From Exotic Spheres to Stable Homotopy Theory

Albert Yang

Department of Mathematics
University of Pennsylvania

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Outline

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- Freedman (1982) proved it for $n = 4$ via intersection forms.
- Perelman (2003) proved for $n = 3$ case.

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Does the same result hold for **smooth** manifolds with the homotopy type of the sphere?

- True for $n = 2, 3$, proved by Moise (1952).
- In general, the answer is **NO**!
- Milnor (1956) constructed an "exotic sphere" that is homeomorphic to S^7, but not diffeomorphic to S^7.

Milnor's Exotic Sphere

Definition

A **structure group** of the fiber bundle $F \to E \to X$ is a group G acting homeomorphically on F such that for any trivialization U_i, U_j, the transition

$$\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F,$$

sends (x, y) to $(x, g_{ij}(x)y)$, for some continuous $g_{ij} : U_i \cap U_j \to G$.

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Regard S^3 as the unit quaternion so that it can be seen as a group with the group operation given by multiplication.
Consider the double cover of $SO(4)$, given by

$$p : S^3 \times S^3 \to SO(4), \quad (x, y) \mapsto (\phi_{x,y} : v \mapsto xvy^{-1}),$$

where $\phi_{x,y}$ can be viewed as an isometry from $\mathbb{R}^4 \to \mathbb{R}^4$.

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For each $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$, define

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Let $f_{m,n} = p \circ \psi_{m,n} : S^3 \to SO(4)$. Define

$$E_{f_{m,n}} = (\mathbb{R}^4 \times S^3) \sqcup (\mathbb{R}^4 \times S^3)/\sim, \quad (u, x) \sim (u, f_{m,n}x).$$

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Then we get a S^3-bundle over S^4 with structure group $SO(4)$:

$$S^3 \to E_{f_{m,n}} \to S^4.$$

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- $E_{f_{m,n}}$ is homeomorphic to S^7 when $m + n = \pm 1$ by Morse theory.
- $E_{f_{m,n}}$ is NOT diffeomorphic to S^7 when $(m - n)^2 \neq 1 \mod 7$ by Hirzebruch signature theorem.

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- $n = 4$, still open.
- For $n \geq 5$, we can do it by studying the **stable homotopy groups of the spheres**!

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Smooth Structures

Fix $n \geq 5$.

Definition

Let Θ_n be the set of homotopy n-spheres up to diffeomorphism. Together with the connecting sum as an operation, it is an abelian group.

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- The key is to study the group Θ_n.
- $|\Theta_n|$ is actually the number of smooth structures.
- (Kervaire-Milnor, 1963) Two steps to tackle the problem:
  1. Classify the homotopy spheres up to framed cobordism.
  2. Classify the homotopy spheres that bound framed manifolds.

Step 1: Framed Manifolds

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Note $T\mathbb{R}^{n+k} = \mathbb{R}^{n+k} \times \mathbb{R}^{n+k}$. The tangent bundle of M^k is included in the restriction of $T\mathbb{R}^{n+k}$ to M^k, i.e. $TM^k \subset M^k \times \mathbb{R}^{n+k}$.

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A **framing** on M^k in $\mathbb{R}^{n+k}$ is a vector space isomorphism

$$f : M^k \times \mathbb{R}^n \to N_{\mathbb{R}^{n+k}/M^k}.$$

It exists iff the normal bundle is trivial.

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It exists iff the normal bundle is trivial. M^k is a **framed** k**-manifold** if it admits a fixed framing f.

Step 1: Framed Cobordism

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Let M^k, N^k be two framed k-manifolds in $\mathbb{R}^{n+k}$. A **framed cobordism** between M^k, N^k is a $(k+1)$-dimensional submanifold W^{k+1} of $\mathbb{R}^{n+k} \times [0,1] \subset \mathbb{R}^{n+k+1}$ such that

$$\partial W^{k+1} = (M^k \times \{0\}) \cup (N^k \times \{1\}),$$

with a framing on W^{k+1} restricts to ones on $M^k \times \{0\}$, $N^k \times \{1\}$.

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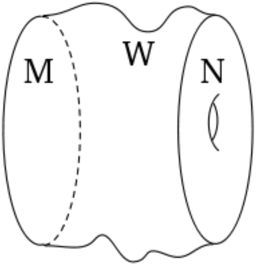

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Theorem (Pontryagin-Thom)

For $k \geq 0$, $n \geq 1$, $\Omega_k^{fr}(\mathbb{R}^{n+k}) \cong \pi_{n+k}(S^n)$.

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Corollary (Freudenthal, 1938)

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When n is sufficiently large, $\pi_{n+k}(S^n)$ depends only on k. Taking the limit, we have the stable homotopy groups of the spheres (called the **stable stems**):

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2. $\pi_1^s = \pi_2^s = \mathbb{Z}/2$.

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Homotopy spheres can be framed. A homotopy k-sphere Σ^k with two different framings F_1, F_2 satisfies

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- Starting Point: **twisted framing** on spheres.

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- If $M^k = S^k$ has a framing F that extends to a $(k+1)$-disk, then $[S^k, F] = \varnothing$.

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- If $M^k = S^k$ has a framing F that extends to a $(k+1)$-disk, then $[S^k, F] = \varnothing$.
- $[S^k, F]$ is non-trivial iff F is twisted, hence determined by an element in $\pi_k(SO(n))$.

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Theorem (Bott, 1959)

$\pi_k(SO)$ is 8-periodic. In particular, one has

k	0	1	2	3	4	5	6	7
$\pi_k(SO)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

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Theorem (Adams 1966, Quillen 1971, Sullivan 1974)

The image of J is a direct summand of π_n^s, and is cyclic for all n. In particular,

1. *If $n \equiv 0, 1 \mod 8$, then $|\mathrm{im}\, J| = 2$.*
2. *If $n \equiv 3, 7 \mod 8$, then $|\mathrm{im}\, J|$ is the denominator of $B_{2k}/(4k)$, where B_{2k} is the Bernoulli number.*
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The Bernoulli number is defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.$$

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The following is a list of some Bernoulli numbers:

k	2	4	6	8	10	12	14
B_k	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$

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Facts

- All homotopy spheres admit (stable) framings.
- $\Omega_n^{fr} \cong \pi_n^s$.
- Elements in Ω_n^{fr} satisfies $[\Sigma^n, F_1] - [\Sigma^n, F_2] = [S^n, F]$.
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We have a homomorphism:

$$\Theta_n \to \pi_n^s / \mathrm{im}\, J.$$

The kernel of this map is denoted by Θ_n^{bp}, which consists of the homotopy spheres that bound framed manifolds.

Kervaire-Milnor Theory

Theorem (Kervaire-Milnor, 1963)

1. *If* $n \neq 2 \mod 4$, *then there is an exact sequence*

$$0 \to \Theta_n^{bp} \to \Theta_n \to \pi_n^s/J \to 0.$$

2. *If* $n \equiv 2 \mod 4$, *then there is an exact sequence*

$$0 \to \Theta_n^{bp} \to \Theta_n \to \pi_n^s/J \xrightarrow{\Phi} \mathbb{Z}/2 \to \Theta_{n-1}^{bp} \to 0,$$

where Φ *is the Kervaire invariant.*

3. *If* n *is even, then* $\Theta_n^{bp} = 0$.
4. *If* $n = 4k - 1$, *then*

$$\Theta_n^{bp} \cong \mathbb{Z}/(2^{2k-2}(2^{2k-1} - 1)c_k),$$

where c_k *is the numerator of* $4B_{2k}/k$.

Kervaire-Milnor Theory

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- Central to homotopy theory, but extremely hard.

Outline

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- **Milestone**: Introduction of the stable homotopy category, by Spanier and Whitehead (1962), and Boardman (1965).

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- Barratt, Mahowald, Tangora, Bruner, Nakamura, etc. computed differentials and π_k^s at mod 2, 3, 5 via MaySS, Toda brackets, Massey products, power operations, etc. in 1960-1980s. They determined π_k^s up to $n = 45$ at mod 2, $n = 108$ at mod 3, and $n = 999$ at mod 5.

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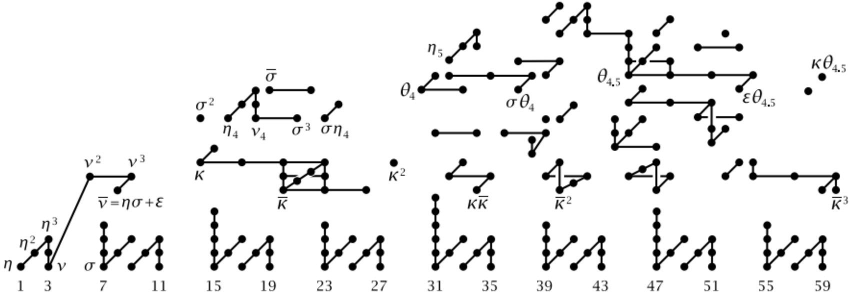
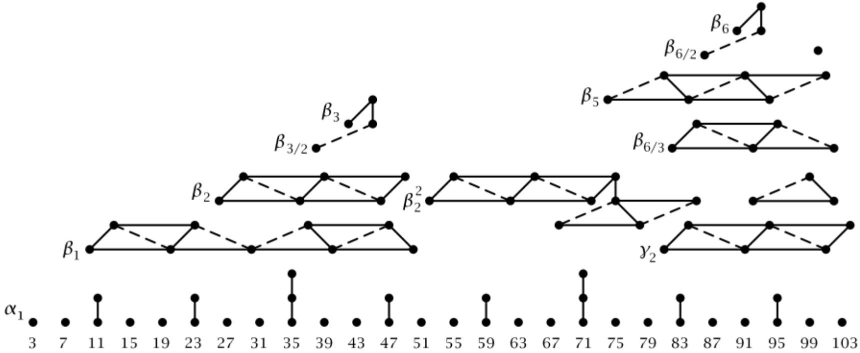
- More results later...
- **Most recent**: Isaksen (2019) and Isaksen-Xu-Wang (2020, 2023) used the motivic Adams spectral sequences to determine π_k^s up to $n = 90$ at mod 2.

Here's the picture of the 2-primary parts of π_k^s from Hatcher, for $i \leq 60$.

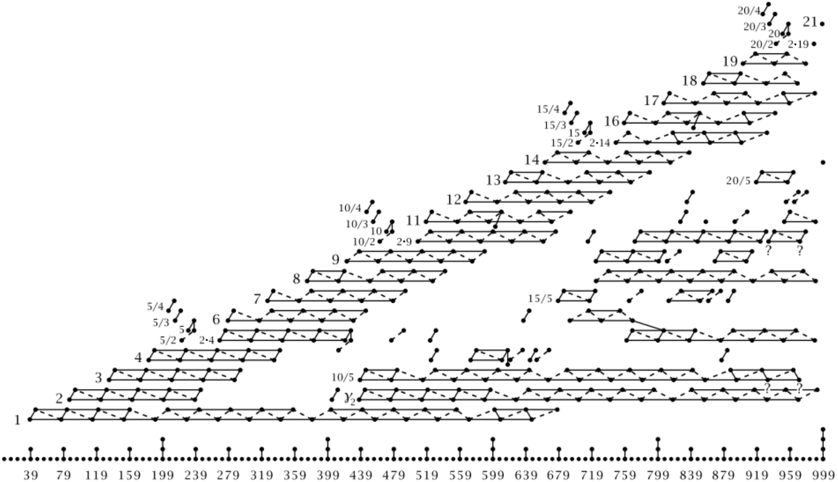


Computational Results

Here's the picture of the 3-primary parts of π_k^s from Hatcher, for $i \leq 108$.



Here's the picture of the 5-primary parts of π_k^s from Hatcher, for $i \leq 999$.



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The ***only even-dimensional spheres below dimension 140*** *which have unique smooth structures are S^2, S^6, S^{12}, S^{56} and perhaps S^4.*

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Theorem (Behrens-Hill-Hopkins-Mahowald)

*The **only even-dimensional spheres below dimension 140** which have unique smooth structures are S^2, S^6, S^{12}, S^{56} and perhaps S^4.*

Conjecture

S^n has a unique smooth structure if either $n \leq 6$, or $n = 12, 56, 61$.

More Results

In general, below dimension 90, one has (picture from Isaksen-Wang-Xu's paper "Stable homotopy groups of spheres: from dimension 0 to 90")

k	v_1-torsion at the prime 2	v_1-torsion at odd primes	v_1-periodic	Group of smooth structures
1	·	·	2	·
2	·	·	2	·
3	·	·	$8\cdot 3$	·
4	·	·	·	?
5	·	·	·	·
6	2	·	·	·
7	·	·	$16\cdot 3\cdot 5$	$\underline{b_2}$
8	2	·	2	2
9	2	·	2^2	$\underline{2}\cdot 2^2$
10	·	3	2	$2\cdot 3$
11	·	·	$8\cdot 9\cdot 7$	$\underline{b_3}$
12	·	·	·	·
13	·	3	·	3
14	$2\cdot 2$	·	·	2
15	2	·	$32\cdot 3\cdot 5$	$\underline{b_4}\cdot 2$
16	2	·	2	2
17	2^2	·	2^2	$\underline{2}\cdot 2^3$
18	8	·	2	$2\cdot 8$
19	2	·	$8\cdot 3\cdot 11$	$\underline{b_5}\cdot 2$
20	8	3	·	$8\cdot 3$
21	2^2	·	·	$\underline{2}\cdot 2^2$
22	2^2	·	·	2^2
23	$2\cdot 8$	3	$16\cdot 9\cdot 5\cdot 7\cdot 13$	$\underline{b_6}\cdot 2\cdot 8\cdot 3$
24	2	·	2	2
25	·	·	2^2	$\underline{2}\cdot 2$
26	2	3	2	$2^2\cdot 3$
27	·	·	$8\cdot 3$	$\underline{b_7}$

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28	2	·	·	2
29	·	3	·	3
30	2	3	·	3
31	2^2	·	$64{\cdot}3{\cdot}5{\cdot}17$	$\underline{b_8}{\cdot}2^2$
32	2^3	·	2	2^3
33	2^3	·	2^2	$\underline{2}{\cdot}2^4$
34	$2^2{\cdot}4$	·	2	$2^3{\cdot}4$
35	2^2	·	$8{\cdot}27{\cdot}7{\cdot}19$	$\underline{b_9}{\cdot}2^2$
36	2	3	·	$2{\cdot}3$
37	2^2	3	·	$\underline{2}{\cdot}2^2{\cdot}3$
38	$2{\cdot}4$	$3{\cdot}5$	·	$2{\cdot}4{\cdot}3{\cdot}5$
39	2^5	3	$16{\cdot}3{\cdot}25{\cdot}11$	$\underline{b_{10}}{\cdot}2^5{\cdot}3$
40	$2^4{\cdot}4$	3	2	$2^4{\cdot}4{\cdot}3$
41	2^3	·	2^2	$\underline{2}{\cdot}2^4$
42	$2{\cdot}8$	3	2	$2^2{\cdot}8{\cdot}3$
43	·	·	$8{\cdot}3{\cdot}23$	$\underline{b_{11}}$
44	8	·	·	8
45	$2^3{\cdot}16$	$9{\cdot}5$	·	$\underline{2}{\cdot}2^3{\cdot}16{\cdot}9{\cdot}5$
46	2^4	3	·	$2^4{\cdot}3$
47	$2^3{\cdot}4$	3	$32{\cdot}9{\cdot}5{\cdot}7{\cdot}13$	$\underline{b_{12}}{\cdot}2^3{\cdot}4{\cdot}3$
48	$2^3{\cdot}4$	·	2	$2^3{\cdot}4$
49	·	3	2^2	$\underline{2}{\cdot}2{\cdot}3$
50	2^2	3	2	$2^3{\cdot}3$
51	$2{\cdot}8$	·	$8{\cdot}3$	$\underline{b_{13}}{\cdot}2{\cdot}8$
52	2^3	3	·	$2^3{\cdot}3$
53	2^4	·	·	$\underline{2}{\cdot}2^4$
54	$2{\cdot}4$	·	·	$2{\cdot}4$

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k	v_1-torsion at the prime 2	v_1-torsion at odd primes	v_1-periodic	Group of smooth structures
55	·	3	$16{\cdot}3{\cdot}5{\cdot}29$	$\underline{b_{14}}{\cdot}3$
56	·	·	2	·
57	2	·	2^2	$\underline{2}{\cdot}2^2$
58	2	·	2	$\underline{2^2}$
59	2^2	·	$8{\cdot}9{\cdot}7{\cdot}11{\cdot}31$	$\underline{b_{15}}{\cdot}2^2$
60	4	·	·	$\underline{4}$
61	·	·	·	·
62	2^4	3	·	$2^3{\cdot}3$
63	$2^2{\cdot}4$	·	$128{\cdot}3{\cdot}5{\cdot}17$	$\underline{b_{16}}{\cdot}2^2{\cdot}4$
64	$2^5{\cdot}4$	·	2	$\underline{2^5}{\cdot}4$
65	$2^7{\cdot}4$	3	2^2	$\underline{2}{\cdot}2^8{\cdot}4{\cdot}3$
66	$2^5{\cdot}8$	·	2	$\underline{2^6}{\cdot}8$
67	$2^3{\cdot}4$	·	$8{\cdot}3$	$\underline{b_{17}}{\cdot}2^3{\cdot}4$
68	2^3	3	·	$\underline{2^3}{\cdot}3$
69	2^4	·	·	$\underline{2}{\cdot}2^4$
70	$2^5{\cdot}4^2$	·	·	$\underline{2^5}{\cdot}4^2$
71	$2^6{\cdot}4{\cdot}8$	·	$16{\cdot}27{\cdot}5{\cdot}7{\cdot}13{\cdot}19{\cdot}37$	$\underline{b_{18}}{\cdot}2^6{\cdot}4{\cdot}8$
72	2^7	3	2	$\underline{2^7}{\cdot}3$
73	2^5	·	2^2	$\underline{2}{\cdot}2^6$
74	4^3	3	2	$2{\cdot}4^3{\cdot}3$
75	2	9	$8{\cdot}3$	$\underline{b_{19}}{\cdot}2{\cdot}9$
76	$2^2{\cdot}4$	5	·	$\underline{2^2}{\cdot}4{\cdot}5$
77	$2^5{\cdot}4$	·	·	$\underline{2}{\cdot}2^5{\cdot}4$
78	$2^3{\cdot}4^2$	3	·	$\underline{2^3}{\cdot}4^2{\cdot}3$
79	$2^6{\cdot}4$	·	$32{\cdot}3{\cdot}25{\cdot}11{\cdot}41$	$\underline{b_{20}}{\cdot}2^6{\cdot}4$
80	2^8	·	2	$\underline{2^8}$

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81	$2^3{\cdot}4{\cdot}8$	3^2	2^2	$\underline{2}{\cdot}2^4{\cdot}4{\cdot}8{\cdot}3^2$
82	$2^5{\cdot}8$	$3{\cdot}7$	2	$2^6{\cdot}8{\cdot}3{\cdot}7$ or $2^4{\cdot}4{\cdot}8{\cdot}3{\cdot}7$
83	$2^3{\cdot}8$	5	$8{\cdot}9{\cdot}49{\cdot}43$	$\underline{b_{21}}{\cdot}2^3{\cdot}8{\cdot}5$
84	2^6 or 2^5	3^2	·	$2^6{\cdot}3^2$ or $2^5{\cdot}3^2$
85	$2^6{\cdot}4^2$ or $2^5{\cdot}4^2$ or $2^4{\cdot}4^3$ or $2^7{\cdot}4$	3^2	·	$2^6{\cdot}4^2{\cdot}3^2$ or $2^5{\cdot}4^2{\cdot}3^2$ or $2^4{\cdot}4^3{\cdot}3^2$ or $2^7{\cdot}4{\cdot}3^2$
86	$2^4{\cdot}8^2$ or $2^2{\cdot}4{\cdot}8^2$	$3{\cdot}5$	·	$2^4{\cdot}8^2{\cdot}3{\cdot}5$ or $2^2{\cdot}4{\cdot}8^2{\cdot}3{\cdot}5$
87	$2^5{\cdot}4$	·	$16{\cdot}3{\cdot}5{\cdot}23$	$\underline{b_{22}}{\cdot}2^5{\cdot}4$
88	$2^4{\cdot}4$	·	2	$2^4{\cdot}4$
89	2^3	·	2^2	$\underline{2}{\cdot}2^4$
90	$2^3{\cdot}8$ or $2^2{\cdot}8$	3	2	$2^4{\cdot}8{\cdot}3$ or $2^3{\cdot}8{\cdot}3$

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Thank you!