the condition that $\beta = \mu((\xi_i))$. □

Therefore, the components of the critical loci are indexed by elements $\beta \in \mathfrak{n}$ satisfying $\beta_{i'} = 0$ for $i' \in I' \subset I$ and $\beta_j = c_j$ for $j \in (I')^c$; upon restricting to some positive Weyl chamber, these shall give indices for the Morse stratification $C_{I'}$.

To describe the strata, we now take a closer look at the gradient flow of $f = ||\mu||^2$. Use the standard Riemannian metric $\sum_i dz_i d\bar{z}_i$. The gradient ∇_f is given by

$$\nabla_f = 2 \sum_i (\lambda_i - \pi|z_i|^2)(z_i \partial \bar{z}_i + \bar{z}_i \partial z_i).$$

Therefore, given some $I' \subset I$, the stratum $S_{I'}$ consist of points (z_i) with only the I'-entries vanishing have their gradient flows converging to $C_{I'}$.

[4]Since N is abelian, any constant vector $\underline{c}$ will do, but for convenience we shall assume that $\underline{c}$ has positive entries.

The regularity property in lemma 4 is still applicable in our setting, hence the stratification is again N-equivariantly perfect, leading to the restricting map $H^*_N(\mathbb{C}^d) \to H^*_N(\mu^{-1}(0))$ being surjective.

As $\mathbb{C}^d$ is contractible, the homotopy quotients $EN \times_N \mathbb{C}^d$ and $EN \times_N \mathrm{pt} \cong BN$ are homotopy equivalent, so $H^*_N(\mathbb{C}^d, \mathbb{C})$ is isomorphic to $H^*_N(\mathrm{pt}, \mathbb{C})$, just as in the ordinary theory[5] Thus multiplicative generators of $H^*_N(\mathrm{pt}, \mathbb{C}) \cong \mathbb{C}[u_1, \ldots, u_{d-n}]$ should correspond to a set of generators of $H^*(X_\Delta, \mathbb{C})$. The elements $u_1, \ldots, u_{d-n} \in H^2_N(\mathrm{pt})$ turn out to be identified with the Chern classes of the line bundles associated to the N-representation on the one-dimensional subspaces $\langle e_i \rangle$, and [9] §9.8 determines relations among the generators by applying the Duistermaat-Heckmann theorem.

6 The Kähler case and GIT

6.1 The Kähler quotient

Let M be a Kähler manifold, Γ be a complex Lie group which is the complexification of a maximal compact subgroup K so that $\mathrm{Lie}(\Gamma) = \mathfrak{k} \oplus i\mathfrak{k}$.

Suppose that K has a Hamiltonian action on M that preserves its Kähler form. Because we are now equipped with a distinguished Riemannian metric arising from the Kähler structure, there is an alternative, nicer description of the stratification constructed above.

In the following, $T \subseteq K$ (so that the complexification $T_\mathbb{C} \subseteq \Gamma$ is again a maximal torus), $\mu, \{\beta \in \mathcal{B}\}, Z_\beta, C_\beta, S_\beta$ are the same as above for the K-action on M;

Definition 9. (i) $Z_\beta^{\min}$ is the subset of points $x \in Z_\beta$ such that the gradient flow of $||\mu - \beta||^2$ from x has limit points in $Z_\beta \cap \mu^{-1}(\beta)$;

(ii) $Y_\beta^{\min}$ is the subset of points $y \in Y_\beta$ such that the gradient flow of μ_β from y has limit points in $Z_\beta^{\min}$;

(iii) Denote the Morse stratum $S_0 \subset M$ associated to $f = ||\mu||^2$ as $M^{\min}$;

(iv) Let B be the Borel subgroup of Γ associated to the positive Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$, and let P_β be the subset of $g \in \Gamma$ such that the curve $(\exp it\beta)g(\exp it\beta)^{-1}$: has a limit in Γ as $t \to \infty$.

Firstly, there are the standard analogous results about the stratification. Note that although the stratification is still based on K, the equivariant cohomology is over the complex group Γ.

Theorem 4 ([13], Theorem 7.4)**.** (i) $S_\beta = \Gamma \cdot (Y_\beta^{\min})$;

(ii) The stratification $\{S_\beta\}_{\beta \in \mathcal{B}}$ induces short exact sequences (as in corollary 1) for Γ-equivariant cohomology:

$$0 \to H_\Gamma^{n-d(\beta)}(S_\beta) \to H^n_\Gamma(U_\beta) \to H^n_\Gamma(U'_\beta) \to 0.$$

(iii) $S_\beta \cong \Gamma \times_{P_\beta} Y_\beta^{\min}$ so that

$$H^*_\Gamma(S_\beta) \cong H^*_\Gamma(Y_\beta^{\min}).$$

[5]The radial homotopy $\mathbb{C}^d \times I \to \mathbb{C}^d$ given by $((z_i), t) \mapsto (z_i / f(t))$ for some suitable real-valued function f is N-equivariant, so it induces a map of the quotients.

One might wonder the relation between a suitable quotient by Γ and the symplectic quotient by $K \subset G$. This is the content of the following theorem:

Theorem 5 ([13], Theorem)**.** If the stabilizer of every $x \in \mu^{-1}(0)$ in K is finite, then $M^{\min}/\Gamma$ is homeomorphic to $M /\!\!/ K = \mu^{-1}(0)/K$.

6.2 GIT

When M is a nonsingular complex projective variety, and Γ is a reductive complex Lie group acting linearly on M, there are two notions of quotients: the symplectic (Kähler) quotient and the GIT quotient $\tilde{M} := \mathrm{Proj}(H^0(M, \mathcal{O})^\Gamma)$ from algebraic geometry. It turns out that under regularity conditions, the two are identified with each other, so the GIT quotient can be approached using symplectic methods. Firstly, let us recall some basic notions in GIT.

Definition 10. Let $M \subset \mathbb{CP}^n$ be a nonsingular complex projective variety, and let Γ be a connected reductive complex group acting on M linearly, ie., via a homomorphism $\varphi : \Gamma \to GL(n+1)$.

$x \in M$ is *semistable* if there exists a homogeneous polynomial $F \in \mathbb{C}[x_0, \ldots, x_n]$ such that F is invariant under the Γ-action and $F(x) \neq 0$.

A *stable* point is a semistable point such that:

(i) with polynomial F as above, the orbits of $\{y \in M : F(y) \neq 0\}$ are closed;

(ii) the point has finite stabilizer under the Γ-action.

Denote the set of semistable points as M^{ss} and the set of stable points as M^s.

Lemma 9. ([15], Theorem 2.1) A point is semistable under the Γ-action if and only if it is semistable under the action of every one-parameter subgroup $\beta : \mathbb{C}^* \to \Gamma$.

Recall that in the symplectic case, one-parameter subgroups T_β are used to index the critical points and the stratification. Therefore, the action of one-parameter subgroups provides a starting point to relate the symplectic quotient and GIT.

Lemma 10. Let $\Gamma = \mathbb{C}^*$ act on M via $\beta : \mathbb{C}^* \to GL(n+1)$, $z \mapsto \mathrm{diag}(z^{r_0}, \ldots, z^{r_n})$ for $r_0, \ldots, r_n \in \mathbb{Z}$.

(i) ([15], Proposition 2.2) A point $x = [x_0 : \cdots : x_n] \in M$ is semistable if and only if $\min\{r_j : x_j \neq 0\} \leq 0 \leq \max\{r_j : x_j \neq 0\}$;

(ii) In this case M^{ss} coincides with $M^{\min} := S_0$, where S_0 is the Morse stratum associated to $f = ||\mu||^2$;

Proof. (Sketch for (ii)) According to theorem 3 (i), $x \in M^{\min}$ if and only if $0 \in \mu(\overline{\Gamma \cdot x})$. Because $\Gamma = \mathbb{C}^* = \mathbb{CP}^1 \setminus \{0, \infty\}$, Γ-action can be extended to the whole of $\mathbb{CP}^1$ via: 0 gives the map that only keeps the x_i coordinate where $r_i = \min\{r_j : x_j \neq 0\}$ and maps the rest of the coordinates to 0; ∞ only keeps the x_i coordinate where $r_i = \max\{r_j : x_j \neq 0\}$.

Therefore, $\overline{\Gamma \cdot x}$ can be identified with the image of x under the action of $\mathbb{CP}^1$, which agrees with the criterion for semistability in (i). □

We return to the general setting as in definition 10, and relate $M^{\min}$ with the minimum stratum of the one-parameter subgroup action.

Lemma 11 ([13], Lemma). Let $x \in M$ then $0 \in \mu(\overline{\Gamma x})$ if and only if $0 \in \mu(\overline{\beta(\mathbb{C}^*)x})$ for every one-parameter subgroup $\beta : \mathbb{C}^* \to \Gamma$ that arises as the complexification of some real one-parameter subgroup $\tilde{\beta} : S^1 \to K$.

The following theorem relates the Kähler (symplectic) quotient to the GIT quotient, ie., it connects the differetial-geometric and the algebraic point of view of a "moduli space", and allows one to study GIT using symplectic methods.

Theorem 6 ([13], Theorem). (i) With the same setting as in definition 10, suppose that Γ has a maximal compact subgroup K such that $\varphi(K) \subseteq U(n+1)$, then $M^{ss} = M^{\min}$;

(ii) If the stabilizer of every semistable point in Γ is finite, then $M^{ss} = M^s$, and M^{ss}/K is homeomorphic to the GIT quotient $\tilde{M}$.

In the following, we sketch two applications of the theory in moduli problems.

6.3 Example: n-tuples on $\mathbb{CP}^1$

We quickly sketch the key example in [13], listing some of the key ingredients that showed up in earlier discussion.

The compact Lie group $\Gamma = SU(2)$ acts on $(\mathbb{CP}^1, \omega_{FS})$ via $[v] \mapsto [Av]$ for $v \in \mathbb{C}^2 \setminus \{0\}, A \in SU(2)$. This is a Hamiltonian action with moment map given by $\tilde{\mu} : \mathbb{CP}^1 \to \mathfrak{su}(2)^*$:

$$\langle \tilde{\mu}([v]), \zeta \rangle = \frac{\bar{v}^t \zeta v}{2\pi ||v||^2}.$$

The diagonal action of $SU(2)$ on $(\mathbb{CP}^1)^n$, ie., the configuration space of n ordered points on a Riemann sphere, is hence also Hamiltonian with moment map $\mu : (\mathbb{CP}^1)^n \to \mathfrak{su}(2)^*$:

$$\langle \mu([v_1], \ldots, [v_n]), \zeta \rangle = \sum_{i=1}^{n} \frac{\bar{v}_i^t \zeta v_i}{2\pi ||v_i||^2}.$$

If one identifies $\mathfrak{su}(2)$ with $\mathbb{R}^3$, and $\mathbb{CP}^1$ with S^2, then up to scalar multiplication, the moment map is $(v_1, \ldots, v_n) \mapsto v_1 + \cdots + v_n$.

Applying the standard inner product on $\mathbb{R}^3$, $f = ||\mu||^2$ takes the minimum on $\mu^{-1}(0)$, which is the set of configurations with "centre of mass" at the origin. Outside $\mu^{-1}(0)$, the critical points are configurations with r coordinates on some $v \in S^2$ while the rest on the antipodal point $-v$. In particular, one sees that the norm of the moment map fails to be a classical Morse function, since the critical points are not isolated points in $(\mathbb{CP}^1)^n$.

Now we examine the stratification. Given any $\beta \in \mathfrak{su}(2)$, its one-parameter subgroup $T_\beta \cong S^1$ is a maximal torus of Γ, so $T = T_\beta$. The fixed point set of T_β-action are the configurations with coordinates in $\{0, \infty\}$. Identify $\mathfrak{t} \cong \mathbb{R}$ with the standard inner product. For a point x with r coordinates at 0 (south pole) and $n - r$ coordinates at ∞ (north pole), $\mu_\beta(x) = 2r - n$.

If $\mathfrak{t}_+ = \mathbb{R}^+$ is taken as the positive Weyl chamber, then

$$\mathcal{B} = \{2r - n : r \in \{0, 1, ..., n\}, 2r - n \geq 0\}.$$

Given $\beta = 2r - n$, $Z_\beta = Z_\beta \cap \mu^{-1}(\beta)$ consists of points with r coordinates at 0 and the rest at ∞. The $SU(2)$-orbits of Z_β, namely C_β, are then points with r coordinates coinciding, and the rest coinciding at a different coordinate.

7 Moduli of holomorphic bundles

When there is an equivariantly perfect stratification that leads to the surjection (1), a set of generators of $H^*_K(M)$ is mapped to a set of generators for the cohomology of the moduli space $H^*(M /\!\!/ K)$. This is particularly useful when M is contractible, so that $H^*_K(M) \cong H^*(BK)$ and thus it is feasible to find a set of generators for $H^*(BK)$.

This is the case in the work of Atiyah-Bott [1] on the moduli space of stable holomorphic vector bundles over a Riemann surface Σ. In fact they aimed to study the moduli space of the following objects, which can be described in three equivalent ways:

(i) Unitary representations of $\pi_1(\Sigma)$;

(ii) Unitary connections on a Hermitian vector bundle $E \to \Sigma$ with a certain central curvature;

(iii) Stable holomorphic vector bundles over Σ.

It turns out that the second viewpoint is precisely an infinite-dimensional analogue of the theory presented above, while the third viewpoint is easier for formulating an equivariantly perfect stratification. We shall introduce both of them and begin by setting various notations.

Definition 11. Let Σ be a Riemann surface with some Kähler volume form vol_Σ, and $E \to \Sigma$ a smooth complex vector bundle of rank n and degree k. Define $\mu(E) = k/n$.

Fix a Hermitian metric h: this is a smoothly-varying collection of Hermitian metrics h_x on the fibres E_x for $x \in \Sigma$.

Let $\mathrm{Aut}(E)$ denote the group of bundle automorphisms of E, $U(E) \subset \mathrm{Aut}(E)$ the automorphisms that preserve the Hermitian metric, $\mathrm{End}(E)$ denote the group of bundle endomorphisms of E, and $\mathfrak{u}(E) \subset \mathrm{End}(E)$ the skew-adjoint endomorphisms.

Definition 12. Let $\mathscr{C}$ be the space of all holomorphic structures on E, with elements represented by their Dolbeault operators[6]. A holomorphic bundle E is semistable if for all holomorphic subbundles $D \subset E$, $\mu(D) \le \mu(E)$; it is stable if all such inequalities are strict.

Definition 13. Let $\mathscr{A}$ be the space of all unitary connections[7] on E. Elements in $\mathscr{A}$ are represented by their covariant derivatives; when a connection is represented by some connection 1-forms A, its covariant derivative is denoted by d_A.

Given $d_A \in \mathscr{A}$, its curvature is $F_A := d_A \circ d_A \in \Omega^2(\Sigma, \mathfrak{u}(E))$. In local coordinates, $F_A = dA + \frac{1}{2}[A, A]$, where A is a collection of connection 1-forms.

The unitary gauge group $\mathscr{G}$ of E is defined as the group of bundle automorphisms that preserves the fibres and induce unitary maps on them. It is an infinite dimensional Lie group with Lie algebra identified[8] with $\Omega^0(\Sigma, \mathfrak{u}(E))$. From [1] §9, $\mathrm{Aut}(E)$ can be identified with the complexification $\mathscr{G}^{\mathbb{C}}$ of the gauge group.

[6] According to [8] §2.2.2, any Dolbeault operator $\bar{\partial}$ comes from a unique holomorphic structure on E, so there is a one-to-one correspondence between the two.

[7] This is a connection ∇ such that for any two sections $s_1, s_2 \in C^\infty(E)$, $dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$, which is equivalent to the connection 1-forms taking coefficients in $\mathfrak{u}(n)$ under some unitary trivialization.

[8] Take a one-parameter family of gauge transformations and differentiating yields a matrix in $\mathfrak{u}(n)$ on fibres that glue together to give a section of $\mathfrak{u}(E)$.

The gauge group acts[9] on $\mathscr{C}$: for $g \in \mathscr{G}$,

$$\bar{\partial} \in \mathscr{C}, g \cdot \bar{\partial} = g \circ \bar{\partial} \circ g^{-1}.$$

For $\nabla_A \in \mathscr{A}, s \in C^\infty(E)$, $\mathscr{G}$ acts by:

$$(g \cdot \nabla_A)s = g(\nabla_A(g^{-1} \cdot s)).$$

Expanding[10], $(g \cdot \nabla_A) = \nabla_A - (\nabla_A g)g^{-1}$.

Then, $\mathscr{G}$ acts on the curvature via $g \cdot F_A = gF_A g^{-1}$.

The following results set up an equivalence between items (ii) and (iii) in the above list:

Proposition 3. (i) $\mathscr{C}$ is non-empty, and is an affine space modeled on $\Omega^{(0,1)}(\Sigma, \mathrm{End}(E))$, and $\mathscr{A}$ is a non-empty affine space modeled on $\Omega^1(\Sigma, \mathfrak{u}(E))$, consequently their tangent spaces can be identified with the corresponding vector spaces;

(ii) Projecting any unitary connection via $\pi^{(0,1)} : \Omega^1(\Sigma, E) \to \Omega^{(0,1)}(\Sigma, E)$ gives a holomorphic structure on E ([1], §5);

(iii) Given any holomorphic structure $\bar{\partial}$, there exists a unique connection ∇ such that $\pi^{(0,1)} \circ \nabla = \bar{\partial}$; these two operations set up an isomorphism between $\mathscr{C}$ and $\mathscr{A}$ as affine spaces.

From the complex analytic point of view, there exists a $\mathscr{G}$-equivariantly perfect stratification of the space of holomorphic structures $\mathscr{C}$, with the bottom strata corresponding to the semi-stable holomorphic bundles $\mathscr{C}_{ss}$. In particular, the analogue of Kirwan surjectivity (1) holds, and generators of $H^*_{\mathscr{G}}(\mathscr{C})$ restrict to generators of $H^*_{\mathscr{G}}(\mathscr{C}_{ss}) \cong H^*(N(n,k))$, where $N(n,k) := \mathscr{C}_{ss}/\mathscr{G}$ is the moduli space of semi-stable bundles[11].

Taking a more differential-geometric perspective, Atiyah and Bott studied the Yang-Mills functional on $\mathscr{A}$. The functional is then associated to the Hamiltonian $\mathscr{G}$-action on $\mathscr{A}$, which can be seen as an infinite-dimensional symplectic manifold. They also conjectured that, just as in finite dimensional case, the gradient flows Yang-Mills functional lead to an equivariantly perfect stratification, which was later proved by Daskalopoulos ([5]).

Now we describe the stratification of holomorphic bundles.

7.1 Stratification

Seshadri ([16]) has shown that for every holomorphic bundle E, there exists a unique filtration

$$0 \subset E_1 \subset \cdots \subset E_r = E$$

such that each $D_i := E_i/E_{i-1}$ is semistable, and

$$\mu(D_1) > \mu(D_2) > \cdots > \mu(D_r).$$

The stratification, formulated by Harder and Narasimhan in [11], is given by the canonical filtrations:

[9] We use "$\cdot$" to denote the group action, and "$\circ$" as composition of maps.

[10] Here ∇_A acts on sections of $\mathrm{End}(E)$ by: given $M \in C^\infty(\mathrm{End}(E)), s \in C^\infty(E)$, $(\nabla_A(M))(s) = \nabla_A(M(s)) - M(\nabla_A(s))$.

[11] [1] mostly focuses on the case when $(n,k) = 1$, for which the $\mathscr{G}$-action on $\mathscr{C}_{ss}$ is free so that $N(n,k)$ is a well-defined space. Later, it will be shown that $N(n,k)$ is the symplectic reduction of $\mathscr{G}$-action on $\mathscr{A}$, so it is in fact a symplectic manifold.

Definition 14. For a holomorphic bundle E of rank n and degree k, denote $n_i = \text{rank } D_i, k_i = \deg D_i$.

Its type is then the vector $\boldsymbol{\mu} \in \mathbb{Q}^n$ where the first n_1 entries are $\mu(D_1)$, the next n_2 entries are $\mu(D_2)$, and so on[12].

Let $P_{\boldsymbol{\mu}}$ be the polygon in $\mathbb{R}^2$ with entries $(0,0), \ldots, (\sum_{i=0}^{j} n_i, \sum_{i=0}^{j} k_i), \ldots (n,k)$. Because $\mu(D_i)$ is strictly decreasing, the polygon is convex. Therefore, there is a well-defined ordering: $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$ if $P_{\boldsymbol{\mu}'}$ lies above $P_{\boldsymbol{\mu}}$.

Let $\mathscr{C}_{\boldsymbol{\mu}}$ be the space of all holomorphic bundles of a fixed type $\boldsymbol{\mu}$, and let $N_{\boldsymbol{\mu}}$ be the conormal bundle of $\mathscr{C}_{\boldsymbol{\mu}} \subset \mathscr{C}$.

Suppose we fix some holomorphic structure on E with type $\boldsymbol{\mu}$, and let $\text{End}'(E)$ denote the bundle endomorphisms of E that preserves its canonical filtration. Now define bundle $\text{End}''(E)$ as the cokernel

$$0 \to \text{End}'(E) \to \text{End}(E) \to \text{End}''(E) \to 0.$$

Thus, $H^1(\Sigma, \text{End}''(E))$ describes infinitesimal variation of holomorphic structures modulo those that preserve the filtration.

Remark 7. (i) For a semistable holomorphic bundle E, its canonical filtration is simply $0 \subset E$, hence the associated polygon is the straight line segment joining $(0,0)$. Therefore the semistable bundles $\mathscr{C}^{ss} \subset \mathscr{C}$ lies at the bottom of all strata, just as $\mu^{-1}(0) \subset M$ in the finite-dimensional case.

(ii) To proceed with the long exact sequence in equivariant cohomology, we need to show that $\mathscr{C}_{\boldsymbol{\mu}} \subset \mathscr{C}$ is locally of finite codimension. This is established in [1], §7, by identifying $N_{\boldsymbol{\mu}}$ with $H^1(\Sigma, \text{End}''(E))$.

Again, to show that the stratification $\mathscr{C} = \bigcup_{\boldsymbol{\mu}} \mathscr{C}_{\boldsymbol{\mu}}$ is equivariantly perfect, it suffices to show that multiplication by the $\mathscr{G}$-equivariant Euler class of $N_{\boldsymbol{\mu}}$ yields an injection $H_{\mathscr{G}}^{*-d(\boldsymbol{\mu})}(\mathscr{C}_{\boldsymbol{\mu}}) \to H_{\mathscr{G}}^{*}(\mathscr{C}_{\boldsymbol{\lambda}})$.

The strategy is the same as in finite-dimensional case: as in lemma 7 (i), we reduce the $\mathscr{G}$-action on $\mathscr{C}_{\boldsymbol{\mu}}$ to a smaller group acting on a subspace, in which a subtorus acts trivially. A result analogus to the localization theorem ([1], Proposition 13.4) then establishes the claim, so that

Proposition 4. (i) The long exact sequence

$$H_{\mathscr{G}}^{*-d(\boldsymbol{\mu})}(\mathscr{C}_{\boldsymbol{\mu}}) \to H_{\mathscr{G}}^{*}(\bigcup_{\boldsymbol{\lambda} \not> \boldsymbol{\mu}} \mathscr{C}_{\boldsymbol{\mu}}) \to H_{\mathscr{G}}^{*}((\bigcup_{\boldsymbol{\lambda} \not> \boldsymbol{\mu}} \mathscr{C}_{\boldsymbol{\lambda}}) \setminus \mathscr{C}_{\boldsymbol{\mu}}) \to H_{\mathscr{G}}^{*-d(\boldsymbol{\mu})+1}(S_{\boldsymbol{\mu}}) \to \cdots$$

splits into short exact sequences:

$$0 \to H_{\mathscr{G}}^{*-d(\boldsymbol{\mu})}(\mathscr{C}_{\boldsymbol{\mu}}) \to H_{\mathscr{G}}^{*}(\bigcup_{\boldsymbol{\lambda} \not> \boldsymbol{\mu}} \mathscr{C}_{\boldsymbol{\mu}}) \to H_{\mathscr{G}}^{*}((\bigcup_{\boldsymbol{\lambda} \not> \boldsymbol{\mu}} \mathscr{C}_{\boldsymbol{\lambda}}) \setminus \mathscr{C}_{\boldsymbol{\mu}}) \to 0,$$

(ii) The restriction map

$$H_{\mathscr{G}}^{*}(\mathscr{C}) \to H_{\mathscr{G}}^{*}(\mathscr{C}_{ss})$$

is surjective.

[12]This makes sense because $\sum_{i=1}^{r} n_i = n$.

Remark 8. In [1], the equivariant cohomology is in fact over the larger group $\mathrm{Aut}(E)$. Still, they observed that $\mathrm{Aut}(E)$ can be identified with the complexification $\mathscr{G}^{\mathbb{C}}$, and $\mathrm{Aut}(E)/\mathscr{G}$ is isomorphic to the contractible space of Hermitian metrics on E. Hence results about $\mathrm{Aut}(E)$-equivariant cohomology can be carried over to $\mathscr{G}$.

When n and k are coprime, Proposition 2.20 in [1] provides a set of (multiplicative) generators of $H^*(B\mathscr{G}, \mathbb{Z})$. As $\mathscr{C}$ is contractible, they correspond to a set of generators of $H^*_{\mathscr{G}}(\mathscr{C}, \mathbb{Z})$, which restrict to generators of $H^*(N(n,k), \mathbb{Z})$.

At the same time, the equivariantly perfect stratification also provides inductive formulas for the $\mathscr{G}$-equivariant Poincaré series of the strata $\mathscr{C}_{\boldsymbol{\mu}}$ ([1], Proposition 7.12, 7.14), leading to the Poincare series of the moduli $N(n,k)$ in [1], §9.

To completely describe the cohomology ring, it remains to determine the relation among the generators. This falls into the theory of “non-abelian localization” introduced in [18] and [12]. For now, we shall turn to the differential-geometric point of view and start by introducing a symplectic structure on $\mathscr{A}$.

7.2 Yang-Mills functional on unitary connections

Let $d_A \in \mathscr{A}$, and identify $T_{d_A}\mathscr{A}$ with $\Omega^1(\Sigma, \mathfrak{u}(E))$. Fibres of $\mathfrak{u}(E)$ can be identified with the vector space of $n \times n$ anti-Hermitian matrices, which admit a positive-definite inner product: $\mathrm{Tr} := (X, Y) \mapsto -\mathrm{Trace}(XY)$. Tr is invariant under coadjoint action from $U(n)$.

The Hodge star operator $*$ acts on $\Omega^*(\Sigma)$ and satisfies $*^2 = -1$. We now extend these structures to bundles $\Omega^*(\mathfrak{u}(E))$:

Definition 15. (i) Given $a \in \Omega^k(\Sigma, \mathfrak{u}(E)), b \in \Omega^{2-k}(\Sigma, \mathfrak{u}(E))$, so that

$$a = \sum_i A_i \otimes \omega_i, b = \sum_j B_j \otimes \mu_j$$

for $A_i, B_j \in \mathfrak{u}(E)$ and $\omega_i \in \Omega^k(\Sigma), \mu_j \in \Omega^{2-k}(\Sigma)$, define

$$\mathrm{Tr}(a, b) := \sum_{i,j} \mathrm{Tr}(A_i, B_j)\omega_i \wedge \mu_j \in \Omega^2(\Sigma);$$

(ii) Define $* : \Omega^k(\Sigma, \mathfrak{u}(E)) \to \Omega^{2-k}(\Sigma, \mathfrak{u}(E))$ by sending

$$a = \sum_i A_i \otimes \omega_i \mapsto \sum_i A_i \otimes *\omega_i.$$

Thus there is a Riemannian metric on $\Omega^k(\Sigma, \mathfrak{u}(E))$ given by

$$(a, a') \mapsto \int_\Sigma \mathrm{Tr}(a \wedge *a');$$

Now define $\omega \in \Omega^2(\mathscr{A})$ by: for tangent vectors $a, b \in T_{d_A}\mathscr{A} \cong \Omega^1(\mathfrak{u}(E))$

$$\omega(a, b)|_{d_A} := \int_\Sigma \mathrm{Tr}(a, b).$$

The following summarizes the Hamiltonian $\mathscr{G}$-action on $\mathscr{A}$:

Proposition 5. (i) ω defines a $\mathscr{G}$-invariant symplectic form, so that $(\mathscr{A}, \omega)$ is an infinite dimensional symplectic manifold;

(ii) Given $\xi \in \Omega^0(\Sigma, \mathfrak{u}(E)) \cong \mathrm{Lie}(\mathscr{G})$, the infinitesimal action on d_A is given by $-d_A\xi \in \Omega^1(\Sigma, \mathfrak{u}(E))$;

(iii) Under the pairing in definition 15 (ii), $* : \Omega^2(\Sigma, \mathfrak{u}(E)) \to (\Omega^0(\Sigma, \mathfrak{u}(E)))^*$ given by

$$\alpha \mapsto (\beta \mapsto \int_\Sigma \mathrm{Tr}(\beta, \alpha))$$

establishes an $\mathscr{G}$-equivariant isomorphism between $\Omega^2(\Sigma, \mathfrak{u}(E))$ and $(\Omega^0(\Sigma, \mathfrak{u}(E)))^* \cong (\mathrm{Lie}(\mathscr{G}))^*$;

(iv) The $\mathscr{G}$-action has moment map at d_A given by

$$F_A \in \Omega^2(\Sigma, \mathfrak{u}(E)) \cong (\mathrm{Lie}(\mathscr{G}))^*.$$

(v) For $a \in \Omega^k(\Sigma, \mathfrak{u}(E))$ define the map

$$a \mapsto ||a||^2 := \int_\Sigma \mathrm{Tr}(a, *a).$$

This defines a $\mathscr{G}$-invariant Riemannian metric on each $\Omega^k(\Sigma, \mathfrak{u}(E))$. In particular, $\mathscr{A}$ is a Kähler manifold and the $\mathscr{G}$-action preserves the Kähler structure. The square norm of the moment map is then $L(d_A) := ||F_A||^2$, the *Yang-Mills* functional of $d_A \in \mathscr{A}$.

Proof. (i) ω is clearly a 2-form. It is closed because the definition does not depend on $d_A \in \mathscr{A}$ so it is constant across $\mathscr{A}$, and non-degenerate by considering $\int_\Sigma \mathrm{Tr}(a \wedge *a) \geq 0$;

ω is $\mathscr{G}$-invariant because Tr is $U(n)$-invariant on each fibre. vol_Σ is also fixed by maps in $\mathscr{G}$, since they project to id_Σ.

(ii) Take a one-parameter family g_t of gauge transformations with $g_0 = \mathrm{id}$ and differentiate $(g_t \cdot d_A) = d_A - (d_A g_t) g_t^{-1}$ at $t = 0$ yields

$$\frac{d}{dt}(d_A - (d_A g_t) g_t^{-1})|_{t=0} = -(d_A g_t)(\frac{d}{dt} g_t^{-1}) - \frac{d}{dt}(d_A g_t) g_t^{-1}|_{t=0} = -d_A(\frac{d}{dt}(g_t)|_{t=0}) = -d_A(\xi).$$

(iii) $*$ establishes an isomorphism as vector spaces. Let U, A, B be $n \times n$ matrices, then $\mathrm{trace}(BU^{-1}AU) = \mathrm{trace}(UBU^{-1}A)$. It follows that for $u \in \mathscr{G}$,

$$u \cdot \alpha \mapsto (\beta \mapsto \int_\Sigma \mathrm{Tr}(\beta, u \cdot \alpha)) = (\beta \mapsto \int_\Sigma \mathrm{Tr}(u^{-1} \cdot \beta, \alpha)),$$

which agrees with the coadjoint action on the linear functional.

(iv) Let $\xi \in \Omega^0(\Sigma, \mathfrak{u}(E))) \cong \mathrm{Lie}(\mathscr{G})$. The vector field generated by ξ has value $d_A\xi$ at $d_A \in \mathscr{A}$. Hence, we wish to show that at $d_A \in \mathscr{A}$, $\iota_{(d_A\xi)}\omega = d(\langle F_A, \xi \rangle)|_A$.

Given any $a \in \Omega^1(\Sigma, \mathfrak{u}(E))) \cong T_{d_A}\mathscr{A}$,

$$\iota_{(d_A\xi)}\omega(a) = -\int_\Sigma \mathrm{Tr}(a, d_A\xi) = \int_\Sigma \mathrm{Tr}(d_A a, \xi),$$

which follows from $\int_\Sigma d(\mathrm{Tr}(a, \xi)) = 0$.

On the other hand, it follows from the local description of the curvature in definition 13 that $F_{A+ta} - F_A = d_A a + \frac{1}{2}t^2[a, a]$ ([1], Lemma 4.5), hence

$$d\Big(\int_\Sigma \mathrm{Tr}(F_A, \xi)\Big)(a)|_A = \frac{d}{dt}\int_\Sigma \mathrm{Tr}(F_{A+ta}, \xi)|_{t=0} = \int_\Sigma \mathrm{Tr}(d_A a, \xi).$$

□

Establishing the symplectic structure and Hamiltonian group action, we shall turn to the critical points of the Yang-Mills functional. Just as in the finite dimensional case, they index the strata, which consists of gradient flows "converging" to components of the critical points.

Proposition 6. (i) The gradient vector field ∇L (as in definition 6) equals $- * d_A * F_A$ at point d_A;

(ii) $d_A \in \mathscr{A}$ is a critical point of J, ie., *Yang-Mills connection*, if and only if $d_A * F_A = 0$, so the set of Yang-Mills connections is $\mathscr{G}$-invariant;

(iii) [[1], §5] If d_A is a Yang-Mills connection, then the eigenvalues of $\frac{i}{2\pi} * F_A \in C^\infty(\mathfrak{u}(E))$ are (locally) constant.

Proof. (ii) and follows from (i): Given some tangent vector a,

$$\frac{d}{dt}(J(A+ta))|_{t=0} = 2\langle F_A, d_A a\rangle = \langle d_A^* F_A, a\rangle,$$

where $\langle , \rangle$ are taken over the Riemannian metrics described in proposition 5 (v), and d_A^* is the adjoint of the linear map d_A. From [1] §4, $d_A^* = - * d_A *$. □

Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\frac{i}{2\pi} * F_A$, and define $\boldsymbol{\mu}(d_A) = (\lambda_1, \ldots, \lambda_n)$. Denote $\mathscr{Y}$ as the set of all Yang-Mills connections and $\mathscr{Y}_{\boldsymbol{\mu}_0}$ as the set of Yang-Mills connections $d_A \in \mathscr{Y}$ such that $\boldsymbol{\mu}(d_A) = \boldsymbol{\mu}_0$. For instance, if $F_A = *2\pi i \frac{k}{n}$, the associated index is $\boldsymbol{\mu} = (k/n, \ldots, k/n)$.

We shall see that the n-tuples $\boldsymbol{\mu}$ index the Morse strata associated to the Yang-Mills functional, which in fact coincides with the Harder-Narasimhan stratification explained earlier. In particular, the index $(k/n, \ldots, k/n)$ corresponds precisely to the strata of stable bundles.

Now we turn to gradient flows, which are crucial in establishing the Morse stratification. For general symplectic actions, the gradient flows for $f = ||\mu||^2$ may fail to converge, approaching the critical locus only asymptotically, an issue that [13], §10 circumvents by appealing to the continuity of Cech cohomology. Still, Yang-Mills functional satisfies suitable regularity properties to establish the following results:

Theorem 7 ([5],§5-6)**.** (i) The Yang-Mills gradient flow has local solutions that can be extended in any finite time;

(ii) The $\mathscr{G}$-orbit of a gradient flow converges to a unique $\mathscr{G}$-orbit of Yang-Mills connection;

(iii) The Yang-Mills gradient flows set up deformation retraction[13] of $(L_1^2(\Sigma) \otimes \mathfrak{u}(E))/\mathscr{G}$ onto $\mathscr{Y}/\mathscr{G}$;

(iv) The partition $\mathscr{Y} = \bigcup_{\boldsymbol{\mu}} \mathscr{Y}_{\boldsymbol{\mu}}$ is $\mathscr{G}$-invariant, and its induced Morse stratum on $\mathscr{A}$ coincides with the Harder-Narasimhan stratification.

In the following we shall quickly go through several results regarding the regularity of the Yang-Mills flow. Meanwhile, the holomorphic notion of stability shall be related to the more differential-geometric concept of curvature, which can be considered analogous to the relationship between GIT and symplectic reduction as seen in section 6.

[13] This is stronger than the cohomological equivalence for general finite-dimensional cases as in [13]. In particular, Daskalopoulos continues to establish isomorphisms between homotopy groups of the strata and critical loci, while §4 of [5] also exhibits satisfactory properties so that the cohomological method is applicable.

According to (i) in proposition 6, the gradient flow equation is

$$\frac{dA_t}{dt} = -d^*_{A_t} A_t. \tag{2}$$

In [6] §1.1, the equation is transformed into a nonlinear parabolic equation: for $g_t \in \mathrm{Aut}(E) \cong \mathscr{G}^{\mathbb{C}}$ and some fixed initial connection A_0, $A_t := g_t(A_0)$ satisfies equation 2 if and only if $h_t := g_t^* g_t$ satisfies[14]

$$h_0 = \mathrm{id}, \frac{\partial h}{\partial t} = -2ih_t\Lambda(F_{A_t} + \bar{\partial}_0(h_t^{-1}(\partial_0 h_t)) - \lambda),$$

where $\Lambda : \Omega^2(\Sigma, \mathrm{End}(E)) \to \Omega^0(\Sigma, \mathrm{End}(E))$ is defined by $f\mathrm{vol}_\Sigma \mapsto f$, and λ is any constant. This is a parabolic equation in h, hence by [10] p. 122, there exists $\epsilon > 0$ such that some smooth solution h_t exists for $t \in [0, \epsilon]$.

Setting up a notion of distance in $\mathscr{G}$ by examining the $\mathscr{G}$-action on the Hermitian metric of E, Donaldson also shows that ([6], Corollary 14, 15) the gradient flow is locally unique and can be continued along any finite time.

To establish the limit of the gradient flow, we pass to the Sobolev (L^2_1) versions of the above bundles[15], and use $L^2_1(\cdot)$ to denote the Sobolev version of the original bundle. In this setting, Uhlenbeck's theorem (item (i) below) provides sequential compactness on the space of $\mathscr{G}$, while the following statement, due to Daskalopoulos, applies the theorem into the Yang-Mills setting.

Theorem 8. (i) ([17], Theorem 7.1) Let A_n be a sequence of L^2_1 connections such that $|F_{A_n}|^2 \leq \epsilon$ for some $\epsilon > 0$. Then there exists a subsequence A'_n and $g'_n \in \mathscr{G}$ such that $g'_n \cdot A'_n$ converges L^2_1-weakly;

(ii) ([5], Proposition 4.1) Let A_n be a sequence of L^2_1 connections such that $|F_{A_n}|^2 \leq \epsilon$ for some $\epsilon > 0$ and $|d^*_{A_n} F_{A_n}|^2 \to 0$ as $n \to \infty$, then there exists a subsequence A'_n and $g'_n \in \mathscr{G}$ such that $g'_n \cdot A'_n$ converges L^2_1-strongly to a smooth connection A_∞ which is a critical point of the Yang-Mills functional.

An important special case is for connections that correspond to a stable holomorphic bundles. In [7], Donaldson relates the notion of stability with curvature and Yang-Mills connections:

Theorem 9. (i) For $d_A \in \mathscr{A}$, define $\tilde{J}(d_A) := |\frac{*F}{2\pi i} + \mu(E)|$. $\tilde{J}$ can be seen as a modification of the Yang-Mills functional J. Then the infimum $\inf_{g \in \mathscr{G}^{\mathbb{C}}} \tilde{J}(g \cdot d_A)$ is attained in the $\mathscr{G}^{\mathbb{C}}$-orbit of d_A;

(ii) d_A corresponds to a stable holomorphic bundle if and only if the $\inf_{g \in \mathscr{G}^{\mathbb{C}}} \tilde{J}(g \cdot d_A)$ is attained at some A_0 in the orbit with $F(A_0) = -2\pi i \mu(E)$, if and only if it is $\mathscr{G}^{\mathbb{C}}$-equivalent to a Yang-Mills connection, which is unique up to $\mathscr{G}$-equivalence.

An application of Uhlenbeck's theorem shows that the infinimum is indeed attained by some connection $d_{A'}$. Let $d_A, d_{A'}$ corresponds to holomorphic bundles E_1, E_2, then elliptic inequalities on the bundle $\mathrm{Hom}(E, E)$ show that $\mathrm{Hom}(E_1, E_2) \neq 0$. A map $\alpha \in \mathrm{Hom}(E_1, E_2)$ can be factored into a "map of bundle extensions", say $0 \to E'_i \to E_i \to E''_i \to 0$, where E'_i, E''_i have smaller

[14] $g_t^* \in \mathrm{Aut}(E)$ denotes the adjoint of g_t, so $g \in \mathscr{G}$ if and only if $g^* g = \mathrm{id}$.

[15] In other words, coefficients with $L^2_1(\Sigma)$ are allowed. We shall assume that $\mathscr{G}$- and $\mathscr{G}^{\mathbb{C}}$-actions are extended to the new bundles and use the L^2- as well as L^2_1-norms of the curvatures. More details can be found in [17], Appendix A.

ranks. By expressing curvature associated to E_i in terms of those of E_i', E_i'', lower and upper bounds are found for $\inf_{g\in\mathscr{G}^{\mathbb{C}}} \tilde{J}(g\cdot d_A)$ and $\inf_{g\in\mathscr{G}^{\mathbb{C}}} \tilde{J}(g\cdot d_{A'})$ are found in terms of the ranks and degrees of the smaller bundles, which, applying induction on the rank of holomorphic bundles, shows that E_1 and E_2 must be $\mathscr{G}^{\mathbb{C}}$-equivalent, which establishes item (i). Item (ii) now follows from differentiating the $\mathscr{G}$-action on the curvature at some $\tilde{J}$-minimizing connection.

In the formulation of [5], there is thus a map $r : \mathscr{A}^s/\mathscr{G} \to \mathscr{Y}/\mathscr{G}$. By considering the Seshadri filtration of semistable bundles, r can be extended to $r : \mathscr{A}^{ss}/\mathscr{G} \to \mathscr{Y}/\mathscr{G}$; as the $\mathscr{G}^{\mathbb{C}}$-orbits of semistable L_1^2-connections contain smooth connections, we have $r : L_1^2(\mathscr{A}^{ss})/\mathscr{G} \to \mathscr{Y}/\mathscr{G}$. Finally, by considering Seshadri filtration of general bundles, there is

$$r : L_1^2(\mathscr{A})/\mathscr{G} \to \bigcup_{\boldsymbol{\mu}}(\mathscr{Y}_{\boldsymbol{\mu}}/\mathscr{G});$$

the map r also induces a $\mathscr{G}^{\mathbb{C}}$-invariant stratification on $L_1^2(\mathscr{A})$ itself: define $L_1^2(\mathscr{A})_{\boldsymbol{\mu}}/\mathscr{G}$ as all $\mathscr{G}$-orbits such that their image under r lies in $\mathscr{Y}_{\boldsymbol{\mu}}/\mathscr{G}$.

Technical analytic arguments are employed to prove (i) of the following theorem, and the next two claims follow from it.

Theorem 10 ([5], §5)**.** (i) r is continuous;

(ii) In $L_1^2(\mathscr{A})/\mathscr{G}$, any Yang-Mills flow within a single $\mathscr{G}^{\mathbb{C}}$-orbit converges to a unique Yang-Mills connection;

(iii) $r : (L_1^2(\mathscr{A}))_{\boldsymbol{\mu}}/\mathscr{G} \to \mathscr{Y}_{\boldsymbol{\mu}}/\mathscr{G}$ is a deformation retraction.

Daskalopoulos also compares the Yang-Mills stratification with the Harder-Narasimhan stratification on $\mathscr{C} \cong \mathscr{A}$:

Theorem 11. The Harder-Narasimhan stratification can be extended to one on $L_1^2(\mathscr{A})$, which we shall denote as $(L_1^2(\mathscr{A}))_{\boldsymbol{\mu}}^{\wedge}$ to distinguish from the Yang-Mills one.

(i) ([5], Proposition 4.12) For each $\boldsymbol{\mu}$, there exists a neighborbood $U_{\boldsymbol{\mu}}$ of $L_1^2(\mathscr{A})_{\boldsymbol{\mu}}/\mathscr{G}$ in $L_1^2(\mathscr{A})/\mathscr{G}$ such that

$$U_{\boldsymbol{\mu}} \cap (L_1^2(\mathscr{A})_{\boldsymbol{\mu}}/\mathscr{G} \subset (L_1^2(\mathscr{A}))_{\boldsymbol{\mu}}^{\wedge}/\mathscr{G};$$

(ii) ([5], Theorem 6.2) $(L_1^2(\mathscr{A}))_{\boldsymbol{\mu}}^{\wedge}$ coincides with $(L_1^2(\mathscr{A}))_{\boldsymbol{\mu}}$.

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