Outline :
1. Waldhausen $S_\bullet$ - construction & K - thy.
2. Modern story of cyclotomic trace, the universal property approach.
3. Statement of Theorem of Dundas - Goodwillie - McCarthy.
4. Idea of Goodwillie calculus.
5. Sketch idea of pf of D-G-M.

I. Waldhausen $S_\bullet$ - construction & alg. K - theory.

Let $\mathcal{C}$ = stable ∞ - cat.

Recall that the Waldhausen $S_\bullet$ - construction assoc. w/ $\mathcal{C}$ is as follows :

$S_0\mathcal{C} \simeq *$

$S_1\mathcal{C} \simeq \mathcal{C}$

$$\begin{tikzcd} * \arrow[r] & X \in \mathcal{C} \arrow[d] \\ & * \end{tikzcd}$$

$S_2\mathcal{C} \simeq \mathrm{Fun}(\Delta^1, \mathcal{C})$

$$\begin{tikzcd} * \arrow[r] & X(0,0) \arrow[r] \arrow[d] & X(0,1) \arrow[d] \\ & * \arrow[r] & X(1,1) \arrow[d] \\ & & * \end{tikzcd}$$

cofib seq $X(0,0) \longrightarrow X(0,1) \longrightarrow X(1,1)$

⋮

$S_n\mathcal{C} \simeq \mathrm{Fun}(\Delta^{n-1}, \mathcal{C})$

$$\begin{tikzcd}
* \arrow[r] & X(0,0) \arrow[r] \arrow[d] & X(0,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & X(0,n) \arrow[d] \\
 & * \arrow[r] & X(1,1) \arrow[r] & \cdots \arrow[r] & X(1,n) \arrow[d] \\
 & & & \ddots & \vdots \\
 & & & & X(n,n) \arrow[d] \\
 & & & & *
\end{tikzcd}$$

w/ each square is cocartesian (pushout).

Then 1) each $S_n\mathcal{C}$ is stable ∞-cat

2) get an algebraic K spectrum $K(\mathcal{C})$ w/

$$K(\mathcal{C})_n = |(S_\bullet^{n}\mathcal{C})^\sim|$$

where $(-)^\sim$ = taking max subgroupoid,

$$S_\bullet^{(n)}\mathcal{C} = \underbrace{S_\bullet \cdots S_\bullet}_{n \text{ times}}\mathcal{C}$$

w/ structure maps induced by

$$\Sigma(-)^\sim \longrightarrow |(S_\bullet\mathcal{C})^\sim|$$

obtained by restriction to 1-skeleton.

So $\Omega^\infty K(\mathcal{C}) \simeq \Omega|(S_\bullet\mathcal{C})^\sim|$

<u>Rk</u>. Should be $\Omega|wS_\bullet\mathcal{C}^\sim|$ instead. $w\mathcal{C}$ = cat w/ w.e. usually model cat. Let $\mathcal{C}$ = ordinary model cat. Then

$$\mathcal{C} \longmapsto N(\text{Fib Replacement}(DK(\mathcal{C}, w\mathcal{C})))$$

where DK = Dwyer - Kan simplicial localization.

Prp 1) K via Q = K via $S_\bullet$

2) $K(\mathcal{C}) = K(\mathrm{Sp}\,\mathcal{C})$.

$\rightsquigarrow$ $K : \mathrm{Cat}_\infty^{st} \to \mathrm{Sp}$. lax symmetric monoidal

∞-cat of small stable ∞-cats,
mor = exact functors.
(preserves finite lim/colims)

Notion $\mathrm{Cat}_\infty^{perf} \subseteq \mathrm{Cat}_\infty^{st}$ idempotent - complete small stable ∞-cats

$\mathrm{Idem} : \mathrm{Cat}_\infty^{st} \rightleftarrows \mathrm{Cat}_\infty^{perf} : \mathrm{Forget}$

Def 1) $f : \mathcal{C} \to \mathcal{D} \in \mathrm{Cat}_\infty^{st}$ is Morita equiv if

$$\mathrm{Idem}\, f : \mathrm{Idem}\,\mathcal{C} \xrightarrow{\sim} \mathrm{Idem}\,\mathcal{D}.$$

2) $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{E}$ is exact (Karoubi) in $\mathrm{Cat}_\infty^{perf}$ if

- f fully faithful
- $\mathcal{D}/\mathcal{C} \simeq \mathcal{E}$
- $g \circ f \simeq 0$.

In Cat_∞^{st} : taking $\mathrm{Idem}(-)$.

Def Let $F : \mathrm{Cat}_\infty^{st} \to \mathrm{Sp}$.

1) F is additive invariant if

① F inverts Morita equiv.

② F preserves filtered colims.

③ F : split exact $\longmapsto$ split cofib seq.

2) F is localizing invariant if ① ② and

③' F : exact $\longmapsto$ cofib seq.

Write $PSh^{add}_{Sp}(Cat^{st}_\infty)$: functors $F : ((Cat^{st}_\infty)^{\omega})^{op} \to Sp$ s.t.

③ is satisfied.

$$\text{Forget} : PSh^{add}_{Sp}(Cat^{st}_\infty) \rightleftarrows PSh_{Sp}((Cat^{st}_\infty)^{\omega}) : L^{add}$$

Consider

$$\mathcal{M}_{add} : Cat^{st}_\infty \xrightarrow{\text{Yoneda}} PSh_{Sp}((Cat^{st}_\infty)^{\omega}) \xrightarrow{L^{add}} PSh^{add}_{Sp}(Cat^{st}_\infty)$$

$$\mathcal{C} \longmapsto \mathcal{M}_{add}(\mathcal{C})$$

additive non-comm. motive.

Thm (Blumberg - Gepner - Tabuada)

$\forall\ \mathcal{C} \in Cat^{st}_\infty$,

$$K(\mathcal{C}) \simeq Map(\mathcal{M}_{add}(\mathbb{S}), \mathcal{M}_{add}(\mathcal{C})).$$

$\forall\ F$ additive invariant, then

$$Map(K, F) \simeq F(\mathbb{S}).$$

Similar for localizing invariant.

Prop THH is a localizing invariant $\Rightarrow$ additive invariant.

$$THH(\mathcal{C}) \simeq \operatorname*{colim}_{c_1, \dots, c_n \in \mathcal{C}^{\simeq}} \bigotimes_{i=1}^{n} Map_{\mathcal{C}}(x_i, x_{i+1})$$

Cor $\pi_0 \mathrm{Map}(K, THH) \cong \pi_0 THH(\mathbb{S}) \cong \pi_0 \mathbb{S} \cong \mathbb{Z}$.

This is the set of equiv. classes of map $K \to THH$

" " of lax sym. mon. map corresponds to the unit $1 \in \mathbb{Z}$

$\rightsquigarrow$ get the Dennis trace = unique lax sym. mon. functor

$$K \xrightarrow{tr} THH$$

II. Cyclotomic trace.

Problem : TC Morita invariant, satisfies localization.

BUT NOT preserves filtered colims.

Resolve : $TC(\mathcal{C}) \simeq \lim_n TC^n(\mathcal{C})$, where

$TR^n(\mathcal{C}) \simeq THH(\mathcal{C})^{C_{p^{n-1}}}$ (categorical fixed pts)

$TC^n(\mathcal{C}) = eq\left(TR^n(\mathcal{C}) \underset{res}{\overset{incl}{\rightrightarrows}} TR^{n-1}(\mathcal{C}) \right)$

Prop (Blumberg - Gepner - Tabuada) TC^n is localizing invariant.

Thm (B-G-T) Dennis trace $tr : K(\mathcal{C}) \to THH(\mathcal{C})$

$\rightsquigarrow$ nat. trans. of localizing invariants $K \to TC^n$.

$\longrightarrow$ " " Sp-valued functor $K \to TC^n$

nat. trans. $K \to TC$ $\Leftrightarrow$ data of compatible maps to TC^n.

So
$$\begin{aligned} \mathrm{Map}(K, TC) &\simeq \lim_n \mathrm{Map}(K, TC^n) \\ &\simeq \lim_n TC^n(\mathbb{S}) \\ &\simeq TC(\mathbb{S}) \overset{\text{after } p\text{-completion}}{\simeq} \mathbb{S} \oplus \Sigma\, \mathrm{fib}(\mathbb{S}_{hT} \to \Sigma^{-1}\mathbb{S}). \end{aligned}$$

$$\pi_0 \mathrm{Map}(K, TC) \cong \pi_0(\cdots) \cong \mathbb{Z}_p.$$

$\Rightarrow$ After p-completion, cyclotomic trace is the unique map
$$\mathrm{trc} : K \longrightarrow TC$$
corresponds to $1 \in \mathbb{Z}_p = \pi_0(\mathrm{Map}(K, TC))$ above.

<u>Slogan</u>

$$\begin{tikzcd} K \arrow[rr, "\mathrm{tr}"] \arrow[d, "\mathrm{trc}"'] \arrow[dr] & & THH \\ & TC^n \arrow[ur] & \\ TC \arrow[ur] & & \end{tikzcd}$$

III. Theorem of Dundas - Goodwillie - McCarthy.

<u>Thm</u> Let $B \to A$ map of connected E_1-rings. $\pi_0 B \to \pi_0 A$ surjective w/ nilpotent kernel. Then the diagram is cartesian in Sp:

$$\begin{tikzcd} K(B) \arrow[r, "\mathrm{trc}"] \arrow[d] & TC(B) \arrow[d] \\ K(A) \arrow[r, "\mathrm{trc}"] & TC(A) \end{tikzcd}$$

The idea of pf is the Goodwillie Calculus.

IV. Goodwillie Calculus.

IDEA : Approximate F (preserves filtered colims) by "n-excisive functors" $P_n F$, i.e.

$$F \to \cdots \to P_{n+1} F \to P_n F \to \cdots \to P_0 F$$

similar to "Tayer poly" of F. Each $\mathrm{fib}(P_n F \to P_{n-1} F)$ is determined by n^{th}-derivative of F, denoted $\partial_n F$, which sees the "multilinear part" of F.

In other word, $\partial_n F$ "=" multilinear approximation to F.

Unpack : (1-) excisive = takes pushouts to pullbacks.

(n-) excisive = takes any 2-face (which is a pushout) of an n-cube to pullback.

P_n is left adjoint to $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$

Write $D_n F = \mathrm{fib}(P_n F \to P_{n-1} F)$. It is n^{th} differential of F.

It is determined by $\partial_n F$. Explicitly,

$$\begin{tikzcd} \mathcal{C}^n \arrow[r, "D_n F"] \arrow[d, "\prod \Sigma^\infty_{\mathcal{C}}"'] & \mathcal{D} \\ (\mathrm{Sp}\,\mathcal{C})^n \arrow[r, "\partial_n F"] & \mathrm{Sp}\,\mathcal{D} \arrow[u, "\Omega^\infty_{\mathcal{D}}"'] \end{tikzcd}$$

define $\partial_n F$ through the equivalence

$$\Omega^\infty_{\mathcal{D}} \circ \partial_n F \simeq F \circ \prod \Omega^\infty_{\mathcal{C}_i}.$$

We are mainly interested in the case $n=1$.

FACT $\mathrm{Exc}^1(\mathcal{C},\mathcal{D}) \subseteq \mathrm{Exc}^2(\mathcal{C},\mathcal{D}) \subseteq \mathrm{Exc}^3(\mathcal{C},\mathcal{D}) \subseteq \cdots$

The Goodwillie derivatives $\partial F := \partial_1 F$ "see" the "linear" part of F means that, for example $\mathcal{C} = \mathrm{Top}_*$, $\mathcal{D} = \mathrm{Sp}$.

Then $U, V \subseteq X$ open, $U \cup V \simeq X$, $U \cap V \simeq *$, then

$$\partial F(X) = \partial F(U) \oplus \partial F(V).$$

Thm (Goodwillie) If $\mathcal{C}$ has pushouts, $\mathcal{D}$ has sequential colims and finite lims (they are commutative), then $F : \mathcal{C} \to \mathcal{D}$ admits n^{th} approximation $P_n F$. Moreover, $P_n F$ is universal among all $F \to$ n-excisive functor.

Rk

Differential Calculus	Calculus of Functors
Smooth manifold M	Compactly generated ∞-category $\mathcal{C}$
Smooth function $f : M \to N$	Functor $F : \mathcal{C} \to \mathcal{D}$ which preserves filtered colimits
Point $x \in M$	Object $C \in \mathcal{C}$
Real vector space	Stable ∞-category
Real numbers $\mathbf{R}$	∞-category Sp of spectra
Linear map of vector spaces	Exact functor between stable ∞-categories
Tangent space $T_{M,x}$ to M at x	∞-category of spectrum objects $\mathrm{Sp}(\mathcal{C}_{/C})$
Differential of a smooth function	Excisive approximation of a functor (see Theorem 6.1.1.10)

V. Outline of pf of DGM.

We use the theorem to prove our statement :

<u>Thm</u> (Goodwillie)

If $F, G : \mathcal{C} \to \mathcal{D}$ both preserves filtered colims, $\partial_1 F \simeq \partial_1 G$, and both are "p-analytic", then the following diagram is cartesian :

$$\begin{array}{ccc} F(Y) & \longrightarrow & G(Y) \\ \downarrow & & \downarrow \\ F(X) & \longrightarrow & G(X) \end{array}$$

$\forall$ $(p+1)$-connected $Y \to X$.

▶ <u>pf. of DGM</u> $\mathcal{C} = \mathrm{Cat}_\infty^{st}$, $\mathcal{D} = \mathrm{Sp}$, $F = K$, $G = TC$

both p-analytic. $\partial_1 K \simeq \partial_1 TC \simeq \Sigma^{-1} THH$. Now reduce to trivial square-zero ext $A \ltimes M \to A$. apply Goodwillie thm.

Step 1. Reduce to trivial square - zero extension.

<u>Thm</u> (Goodwillie 86')

If the theorem is true in the special case when $R = A \ltimes M$ (trivial square - zero ext.), $S = A$, then it is true in general.

<u>IDEA of pf.</u> Use the simplicial approximation

$S \longrightarrow$ bar construction $S_\bullet$ $\quad |S_\bullet| \simeq S$

$R \to \cdots \quad R_\bullet \qquad |R_\bullet| \simeq R$

Can prove the question levelwisely. i.e. focus on

$K(R_r \to S_r)$. Each S_r is free associative. by assumption

$R_r \to S_r$ split surjective $\Rightarrow R_r = M_r \ltimes A_r$

Step 2. Goodwillie derivatives of K & TC.

FACT (Dundas - McCarthy)

$M \in A$-bimod. $\mathcal{P}_A$ = proj A-mod. Then

$$THH(A, M) \simeq \operatorname{colim}_n \Omega^n \Big| \bigoplus_{c \in S_\bullet^{(n)} \mathcal{P}_A} \operatorname{Hom}_{S_\bullet \mathrm{Mod}_A}(c, c \otimes_A M) \Big|$$

Let $\mathcal{P}(A, M)$ be cat of obj: (P, α).

where $P \in \mathcal{P}_A$

$\alpha: P \to P \otimes_A M$ A-linear

mor:
$$\begin{array}{ccc} (P_1, \alpha_1) & & P_1 \longrightarrow P_1 \otimes_A M \\ \downarrow f & & f_P \downarrow \qquad \qquad \downarrow f_P \otimes 1 \\ (P_2, \alpha_2) & & P_2 \longrightarrow P_2 \otimes_A M \end{array}$$

Define $$K(\mathcal{P}(A, M)) = \operatorname{colim}_n \Omega^n \Big| \Big(\coprod_{c \in S_\bullet^{(n)} \mathcal{P}_A} \operatorname{Hom}_{S_\bullet \mathrm{Mod}_A}(c, c \otimes_A M) \Big) \Big|$$

and $\tilde{K}(A, M) := \operatorname{hofib}(K(\mathcal{P}(A, M)) \to K(A))$

$= \partial K(A, M)$.

e.g. $K(\mathcal{P}(A, 0)) = \operatorname{colim}_n \Omega^n \Big| \Big(\coprod_{c \in S_\bullet^{(n)} \mathcal{P}_A} * \Big) \Big| \simeq K(A)$.

FACT (Dundas - McCarthy) $K(\mathcal{P}(A, M)) = K(A \ltimes M)$.

Like in Step 1, we can also use the simplicial approximation $M_\bullet \longrightarrow M$. $|M_\bullet| \simeq M$. Suppose M is m-connected, i.e. $\pi_*|M_\bullet| = 0$, $* \le m$. Consider the diagram

$$\begin{tikzcd}
\tilde{K}(A, M) \arrow[rr, dashed] \arrow[d, "\simeq"'] & & THH(A, M) \arrow[d, "\simeq"] \\
\Omega^p fib^{(p)} \arrow[r, "\alpha_p"] & \Omega^p cof^{(p)} \arrow[r, "\beta_p"] & \Omega^p THH^{(p)}
\end{tikzcd}$$

where

$$\begin{cases}
fib^{(p)} = fib\left(\left| \coprod_{c \in S_\bullet^{(p)} \mathcal{P}_A} Hom_{S_\bullet^{(p)} Mod_A}(c, c \otimes_A M) \right| \longrightarrow | S_\bullet^{(p)} \mathcal{P}_A | \right) \\
cof^{(p)} = cofib\left(| S_\bullet^{(p)} \mathcal{P}_A | \longrightarrow \left| \coprod_{c \in S_\bullet^{(p)} \mathcal{P}_A} Hom_{S_\bullet^{(p)} Mod_A}(c, c \otimes_A M) \right| \right) \\
THH^{(p)} = \left| \bigoplus_{c \in S_\bullet^{(p)} \mathcal{P}_A} Hom_{S_\bullet^{(p)} Mod_A}(c, c \otimes_A M) \right| .
\end{cases}$$

Prop (Dundas - McCarthy 94')

α_p is at least $(p-3)$-connected. β_p is $2m$-connected.

IDEA of pf. Consider the diagram

$$\begin{tikzcd}
\left| \left(\coprod_{c \in S_\bullet^{(p)} \mathcal{P}_A} Hom_{S_\bullet Mod_A}(c, c \otimes_A M) \right) \right| \arrow[r] \arrow[d] & | S_\bullet^{(p)} \mathcal{P}_A | \arrow[d] \\
\left| \left(\bigvee_{c \in S_\bullet^{(p)} \mathcal{P}_A} Hom_{S_\bullet Mod_A}(c, c \otimes_A M) \right) \right| \arrow[r] & *
\end{tikzcd}$$

Then use Blakers - Massey theorem.

- Take $p \gg 2m$, $p \to \infty$, get

$$\partial K(A, M) = THH(A, M).$$

For TC, the story is similar. Consider

$$\widetilde{TC}(A, M) := \mathrm{hofib}(TC(A \ltimes M) \longrightarrow TC(A))$$
$$= \partial TC(A, M).$$

The following from Hesselholt (1994): Stable TC is THH.

Also use the simplicial approximation. Note the underlying space is $TC(A \oplus M)$. Suppose $M \leftarrow M_\bullet$ is m-connected.

Recall $THH(A, M) = |N_\bullet^{cyc}(A \oplus M)|$ where the cyclic bar construction $N_n^{cyc}(A \oplus M) \simeq (A \oplus M)^{\otimes n+1}$

$$\simeq \bigvee_{S \subset [n]} A^{\otimes([n]-S)} \otimes M^{\otimes S}$$

Write $T_{a,n}(A, M) = \bigvee_{\substack{S \subset [n] \\ |S| = a}} A^{\otimes([n]-S)} \otimes M^{\otimes S}$

e.g. $T_{0,\bullet}(A, M) = THH(A)$.

<u>Prop</u> (Hesselholt 94').

1) $|T_{1,\bullet}(A, M)| \simeq S^1_+ \wedge THH(A, M)$

2) cyclotomic structure map is given by

$$R_p : T_{a,\bullet}(A, M)^{C_{p^r}} \longrightarrow T_{a/p, \bullet}(A, M)^{C_{p^{r-1}}}$$

which induces

$$\widetilde{TC}(A, M)^\wedge_p \simeq \left(\operatorname*{holim}_{R_p} \left(\bigvee_{s=0}^{\infty} T_{p^s, \bullet}(A, M) \right)^{C_{p^r}} \right)^\wedge_p$$

3) By checking the connectivity. and look at the free S^1-action.

$$\text{RHS} \simeq \left(\operatorname{holim}_{R_p} \left(T_{1,\bullet}(A, M)\right)^{C_{p^r}}\right)^{\wedge}_p$$

$$\simeq \left(\operatorname{holim}_r \left(S^1/C_{p^r +} \wedge THH(A, M)\right)\right)^{\wedge}_p .$$

which is a consequence of (1).

By Proposition, and $S^1/C_{p^r} \simeq S^1$. one gets

$$\partial TC(A, M)^{\wedge}_p = \widetilde{TC}(A, M)^{\wedge}_p \simeq \left(\Sigma\, THH(A, M)\right)^{\wedge}_p .$$

Step 3. Analytcity of K & TC.

<u>Thm</u> (Goodwillie for K. 92'; McCarthy for TC. 97')

K and TC are (-1)-analytic.

Step 4. Approximate $B \to A$ in the argument again by simplicial rings $B_\bullet$, $A_\bullet$, respectively. then use the p-completed case (a.k.a. McCarthy's theorem) and use some complicated examination on connectivity yields the desired result (a.k.a Dundas' theorem).