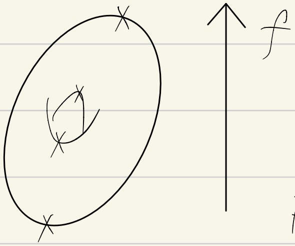
Morse homology



$f: M \to \mathbb{R}$ (→ cmpct Rmn mnfd) Morse function, crit(f) are indexed (by writing $f = \pm x_i^2$ locally).

$p, q \in \mathrm{Crit}(f)$: $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$,

$\mathcal{M}(p,q) = \{ r \in M : $ q — r — p, $-\infty$ — 0 — ∞ $\}$

Fact: Generically $\mathcal{M}(p,q)$ finite, define $C_k^{Mor}(M) = \mathbb{Z}\{\text{ind-}k \text{ crit pts}\}$.

$$\partial p = \sum_{\mathrm{ind}(q) = \mathrm{ind}(p) - 1} \# \widehat{\mathcal{M}}(p,q) \cdot (-1)^{\varepsilon_{\mathrm{ind}(p)}} \cdot q$$

Spoiler: Let $L_0, L_1 \subset M$ be "nice" transverse Lagrangians in "nice" M,

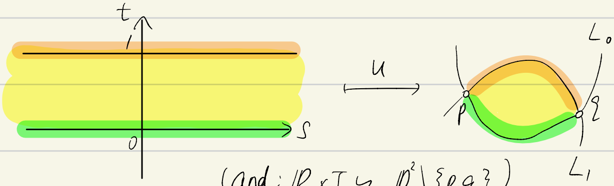
$p, q \in L_0 \cap L_1$, define $CF(L_0, L_1) = \mathbb{Z}\{L_0 \cap L_1\}$ ← grading later!

$$\partial p = \sum_{\substack{q \in L_0 \cap L_1, \\ \beta : \mathrm{ind}(\beta) = 1}} \left(\# \widehat{\mathcal{M}}(p, q; \beta, J) \right) T^{\omega(\beta)} q,$$

Goal: make sense of terms in this formula:

① $\beta \in \pi_2(M, L_0 \cup L_1)$.

$$\mathcal{M}(p,q;\beta,J) = \left\{ \begin{array}{l} u : \mathbb{R} \times I \to M : \ (s,t) \mapsto u(s,t), \quad \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} = 0 \ (J\text{-holomorphic}), \\ [u(\mathbb{R} \times I)] = \beta \in \pi_2(M, L_0 \cup L_1), \\ \lim_{s \to \infty} u(s,t) = p, \ \lim_{s \to -\infty} u(s,t) = q, \ u(s,0) \in L_0, \ u(s,1) \in L_1, \\ \text{energy} \to E(u) := \int_{\mathbb{R} \times I} |du|^2 < \infty. \end{array} \right\}$$



(and: $\mathbb{R} \times I \cong D^2 \setminus \{p, q\}$) biholom

② $\widehat{\mathcal{M}}(p,q,\beta,J) = \mathcal{M}(p,q,\beta,J)/\mathbb{R}$ ← translation on s-axis $u(s,t) \rightsquigarrow u(s+\alpha,t)$, proper + free action.

$\widehat{\mathcal{M}}$ 0-dim, cmpct, ori when $\mathrm{ind}(\beta)=1$, J generic, + nice M

③ Suppose $u: \mathbb{R}\times I \to M$ has $[u]=\beta$, then

$$\mathrm{ind}\,\beta = \mathrm{ind}[u] = \mathrm{ind}\, D_{\bar\partial_J,u}$$ ← Fredholm operator, computed using Maslov index (also relevant for grading).

Similarly $\omega(\beta) := \int_{\mathbb{R}\times I} u^*\omega$ for $[u]=\beta$.

④ T as in $T^{\omega(\beta)}$ is the formal variable in Novikov field: → Fact: alg closed when $k=\mathbb{C}$.

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} : a_i \in k,\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} \lambda_i = \infty \right\}.$$

"nodal discs are sequentially cmpct"

⑤ Is the expression a finite sum? Guaranteed by Gromov compactness.

Morse homology	Lagrangian Floer cohomology
• Critical pts of $F: M \to \mathbb{R}$ $C_*^{Mor}(M)$	• $L_0 \cap L_1$, L_0, L_1 transverse Lagrangians $CF^*(L_0, L_1)$
• Get ∂ by counting gradient flows between crit pts.	• Get ∂ by counting (punctured) J-holom discs between two points
• $H_*(M)$ hmtpy invariant	• $HF^*(L_0,L_1)$ inv under Hamiltonian isotopy ($\phi: M\times I \to M$, ϕ_t symplectmphsm, $\iota_{X_t}\omega$ exact), L_0, L_1 Hmtn isotopic $\Rightarrow HF^*(L_0,L_1) \cong H^*(L_0)$, L_1, L_1' .... $\Rightarrow HF^*(L_0,L_1) \cong HF^*(L_0,L_1')$.

- $\sum$ indices of crit(F) $\geq \sum \operatorname{rank} H_i(M;\mathbb{Q})$.

- (Arnold conjecture) $L \subset (M,\omega)$ Lagrangian, st $\forall$ disc $D \subset L$, $\int_D \omega = 0$.
  Let $H: M \times I \to \mathbb{R}$ be time-dpdnt Hamiltonian, $H_t : M \times \{t\} \to \mathbb{R} \rightsquigarrow X_t \in C^\infty(TM)$, Consider curve $\alpha(t)$ st $\alpha'(t) = X_t(\alpha(t))$, let $\psi \in \operatorname{Diff}(M)$ be the $t=1$ flow.
  $\psi(L), L$ transverse
  $\Rightarrow |\psi(L) \cap L| \geq \sum_i \dim H^i(L;\mathbb{Z}/2)$
  ($|\psi(L) \cap L| = |\operatorname{Fix}(\psi|_L)|$)

Many versions, some still open. This one established by Floer using HF. See also recent work by Abouzaid-Blumberg.

Moduli $\mathcal{M}$ of punctured J-holom discs plays an important role:
- $\dim \widehat{\mathcal{M}} = 0$ in suitable cases $\Rightarrow \# \widehat{\mathcal{M}}$ makes sense
- "Gluing operations" on discs $\Rightarrow$ product structures $\mu_k : CF^{\otimes n} \to CF$,
- Orientation $\Rightarrow \mu_k$ fit into defn of A_∞-Cat structure
  $\Rightarrow$ Fukaya cat $\operatorname{Fuk}(M,\omega)$,
  obj = {"nice $L \subset M$"},
  $\operatorname{Hom}(L_0, L_1) =$ "$CF(L_0, L_1)$"
- Invariance under $J \Rightarrow HF^*$, Fuk independent of J.

A lot of techical difficulties regarding $\mathcal{M} \Rightarrow$ regarding HF^*, Fuk, and so on. (Ex: What's $CF(L,L)$??)

Way out:
① Focus on nice M, nice $L \subset M$.
② Set up more powerful machinery (!)

References

Floer's papers:

[FRel] "Relative Morse Index for Symplectic Action." Viterbo ("Maslov") index = Fredholm index = dim of moduli.

[FUnr] "The unregularized gradient flow of the symplectic action." Implicit function theorem for moduli.

[FWit] "Witten's Complex and Infinite-Dimensional Morse Theory." Conley index, punctured J-holom discs = "gradient flows."

Expository

[A] Auroux, "A Beginner's Introduction to Fukaya Categories."

[S] Smith, "A Symplectic Prolegomenon."

Ono, Lectures on Lagrangian Floer Theory, video link at https://hackmd.io/@nYzitppIRA2rAo3R9To9FA/ryaYvna5M?type=view

Pascaleff's 595 lecture notes, L9-18, https://faculty.math.illinois.edu/~jpascale/courses/2018/595/ follows [SeiPL]

Pascaleff's M 392C (Lagrangian Floer Homology) lecture notes, L9-14, https://faculty.math.illinois.edu/~jpascale/courses/2014/m392c/

Texts

[MSJ] "J-holomorphic Curves and Quantum Cohomology"; [MSJ12] "J-holomorphic Curves and Symplectic Topology." 2012 edition. Treats the closed case, new version significantly expanded

[FOOO] Fukaya, Oh, Ohta, Ono, "Lagrangian Intersection Floer Theory."

[W] Wendl, "Lectures on Symplectic Field Theory." Punctured case

[Aud] Audin, Damian. "Morse Theory and Floer Homology." Closest to a textbook

[SeiPL] Seidel, "Fukaya categories and Picard-Lefschetz theory"

Papers

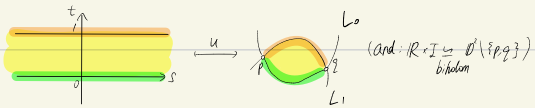
[Sei] Graded Lagrangian submanifolds https://arxiv.org/abs/math/9903049

[IS] S. Ivashkovich , V. Shevchishin. Gromov Compactness Theorem for Stable Curves http://arxiv.org/abs/math/9903047v1

[G] Gromov, "Pseudo holomorphic curves in symplectic manifolds."

Outline: • $\mathcal{M}(p,q,\beta,J)$: set up, dim/index, ori and compactness

• HF: defn, higher products • Technical difficulties



$\mathcal{M}(p,q,\beta,J)$.

• Basic set up: $\mathcal{X} = W^{k,p}(\mathbb{R}\times I, M)$, $\mathcal{E}_u = W^{k-1,p}(\Lambda^{0,1}\otimes u^*TM)$,

$\mathcal{X} \xrightarrow{s} \mathcal{E}$ given by $u \longmapsto \bar{\partial}_J u$ section. $\mathcal{E}\to\mathcal{X}$.

Implicit function thm: $\mathcal{M} = s^{-1}(0)$ mnfd if s transverse with $\mathcal{X}\xrightarrow{0}\mathcal{E}$.

Fact: guaranteed when linearization ← Fact: J elliptic $\Rightarrow$ D_u Fredholm.

$$D_u: T_u\mathcal{X} \to T_{(u,\bar{\partial}_J u)}\mathcal{E} \to \mathcal{E}_u \quad \text{surjective.}$$

See notes from Jun 24. Recall: J-holom $\Leftrightarrow$ $\bar{\partial}_J u = s(u) = 0$

Thm: For a countable $\cap$ of open dense in $\{J\}$, D_u surj. for all $u\in\mathcal{M}$. $\dim\mathcal{M} = \operatorname{ind} D_u = \dim\ker D_u - \dim\operatorname{coker} D_u$.

$\mathcal{J} = \{\omega\text{-cmptbl al cx } J\} \subset W^{k,p}(\operatorname{End}(TM))$, key idea: Sard-Smale thm.

• Computing ind D_u.

For $u\in\mathcal{M}$, use the following construction ([FRel], §5):

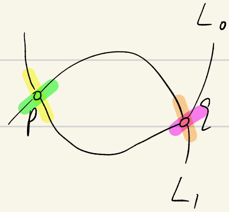
• $T_pL_0, T_pL_1 \subset T_pM$ both Lagrangian $\Rightarrow$ $\exists\,\phi\in Sp(2n;\mathbb{R})$: {Lagr subspaces of $\mathbb{R}^{2n}$}

$\phi(T_pL_0) = \mathbb{R}^n$, $\phi(T_pL_1) = (i\mathbb{R})^n$. $\lambda_p := \phi^{-1}((e^{-i\pi t/2}\mathbb{R})^n)_{t\in I} \in Map(I,\mathcal{L}(n))$.

• $\mathbb{R}\times I$ contractible $\Rightarrow$ u^*TM trivial $\Rightarrow$ for $i=\{0,1\}$, $u^*|_{\mathbb{R}\times\{i\}} TL_i \subset u^*TM$ a path of Lgr from T_pL_i to T_qL_i, denote as ℓ_i.

Thm (Floer) $\operatorname{ind}(D_u) = \mu\left(T_qL_0 \xrightarrow[\ell_0^{-1}]{} T_pL_0 \xrightarrow[\lambda_p]{} T_pL_1 \xrightarrow[\ell_1]{} T_qL_1 \xrightarrow[\lambda_q^{-1}]{} T_qL_0\right)$.

Maslov index: $\pi_1(\mathcal{L}(n)) \xrightarrow{\simeq} \mathbb{Z}$.



Cor: hmtpy of map u $\Rightarrow$ hmtpy of loop $\Rightarrow$ $\operatorname{ind}(D_u)$ hmtpy inv.

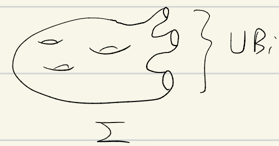
To make this precise, need to specify hmtpy rel certain bndries.

J-holom.

Alternative descriptions: Recall that for closed Riemann surfaces $\Sigma \xrightarrow{u} M$,

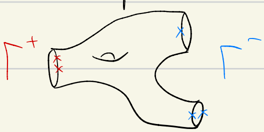
$$\text{ind } D_u = \frac{1}{2}\dim M \cdot \chi(\Sigma) + 2\langle c_1(TM), u_*[\Sigma]\rangle \quad \left(\begin{array}{c}\text{Atiyah-Singer}\\ \text{Riemann-Roch}\end{array}\right),$$

- Fact: for Riemann surfaces w/ boundary, suppose $\partial\Sigma = \sqcup B_i$, $B_i \cong S^1$, $u:(\Sigma, B_i) \to (M, L_i)$. Under trivialization u^*TM, ([MS], Prop 2.6.7) $u^*TL_i|_{B_i}$ defines loops $l_i \in \mathcal{L}(n)$,



$$\text{ind}(D_u) = \chi(\Sigma)\frac{\dim M}{2} + 2\underbrace{\textstyle\sum_i \mu(l_i)}_{\text{"rel Chern \#"}}.$$

For punctures, further correction terms given by



$$\sum_{z\in\Gamma^+}\mu_{CZ}^{(z)} - \sum_{z\in\Gamma^-}\mu_{CZ}^{(z)}$$

← Conley-Zehnder ind (Similar to Maslov index, references at [MS 12], p. 490)

- Rmk: Main result of [FRel] (Thm 1) describes ind D_u as signed count of the spectra of a path of operators in $\text{End}(u^*TM) \to \mathbb{R}\times I$. (ie, the "spectral flow"). "ind = crossings of spec flow" back to Atiyah-Patodi-Singer on Atiyah-Singer. See also [W] § 3-4.

- Bubbling and Gromov compactness.

Recall: $u: \mathbb{R}\times I \to M$, $E(u) := \int_{\mathbb{R}\times I} |du|^2$. (→ energy)

Lma: $E(u) = \int_{\mathbb{R}\times I} 2|\bar\partial_J u|^2 + u^*\omega$. Pf.: Compute using loc coor (s,t), $j\partial_s = \partial_t$.

Rmk: $\int_{\mathbb{R}\times I} u^*\omega$ only depends on $[u]\in H_2$, so J-holom u ($\bar\partial_J u = 0$) minimizes $E(u)$ for fixed $[u]\in H_2$, $E(u) = \int u^*\omega$.

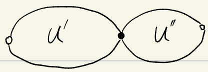
Question: if $\{u^i\}_{i\in\mathbb{N}} \in \mathcal{M}(p,q,\beta,J)$ satisfies $\sup_i E(u^i) < \infty$, how does $\{u^i\}$ converge (within some larger space, $C^0(\mathbb{R}\times I, M)$)?

Fact: ([MSJ], Thm 4.1.3) if further $\sup_i \|du^i\|_{L^\infty(K)} < \infty$ for all cmpct $K\subset \mathbb{R}\times I$, then $\exists$ subsequence converge w/in $\mathcal{M}$ $\begin{cases}\text{uniformly on all derivatives,}\\ \text{on all cmpcts} \subset \mathbb{R}\times I.\end{cases}$

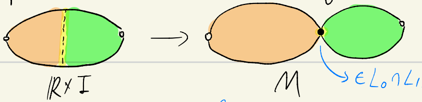
L^∞-bound can be replaced by $W^{1,p}$-bounds for any $p>2$.

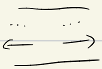
However, $\sup_i E(u) < \infty$ concerns L^2-norm of du, so need to consider cases where $\sup_i \|du^i\|_{L^\infty} = \infty$. ($\sup_i \|du^i\|_\infty < \infty$ stronger condition)

Lma: $\exists\, z^i \in \mathbb{R}\times I : |du^i(z^i)| = \|du^i\|_{L^\infty}$,

- If subsequence i_n, z^{i_n} converges to $\pm\infty\times I$, $\exists\, a^{i_n}_\pm \in \mathbb{R}$, $a^{i_n}_\pm \to \pm\infty$, st $u^{i_n}(\cdot - a^{i_n}_\pm, \cdot)$ converges to J-holom strips u', u'', then u^{i_n} converges to 

Strip breaking

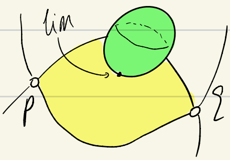
More precisely: u^{i_n} converge to  in C^0 topology,

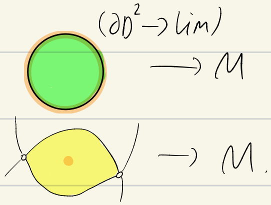


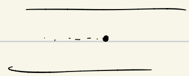
and $W^{1,p}_{loc}$ $(\forall p)$ away from ← i.e., away from singularity.

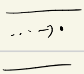
- if $\exists$ subsequence i_n, z^{i_n} converges to $\mathrm{int}(\mathbb{R}\times I)$, $\exists\, \phi^{i_n} \in \mathrm{Aut}(\mathbb{R}\times I)$, st near $\lim_{n\to\infty} z^{i_n}$, $(\phi^{i_n}\circ u^{i_n})$ converges to J-holom $S^2 \to M$.

Sphere bubbling



or gluing 

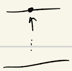


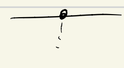


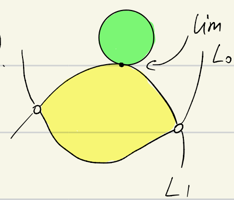
- if $\exists$ subsequence i_n, z^{i_n} converges to $\mathbb{R}\times\partial I$, $\exists\, \phi^{i_n} \in \mathrm{Aut}$ st near $\lim_{n\to\infty} z^{n_i}$, $(\phi^{i_n}\circ u^{i_n}) \to$ J-holom $(D^2, S^1) \to (M, L_1)$.

Disc bubbling

(For reference, see [IS]; the analogue for $S^2 \to M$ is established in [G], §1.5)



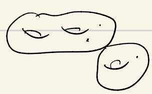




Rmk: • Think of the added curves as boundary of $\bar{\mathcal{M}}$.

• Compare w/ Deligne-Mumford cmptfction of moduli of curves

$\mathcal{M}_{g,n} \subset \bar{\mathcal{M}}_{g,n}$,

$\mathcal{M}_{g,n} = \{$  $\}$, $\bar{\mathcal{M}}_{g,n} = \mathcal{M}_{g,n} \cup \{$  $\}$ ← Stable curves: nodal curves w/ combinatorial constraints (for small/triv Aut grp).

See [FOOO], §2.1.2, [MSJ12] §5-6, [W] §9.3 for more.

- Later we will look at moduli of punctured discs in more detail.

- Orientation

Principal $SO(n)$-bundle: bundle with fibres homeo to $SO(n)$ and free $SO(n)$-action.

(M,g) ori $\Rightarrow F_{SO}(M)_p := \{$ori orthonormal basis of $T_pM\}$ ori.

Recall: $Spin(n)$ is the double cover of $SO(n)$, hence a Lie grp.

Defn: A spin structure on (M,g) is a principal $Spin(n)$-bundle $P \to M$ w/ bundle map $P \to F_{SO}(M) = \{SO(n)\text{-frames of } TM\}$, which is an equivariant double cover. Similar defn for vector bundle w/ metric.

"Chern classes for $\mathbb{R}$-vect bundles"

Fact: (M,g) admits a spin structure iff $w_2(M) \in H^2(M, \mathbb{Z}/2)$ vanishes.

Defn ([FOOO], §8.1.2) (L_0, L_1) is relatively spin if $\exists\, \varphi \in H^2(M, \mathbb{Z}/2)$ st

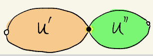
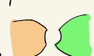
$\varphi|_{L_i} = w_2(L_i) \in H^2(L_i, \mathbb{Z}/2)$. (Special case: L_i both spin, $\varphi = 0$.)

Rel spin structure: ori on L_i, $\mathbb{R}$-vect bundle $V \to M^3$ w/ $w_2(V) = \varphi$, and spin structure on $TL_i \oplus V|_{L_i^2}$ (2- and 3-skeleta)

Thm. ([FOOO], §8.1.14) (L_0, L_1) rel spin $\Rightarrow \mathcal{M}(p, q, \beta, J)$ ori.

[W], §11

Idea: defn D_u ori if $\det D_u = \det(\operatorname{coker} D_u) \otimes \det \ker D_u$ ori. This gives ori on $T_u\mathcal{M}$. Globalize this to $\det \to \mathcal{M}$, $(\det)_u := \det D_u$.

Rmk: We will also be interested in orienting $\partial\bar{\mathcal{M}}$, in particular strip breaking (
), idea: chop off the ends and glue to $\tilde{u}_R$ , ($\pm(R,\infty) \times I$ chopped off)

compare $\det D_{u'} \otimes \det D_{u''}$ with $\det D_{\tilde{u}_R}$ for large R,

so that the broken strip gluing map $\mathcal{M} \times \mathcal{M} \to \bar{\mathcal{M}}$ ori-preserving.