TRIANGULATED CATEGORIES AND DERIVED CATEGORIES

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1. Road Map to Derived Categories

We assume the categories are always additive, e.g. Mod_k-enriched, for k commutative. For the most of cases, it is convenient to assume that the categories are abelian.

Let $\mathcal{A}$ be an abelian category. There are two ways to obtain the derived category $D(\mathcal{A})$ of $\mathcal{A}$. Start with the category of chain complexes of $\mathcal{A}$, denoted by $\mathrm{Ch}(\mathcal{A})$. It is a differential graded category. The first way is to take its homotopy category $K(\mathcal{A})$, which is a triangulated category, then applies the Verdier quotient (which is a localisation) to obtain $D(\mathcal{A})$. The second way is to take the Keller-Drinfeld quotient to get $\mathrm{Ch}(\mathcal{A})/\{\mathrm{Acyclic}\}$, then take the homotopy category of it, which is $D(\mathcal{A})$. In fact, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Ch}(\mathcal{A}) & \xrightarrow{\text{(Keller-)Drinfeld quotient}} & \mathrm{Ch}(\mathcal{A})/\{\mathrm{Acyclic}\} \\ {\scriptstyle\text{homotopy category}}\downarrow & & \downarrow{\scriptstyle\text{homotopy category}} \\ K(\mathcal{A}) & \xrightarrow{\text{Verdier quotient}} & D(\mathcal{A}) \end{array} \tag{1}$$

We can generalize the starting point $\mathrm{Ch}(\mathcal{A})$ to $\mathcal{A}_\infty$-category, and obtain the category of twisted complexes by taking Verdier quotient of $H^0\mathcal{A}$. This is often used in HMS. However, one should be warned that the left vertical arrow in (1) usually does not exist because $H^0\mathcal{A}$ is not necessarily triangulated.

Set $\mathcal{A}$ to be the category of coherent sheaves of a algebraic variety of X, and everything in (1) is taken to be "bounded". That is, for $M \in D^\sharp(\mathcal{A})$ ($D^\sharp(\mathcal{A})$ is "bounded" $D(\mathcal{A})$), $M^i = 0$ for $|i| \gg 0$. In HMS, the lower horizontal arrow in (1) usually refers to the B**-side** (be careful that the left vertical arrow still does not exist in general). On the other hand, the upper right corner of (1) corresponds to the A**-side**. In this approach, we choose a symplectic manifold (M,ω), and a Fukaya category $\mathcal{F}(M,\omega)$. Note that a Fukaya category is an $\mathcal{A}_\infty$-category. After moving forward along the arrows, we reach the derived Fukaya category $D\mathcal{F}(M,\omega)$, which is $H^0(\mathrm{Tw}F(M,\omega))$, where $\mathrm{Tw}F(M,\omega)$ is a twisted $\mathcal{A}_\infty$-category coming from the twisted complexes. We can prove that A-side and B-side essentially provide the same information.

We will introduce the notions in (1) in the following lectures. For reference, the reader is encouraged to review [1] for a detailed discussion.

2. Triangulated Categories

Definition 2.1. $\mathcal{C}$ is a **triangulated category**, if it consists of the following data:

(1) an auto-morphism $T:\mathcal{C}\to\mathcal{C}$,
(2) a collection of **distinguished triangles**, each of which is a sequence of morphisms $A\to B\to C\to T(A)$ such that the following axioms are satisfied:

TR1 (a) If $u:X\to Y$, then there exists a $Z\in\mathcal{C}$ s.t. $X\xrightarrow{u}Y\to Z\to T(X)$ is a distinguished triangle;
(b) for every $X\in\mathcal{C}$, $X\xrightarrow{\mathrm{id}}X\to 0\to T(X)$ is a distinguished triangle, where 0 is the zero object of $\mathcal{C}$;
(c) if triangles $\Delta\cong\Delta'$ and Δ is distinguished, then Δ' is distinguished.

TR2 If $X\to Y\to Z\to TX$ is a distinguished triangle, then $T^{-1}Z\to X\to Y\to Z$ and $Y\to Z\to TX\to TY$ are distinguished triangles.

TR3 Given two distinguished triangles $X\xrightarrow{f}Y\xrightarrow{g}Z\xrightarrow{h}TX$ and $X'\xrightarrow{f'}Y'\xrightarrow{g'}Z'\xrightarrow{h'}TX'$, and morphisms $\alpha:X\to X'$ and $\beta:Y\to Y'$ with $f'\circ\alpha=\beta\circ f$, there exists a morphism (necessarily unique) $\gamma:Z\to Z'$ giving rise to a morphism of distinguished triangles:

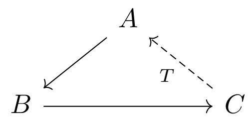
$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T\alpha} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX' \end{array}$$

TR4 (Octahedral axiom) Given three distinguished triangles

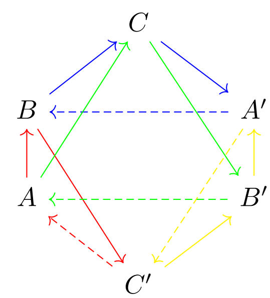
$$A\xrightarrow{u}B\xrightarrow{v}C\xrightarrow{\partial}TA,$$
$$B\xrightarrow{v}C\xrightarrow{x}A'\xrightarrow{i}TB,$$
$$A\xrightarrow{v\circ u}C\xrightarrow{y}B'\xrightarrow{\delta}TA,$$

there exists a distinguished triangle $C'\xrightarrow{f}B'\xrightarrow{g}A'\xrightarrow{T(j)\circ i}TC'$, such that $\partial=\delta\circ f$, $x=g\circ y$, $y\circ v=f\circ j$, and $u\circ\delta=i\circ g$.

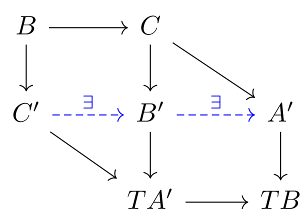
Usually, we express the distinguished triangle $A \to B \to C \to TA$ as the diagram:



We use the dashed arrow $C \to A$ because the diagram is not necessarily commutative. The octahedral axiom gets its name because it can be expressed as



In the diagram, each face enclosed by same-colored triangle is exact and other faces commute. An even better way to visualize this is to focus on each block. For example,



Then octahedral axiom urges each path from B to TB in the diagram (distinguished triangles), there exists $C' \to B'$, and $B' \to A'$ s.t. all paths are distinguished triangles, and the whole diagram commutative.

Definition 2.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from a triangulated category to an abelian category. F is called a **cohomolgical functor** if for every distinguished triangle $A \to B \to C \to TA$, $F(A) \to F(B) \to F(C)$ is exact.

Remark 2.3. By TR2, a cohomolgical functor gives a long exact sequence:

$$\cdots \to F(A) \to F(B) \to F(C) \to F(TA) \to \cdots$$

Proposition 2.4. *Let $\mathcal{C}$ be triangulated.*

(1) If $A \xrightarrow{f} B \xrightarrow{g} C \to TA$ is a distinguished triangle, then $g \circ f = 0$.
(2) For any $W \in \mathcal{C}$, the functors $\hom_{\mathcal{C}}(W, -)$ and $\hom_{\mathcal{C}}(-, W)$ are cohomological.

Proof. (1) Apply TR1.(b) and TR3 to get a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\text{id}} & A & \longrightarrow & 0 & \longrightarrow & TA \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \text{id}} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & TA \end{array}$$

Then $g \circ f = 0$.

(2) Let $A \xrightarrow{f} B \xrightarrow{g} C \to TA$ be a distinguished triangle. We want to show $\hom_{\mathcal{C}}(W, A) \xrightarrow{f\circ -} \hom_{\mathcal{C}}(W, B) \xrightarrow{g\circ -} \hom_{\mathcal{C}}(W, C)$ is exact. That is, for all $\phi : W \to B$ with $g \circ \phi = 0$, there exists $\psi : W \to A$ such that $f\psi = \phi$. Apply TR1.(b), TR2, and TR3 to get a commutative diagram

$$\begin{array}{ccccccc} W & \xrightarrow{\text{id}} & W & \longrightarrow & 0 & \longrightarrow & TW \\ {\scriptstyle \exists\psi}\downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & TA \end{array}$$

The result follows. The proof for $\hom_{\mathcal{C}}(-, W)$ is basically the same.

□

Let $\mathcal{A}$ be abelian and cocomplete. Denote the category of chain complexes associated with $\mathcal{A}$ by $\text{Ch}(\mathcal{A})$. This is again an abelian category because the zero object is $\cdots \to 0 \to 0 \to 0 \to \cdots$, and the kernel of a morphism $f_\bullet : A_\bullet \to B_\bullet$ is the complex of the kernels $\ker(f_i)$ for $i \in \mathbb{Z}$. Similarly we can figure out the cokernels $\text{coker}\,(f_i)$ for $i \in \mathbb{Z}$. $\mathcal{A}$ is a full subcategory of $\text{Ch}(\mathcal{A})$ because we can identify an object $A \in \mathcal{A}$ with a complex $A_\bullet$ with $A_0 = A$ and $A_i = 0$ for all $i \neq 0$.

$\text{Ch}(\mathcal{A})$ has two important features: cohomology and shift.

Definition 2.5. Let $A_\bullet \in \text{Ch}(\mathcal{A})$. A **shift functor** $T : \text{Ch}(\mathcal{A}) \to \text{Ch}(\mathcal{A})$ is given by $T(A_\bullet) = A_\bullet[1]$, where $A_i[1] = A_{i+1}$ and $d_i^{A_\bullet[1]} = -d_{i+1}^{A_\bullet}$. For $f_\bullet : A_\bullet \to B_\bullet$, $Tf_\bullet = f_\bullet[1]$, where $f_i[1] = f_{i+1}$.

It is easy to see T defines an equivalence of abelian categories, whose inverse functor T^{-1} is given by $A_\bullet \mapsto A_\bullet[-1]$. However, $\text{Ch}(\mathcal{A})$ endowed with T does not define a triangulated category. This is because the canonical choices for distinguished triangles, like short exact sequences or mapping cones, do not satisfy the axioms. To fix it, we localize $\text{Ch}(\mathcal{A})$ to its **homotopy category** $K(\mathcal{A})$, whose objects are the same as ones in $\text{Ch}(\mathcal{A})$, but morphisms are

$$\hom_{K(\mathcal{A})}(A, B) = \hom_{\text{Ch}(\mathcal{A})}(A, B)\big/\text{chain homotopies}.$$

Proposition 2.6. *$K(\mathcal{A})$ is additive, but not necessarily abelian.*

The proof of additivity is tedious. One has to check for the definition of additive categories and uses the fact that $\text{Ch}(\mathcal{A})$ is abelian. We encourage the readers to do it by yourself or to see the section 2 of the note The Homotopy Category of Chain Complexes and Triangulated Categories for details. $K(\mathcal{A})$ is not necessarily abelian because the kernel of a chain map in $\text{Ch}(\mathcal{A})$ might not exist in $K(\mathcal{A})$. In fact, we have the following result:

Proposition 2.7. *If $\mathcal{C}$ is abelian and triangulated, then it is semi-simple (i.e. every short exact sequence in $\mathcal{C}$ splits).*

Proof. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in $\mathcal{C}$. By TR1.(a), there exists a distinguished triangle $A \to\xrightarrow{f} B \xrightarrow{u} Z \xrightarrow{v} TA$ for some $Z \in \mathcal{C}$. By TR1.(b), we have the following commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u} & Z & \xrightarrow{v} & TA \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \text{id}} & & & & \downarrow{\scriptstyle Tf} \\ B & \xrightarrow{\text{id}} & B & \longrightarrow & 0 & \longrightarrow & TB \end{array}$$

By TR3, there exists a morphism $\gamma : Z \to 0$, which is unique since 0 is the zero object, such that the diagram commutes

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u} & Z & \xrightarrow{v} & TA \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle Tf} \\ B & \xrightarrow{\mathrm{id}} & B & \longrightarrow & 0 & \longrightarrow & TB \end{array}$$

In particular, $0 = Tf \circ v = Tf \circ T(T^{-1}v) = T(f \circ T^{-1}(v))$, yielding $f \circ T^{-1}(v) = 0$. Since $f \neq 0$, we have $T^{-1}(v) = 0$, yielding $v = 0$. Therefore, the following diagram commutes in $\mathcal{C}$:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{u} & Z & \xrightarrow{0} & TB \\ \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ A & \xrightarrow{\mathrm{id}} & A & \longrightarrow & 0 & \longrightarrow & TA \end{array}$$

Again, by TR3, we can find a morphism $\tilde{f} : B \to A$ to complete the above diagram to a morphism of triangles. In particular, $\tilde{f} \circ f = \mathrm{id}$. This implies that $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits in $\mathcal{C}$. □

Consider $\mathcal{A} =$**AbGrp**. $K(\mathcal{A})$ is triangulated but not semi-simple. This is because in the short exact sequence in $K(\mathcal{A})$: (all concentrated at 0)

$$\begin{array}{ccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \\ \cdots & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f} & & \\ \cdots & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle g} & & \\ \cdots & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \cdots \\ & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

where $f(0) = 0$, $f(1) = 2$ and $g(0) = 0$, $g(1) = 3$, $g(2) = 0$, $g(3) = 3$. Proposition 2.7 tells us the short exact sequence splits. In particular, there exists a map $\tilde{f}$: $\mathbb{Z}/4 \to \mathbb{Z}/2$ such that $\tilde{f} \circ f = \mathrm{id}$ in $K(\mathcal{A})$. However, this implies

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

splits in **AbGrp**, which is impossible because $\mathbb{Z}/4 \ncong \mathbb{Z}/2 \times \mathbb{Z}/2$. Hence $K(\mathcal{A})$ cannot be abelian.

For fun, readers may wish to look at this post for another excellent counterexample. In general, if $\mathcal{A}$ is semi-simple, then $K(\mathcal{A})$ is abelian. See III.2.3 of [3].

Proposition 2.8. *$K(\mathcal{A})$ endowed with T is triangulated.*

In order to show that $K(\mathcal{A})$ is a triangulated category, we need to construct the distinguished triangles. This is where the notion of cone gets involved. A detailed discussion of cone can be found in course notes of C2.2 Homological Algebra. For completeness, we write down the definition of it here.

Definition 2.9. The **cone** of a morphism $f : A \to B$ is $\operatorname{cone}(f) = A[1] \oplus B$. So $\operatorname{cone}(f)_i = A_{i+1} \oplus B_i$, and the differential $d_i^{\operatorname{cone}(f)} : \operatorname{cone}(f)_i \to \operatorname{cone}(f)_{i+1}$ is

$$d_i^{\operatorname{cone}(f)} \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} -d_{i+1}^A & 0 \\ f_{i+1} & d_i^B \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix}.$$

Equivalently, $d_i^{\operatorname{cone}(f)}(a, b) = (-d_{i+1}^A(a), f_{i+1}(a) + d_i^B(b))$. Set the distinguished triangles in $K(\mathcal{A})$ to be of the form

$$A \xrightarrow{f} B \to \operatorname{cone}(f) \to A[1].$$

Proof of Proposition 2.8. TR1 and TR3 are easily checked. We prove TR2 and TR4. For TR2, we want $B \to \operatorname{cone}(f) \to A[1] \to B[1]$ to be distinguished. From the construction of distinguished triangles, $B \xrightarrow{g} \operatorname{cone}(f) \to \operatorname{cone}(g) \to B[1]$ is distinguished. Note that $\operatorname{cone}(g) = B \oplus A[1] \oplus B[1] \cong A[1] \oplus (B \oplus B[1])$. We claim that

$$\begin{array}{ccccccc} B & \longrightarrow & \operatorname{cone}(f) & \longrightarrow & A[1] & \longrightarrow & B[1] \\ \downarrow{=} & & \downarrow{=} & & \downarrow{\phi} & & \downarrow{=} \\ B & \xrightarrow{g} & \operatorname{cone}(f) & \longrightarrow & \operatorname{cone}(g) & \longrightarrow & B[1] \end{array}$$

there exists a morphism ϕ such that the diagram commutes and ϕ is an isomorphism. ϕ is easy to define: for each i,

$$\phi : A[1]_i = A_{i+1} \to \operatorname{cone}(g)_i = (A[1] \oplus B \oplus B[1])_i = A_{i+1} \oplus B_i \oplus B_{i+1},$$

sending a to $(a, 0, f_{i+1}(a))$. The inverse ϕ^{-1} can be given as the projection onto the first factor. The next thing is to check is the commutativity. The commutativity of the part

$$\begin{array}{ccc} A[1] & \longrightarrow & B[1] \\ \downarrow{\phi} & & \downarrow{=} \\ \operatorname{cone}(g) & \longrightarrow & B[1] \end{array}$$

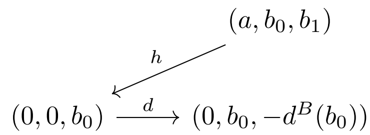
is straightforward. However, efforts should be made to prove the commutativity of the part

$$\begin{array}{ccc} \operatorname{cone}(f) & \longrightarrow & A[1] \\ \downarrow{=} & & \downarrow{\phi} \\ \operatorname{cone}(f) & \longrightarrow & \operatorname{cone}(g) \end{array}$$

since it is not commutative in $\operatorname{Ch}(\mathcal{A})$. It is commutative up to homotopy. To see this, note that $\phi \circ \phi^{-1}$ is homotopic to id by easy computation ($h_i : \operatorname{cone}(g)_i \to \operatorname{cone}(g)_{i+1}$ sending (a, b_0, b_1) to $(0, 0, b_0)$)

$$\begin{array}{ccc} (a, b_0, b_1) & \xrightarrow{d} & (-d^A(a), b_1 + f(a) + d^B(b_0), -d^B(b_1)) \\ & \swarrow{h} & \\ (0, 0, b_1 + f(a) + d^B(b_0)) & & \end{array}$$

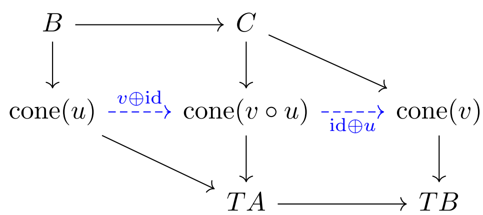
and



and $(\mathrm{id} - \phi \circ \phi^{-1})(a, b_0, b_1) = (0, b_0, f(a) + b_1)$, since $d_i^{\mathrm{cone}(g)} : \mathrm{cone}(g)_i \to \mathrm{cone}(g)_{i+1}$ is (here we regard $\mathrm{cone}(g) = B[1] \oplus A[1] \oplus B$)

$$d_i^{\mathrm{cone}(f)} = \begin{pmatrix} -d_{i+1}^B & 0 \\ g_{i+1} & d_i^{\mathrm{cone}(f)} \end{pmatrix} = \begin{pmatrix} -d_{i+1}^B & 0 & 0 \\ 0 & -d_i^A & 0 \\ 1 & f_{i+1} & d_i^B \end{pmatrix}.$$

One should be careful that $-f : A[1] \to B[1]$ induced by $f : A \to B$ has a changed sign. Hence we proved our claim. To show that TR4 is satisfied, one can look at each block in the form



for $A \xrightarrow{u} B \xrightarrow{v} C$, and check it is commutative. We omit the details here. ☐

Definition 2.10. The **cohomology** $H^i(A_\bullet)$ of a complex $A_\bullet$ is the quotient

$$H^i(A_\bullet) = \frac{\ker(d_i)}{\mathrm{im}\,(d_{i-1})} \in \mathcal{A}.$$

A complex $A_\bullet$ is **acyclic** if $H^i(A_\bullet) = 0$ for all $i \in \mathbb{Z}$. A morphism of complexes $f_\bullet : A_\bullet \to B_\bullet$ is a **quasi-isomorphism** if the induced map $H^i(f_\bullet)$ is an isomorphism for all i.

Proposition 2.11. *For a distinguished triangle*

$$A \xrightarrow{f} B \to \mathrm{cone}(f) \to A[1],$$

f *is a quasi-isomorphism iff* $\mathrm{cone}(f)$ *is acyclic.*

Proof. It is clear that $(\Rightarrow)$ holds. For $(\Leftarrow)$, consider the distinguished triangles:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow \\ 0 & \longrightarrow & A & \xrightarrow{\mathrm{id}} & A & \longrightarrow & 0 \end{array}$$

By TR3 there exists a map $B \to A$ such that the diagram commutes. Hence we obtain a quasi-isomorphism. ☐

Proposition 2.12. $H^i : K(\mathcal{A}) \to \mathcal{A}$ *is a cohomological functor. Any distinguished triangle* $A \to B \to C \to A[1]$ *induces a long exact sequence*

$$\cdots \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots.$$

Proof. Let $A \xrightarrow{f} B \to C \to A[1]$ be a distinguished triangle. Then it is isomorphic to $A \to B \to \text{cone}(f) \to A[1]$. Since

$$0 \to B \to \text{cone}(f) \to A[1] \to 0$$

is exact, it is immediate that

$$H^i(B) \to H^i(\text{cone}(f)) \to H^{i+1}(A)$$

is exact. Hence

$$H^i(B) \to H^i(C) \to H^{i+1}(A)$$

is exact. □

3. Derived Categories

The derived category $D(\mathcal{A})$ is defined to be the localization of $K(\mathcal{A})$ by a multiplicative set of morphisms related to a null system. We first give the definition of the localization.

3.1. **Localization.**

Definition 3.1. Let $\mathcal{C}$ be an arbitrary category, and $S \subset \text{Mor}(\mathcal{C})$ be a set of morphisms. A **localization of** $\mathcal{C}$ **by** S, is the data of a category $S^{-1}\mathcal{C}$ and a functor $F : \mathcal{C} \to S^{-1}\mathcal{C}$ satisfying:

(1) for all $s \in S$, $F(s)$ is an isomorphism,
(2) (Universal property) for any functor $G : \mathcal{C} \to \mathcal{D}$ satisfies (1), there exists a lifted functor $\tilde{F} : S^{-1}\mathcal{C} \to \mathcal{D}$ of F, such that $F \simeq \tilde{F} \circ G$. That is, the diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ \downarrow{\scriptstyle F} & \nearrow{\scriptstyle \tilde{F}} & \\ S^{-1}\mathcal{C} & & \end{array}$$

(3) if $G_1, G_2 : S^{-1}\mathcal{C} \to \mathcal{D}$ are two functors, then the natural map

$$\hom_{\text{Fun}(S^{-1}\mathcal{C},\mathcal{D})}(G_1, G_2) \to \hom_{\text{Fun}(\mathcal{C},\mathcal{D})}(G_1 \circ F, G_2 \circ F)$$

is bijective. That is, $- \circ F$ is fully faithful. This also implies that $\tilde{F}$ in (2) is unique up to unique isomorphism.

Definition 3.2. Let $S \subset \text{Mor}(\mathcal{C})$ be a set of morphisms in the category $\mathcal{C}$. S is said to be **right multiplicative** if

(1) for every $x \in \mathcal{C}$, $\text{id}_x \in S$,
(2) for every two morphisms $f, g \in S$, $f \circ g \in S$,
(3) for every $f : x \to x'$ in S and $g : x \to y$ not necessarily in S, there exists $f' : y \to y'$ in S and $g' : x' \to y'$ not necessarily in S, s.t. the diagram commutes:

$$\begin{array}{ccc} y & \xrightarrow{f'} & y' \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g'} \\ x & \xrightarrow{f} & x' \end{array}$$

(4) for every $f, g : x \to y$ and $h : z \to x$ satisfying $h \in S$ and $f \circ h = g \circ h$, there exists $h' \in S$, $h' : y \to w'$, with $h' \circ f = h' \circ g$.

$$z \xrightarrow{h} x \underset{g}{\overset{f}{\rightrightarrows}} y \overset{h'}{\dashrightarrow} w$$

The existence of localization of a category $S^{-1}\mathcal{C}$ can indeed be obtained by some right multiplicative system §. Define the category S^Y as follows ($Y \in \mathcal{C}$):

$$\begin{aligned}&\mathrm{Obj}(S^Y) = \{s : Y \to Y' \mid s \in S\},\\ &\hom_{S^Y}((s : Y \to Y'), (s' : Y \to Y'')) = \{h : Y' \to Y'' \mid h \circ s = s'\}.\end{aligned}$$

This is equivalent to say $S^Y = S \cap (Y/\mathcal{C})$, where $(Y/\mathcal{C})$ is a comma category. Let X, Y be arbitrary objects in $\mathcal{C}$. Define a new category $\mathcal{C}^r_S$ as follows:

$$\begin{aligned}&\mathrm{Obj}(\mathcal{C}^r_S) = \mathrm{Obj}(C),\\ &\hom_{\mathcal{C}^r_S}(X, Y) = \mathrm{colim}_{(Y \to Y') \in S^Y} \hom_{\mathcal{C}}(X, Y').\end{aligned}$$

We need to check that $\mathcal{C}^r_S$ is indeed a category.

Lemma 3.3. *Let $s : X \to X'$ be in S. s induces an isomorphism*

$$\hom_{\mathcal{C}^r_S}(X', Y) \xrightarrow[-\circ s]{\simeq} \hom_{\mathcal{C}^r_S}(X, Y).$$

Proof. First to show that $- \circ s$ is bijective. It is injective because if $f, g \in \hom_{\mathcal{C}^r_S}(X', Y)$ such that $f \circ s = g \circ s$, by (4) in Definition 3.2, there exists Y'' and $h : Y' \to Y''$ in S with $h \circ \tilde{f} = h \circ \tilde{g}$, such that

$$\begin{array}{ccccccc} X & \xrightarrow{s} & X' & \underset{\tilde{g}}{\overset{\tilde{f}}{\rightrightarrows}} & Y' & \overset{h'}{\dashrightarrow} & Y'' \\ & & & & \uparrow{\scriptstyle t} & & \\ & & & & Y & & \end{array}$$

where $f = \mathrm{colim}_{(t:Y \to Y') \in S^Y}(\tilde{f} : X' \to Y')$, $g = \mathrm{colim}_{(t:Y \to Y') \in S^Y}(\tilde{g} : X' \to Y')$. This implies $f = g$ by the universal property of colimits. Abuse the notation, we can prove the surjectivity by (3) in Definition 3.2 as follows: for every $\phi \in \hom_{\mathcal{C}^r_S}(X, Y)$, $\phi = \mathrm{colim}_t(\tilde{\phi} : X \to Y')$, there exists morphism $X' \to Y''$ and $Y' \xrightarrow{t'} Y''$ such that the diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{s} & X' \\ \downarrow{\scriptstyle \tilde{\phi}} & & \downarrow \\ Y' & \overset{t'}{\dashrightarrow} & Y'' \\ \uparrow{\scriptstyle t} & & \\ Y & & \end{array}$$

So we can define a morphism $\tilde{f} : X' \to Y'$ with $\tilde{f} \circ s = \tilde{\phi}$, and this yields $t' \circ (\tilde{f} \circ s) = t' \circ \tilde{\phi}$. Now we get $f = \mathrm{colim}_t \tilde{f} \in \hom_{\mathcal{C}^r_S}(X', Y)$. □

The composition in $\mathcal{C}_S^r$ is defined through

$$
\begin{aligned}
&\operatorname{colim}_{(Y\to Y')\in S^Y} \hom_{\mathcal{C}}(X,Y') \times \operatorname{colim}_{(Z\to Z')\in S^Y} \hom_{\mathcal{C}}(Y,Z') \\
&\qquad \simeq \operatorname{colim}_{(Y\to Y')\in S^Y} \left(\hom_{\mathcal{C}}(X,Y') \times \operatorname{colim}_{(Z\to Z')\in S^Y} \hom_{\mathcal{C}}(Y,Z')\right) \\
&\qquad \xrightarrow[\text{Lemma 3.3}]{\simeq} \operatorname{colim}_{(Y\to Y')\in S^Y} \left(\hom_{\mathcal{C}}(X,Y') \times \operatorname{colim}_{(Z'\to Z')\in S^Y} \hom_{\mathcal{C}}(Y',Z')\right) \\
&\qquad \to \operatorname{colim}_{(Y\to Y')\in S^Y} \operatorname{colim}_{(Z\to Z')\in S^Y} \hom_{\mathcal{C}}(X,Z') \\
&\qquad \to \operatorname{colim}_{(Z\to Z')\in S^Y} \hom_{\mathcal{C}}(X,Z').
\end{aligned}
$$

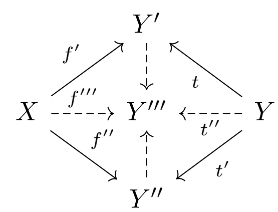
We omit the proof that the composition is associative. Now $\mathcal{C}_S^r$ is a well-defined category. We will show that $\mathcal{C}_S^r$ is essentially the localization $S^{-1}\mathcal{C}$.

Theorem 3.4. *Let S be a right multiplicative system. Then $\mathcal{C}_S^r \simeq S^{-1}\mathcal{C}$.*

Proof. Let $Q_S : \mathcal{C} \to \mathcal{C}_S^r$ be the natural functor associated with $\hom_{\mathcal{C}}(X,Y) \to \operatorname{colim}_{(Y\to Y')\in S^Y} \hom_{\mathcal{C}}(X,Y')$. By Lemma 3.3, for any $Z \in \mathcal{C}_S^r$ and morphism $s : X \to Y$ in S, $\hom_{C_S^r}(Y,Z) \to \hom_{C_S^r}(X,Z)$ is bijective. This indicates that $Q_S(s)$ is invertible. Let $f : X \to Y$ be any morphism in $\mathcal{C}_S^r$. By definition, f is given by an equivalence class of triplets $(Y', t : Y \to Y', f' : X \to Y')$, where $t \in S$ and

$$X \xrightarrow{f'} Y' \xleftarrow{t} Y.$$

$(Y', t, f') \sim (Y'', t', f'')$ iff there exists (Y''', t'', f''') with $t, t', t'' \in S$, and the diagram commutes



Note that in $\mathcal{C}_S^r$, $f = Q_S(t)^{-1} \circ Q_S(f')$. Hence for any two parallel arrows $f, g : X \to Y$ in $\mathcal{C}$, we have that $Q_S(f) = Q_S(g)$ iff there exists a morphism $s : Y \to Y'$ in S such that $s \circ f = s \circ g$ (note the composition is just the composition of cospans). ☐

Let I be a full subcategory of $\mathcal{C}$, and S be a right multiplicative system in $\mathcal{C}$. $I \cap S$ is the family of morphisms in I which belong to S. There is another way to characterize $S^{-1}\mathcal{C}$:

Theorem 3.5. *If for every $X \in \mathcal{C}$, there exists $s : X \to W$ with $W \in I$ such that $s \in S$, then $I \cap S$ is a right multiplicative system, and $(I \cap S)^{-1} I \simeq S^{-1}\mathcal{C}$.*

Proof. We skip the proof of $I \cap S$ being a right multiplicative system. The readers are encouraged to check it by definition. For $W \in I$, $(I \cap S)^W$ is a full subcategory of S^W whose objects are the morphism $i : W \to V$ with $V \in I$ and $i \in I \cap S$. One can check that the natural functor $\Phi : (I \cap S)^{-1} I \to S^{-1}\mathcal{C}$ is cofinal (see Proposition 5.1.7 in [1]), and so it is fully faithful by Definition of $\mathcal{C}_S^r$. Φ is essentially surjective by the assumption. Hence $(I \cap S)^{-1} I \simeq S^{-1}\mathcal{C}$. ☐

Remark 3.6. Dually, we can define the notion of a left multiplicative system S be reversing arrows in Definition 3.2. In this case, we can set $\operatorname{Obj}(\mathcal{C}_S^l) = \operatorname{Obj}(\mathcal{C})$, and the morphisms in $\mathcal{C}_S^l$ to be

$$\hom_{\mathcal{C}_S^l}(X,Y) = \operatorname{colim}_{(X\to X')\in S_X} \hom_{\mathcal{C}}(X',Y),$$

where S_X is the category defined as follows:

$$\begin{aligned}&\mathrm{Obj}(S_X) = \{s : X' \to X \mid s \in S\},\\ &\hom_{S_X}((s : X' \to X), (s' : X'' \to X)) = \{h : X' \to X'' \mid s = s' \circ h\}.\end{aligned}$$

One goes through the proof of Theorem 3.4 and finds $\mathcal{C}_S^l$ and $\mathcal{C}_S^r$ give equivalent categories. So we are free to choose the left or right multiplicative system to get a localization.

3.2. **Localization of triangulated categories.** Let $\mathcal{D}$ be a triangulated category.

Definition 3.7. Let $N \subset \mathcal{D}$ be a set of morphisms in $\mathcal{D}$. It is called a **null system** if

(1) $0 \in N$,
(2) every $x \in N$ iff $Tx \in N$,
(3) for every distinguished triangle $x \to y \to z \to Tx$, if $x, y \in N$, then $z \in N$.

Define

$$S = \{f : x \to y \mid \exists \text{ distinguished triangle } x \to y \to z \to Tx \text{ s.t. } z \in N\}.$$

Exercise 3.8. S is right multiplicative.

Assume $\mathcal{A}$ is an abelian category from now on. In $K(\mathcal{A})$, the null system N is the set of acyclic chains, so S is the set of quasi-isomorphisms. By definition, the derived category of $\mathcal{A}$ is $D(A) = S^{-1}K(\mathcal{A})$.

Corollary 3.9. *$D(\mathcal{A})$ is additive, but not necessarily abelian.*

Let I be the full subcategory of all cochain complexes of injectives of $K^+(\mathcal{A})$, where we write $(-)^+$ to denote "bounded below". I is then additive and triangulated.

Proposition 3.10. *Suppose $\mathcal{A}$ has enough injectives. For every $A \in K^+(\mathcal{A})$, there exists an injective $J \subset I$ such that the morphism $A \to J$ is a quasi-isomorphism.*

Proof. We define J as follows. Let $A_0 = Z_0$. Since $\mathcal{A}$ has enough injectives, one can find a monomorphism from Z_0 to an injective J_0. Define Z_1 to be the pushout of $J_0 \hookleftarrow Z_0 \xrightarrow{d_0} A_1$. Then define Z_2 to be the pushout of $J_1 \hookleftarrow Z_1 \xrightarrow{d_1} A_2$, where $Z_1 \rightarrowtail J_1$ is a monomorphism. Recursively, we define Z_{k+1} to be the pushout of $J_k \hookleftarrow Z_k \xrightarrow{d_k} A_{k+1}$. A diagram demonstrating the process is the following:

$$\begin{array}{ccccccc} A_0 & = & Z_0 & \longrightarrow & A_1 & & \\ & & \downarrow & & \downarrow & & \\ & & J_0 & \xrightarrow{\text{pushout}} & Z_1 & \longrightarrow & A_2 \\ & & & & \downarrow & & \downarrow \\ & & & & J_1 & \xrightarrow{\text{pushout}} & Z_2 & \longrightarrow \\ & & & & & & \downarrow & & \ddots \end{array}$$

It suffices to prove that in the distinguished triangle $A \xrightarrow{f} J \to \text{cone}(f) \to A[1]$, $\text{cone}(f)$ is acyclic. Consider $Z \to J \oplus A[1] \to Z[1]$, i.e. the diagonal of the diagram above. Apply the cohomological functor H^i, we know that

$$\cdots \to H^i(Z) \xrightarrow{\text{id}} H^i(Z) \to H^i(\text{cone}(f)) \to H^{i+1}(Z) \to \cdots$$

is exact. This implies $H^i(\text{cone}(f)) = 0$ for all i. The result follows. □

On the other hand, by Theorem 3.5, $D^+(\mathcal{A}) \simeq (I \cap S)^{-1} I$. Let $A \in I$ be acyclic (i.e. $A \in N$), then $\text{id} : A \to A$ is in S, implying id is homotopic to 0. Hence we have a very nice result:

$$D^+(\mathcal{A}) \simeq (I \cap S)^{-1} I \simeq I.$$

Remark 3.11. In the proof of Proposition 3.10, we implicitly assume that $\mathcal{A}$ has cokernels and coequalizers. In fact, this assumption can be dropped. See Lemma 2.38 in [4].

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