A BRIEF INTRODUCTION TO HODGE THEORY OF COMPACT KÄHLER MANIFOLDS

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Contents

1. Preliminary: Almost Complex Structure

We assume the familiarity of smooth manifolds and vector bundles in this note. We will always assume the manifolds are smooth and finite dimensional unless otherwise stated.

Let V be a real vector space with dimension $2n$. A complex structure on V is an endomorphism $J : V \to V$ such that $J^2 = -\mathrm{id}$. We can complexify V into a $\mathbb{C}$-vector space by tensoring a $\mathbb{C}$. J can be extended to $V \otimes \mathbb{C}$ by $J(v \otimes z) = J(v) \otimes z$ for $v \in V, z \in \mathbb{C}$. As a linear transformation, $J^2 = -\mathrm{id}$ has two eigenvalues i and $-i$. Denote the eigenspace associated with i by $V^{1,0}$, and the one associated with $-i$ by $V^{0,1}$. Now we can write $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$.

Exercise 1.1. Every element in $V^{1,0}$ can be written in the form $v \otimes 1 - Jv \otimes i$. Similarly, every element in $V^{0,1}$ can be written in the form $v \otimes 1 + Jv \otimes i$.

We can apply this idea to the tangent space of a manifold M.

Definition 1.2. Let M be a manifold. An **almost complex structure** on M is a vector bundle isomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{id}$. A manifold admitting an almost complex structure is called an **almost complex manifold**.

This J turns TM into a $\mathbb{C}$-vector bundle, but does not turn M into a complex manifold because it tells us nothing about the local diffeomorphisms. By definition, if M admits an almost complex structure, then M is even-dimensional and orientable (Exercise). A famous theorem by Borel and Serre said that the only spheres admitting an almost complex structure is S^2 and S^6.

Example 1.3 (4-sphere)**.** Recall the first Pontryagin class of a real vector bundle $E \to M$ with a complex structure J is given by the second Chern class: $p_1(E) = -c_2(E \otimes \mathbb{C}) \in H^4(M)$. One can decompose $E \otimes \mathbb{C} = (E, J) \oplus (E, -J)$. Write $\overline{E} = (E, -J)$. By Whitney sum formula and $c_j(\overline{E}) = (-1)^j c_j(E)$,

$$\begin{aligned} c_2(E \otimes \mathbb{C}) &= c_2(E \oplus \overline{E}) = c_2(\overline{E}) + c_1(E)c_1(\overline{E}) + c_2(E) \\ &= 2c_2(E) - (c_1(E))^2. \end{aligned} \tag{1}$$

Suppose S^4 has an almost complex structure J. Apply (1) to $TS^4 \to S^4$, we get

$$p_1(TS^4) = 2c_2(TS^4E) - (c_1(TS^4))^2.$$

Since $H^2(S^4) = 0$, the signature of S^4 is $\sigma(S^4) = 0$. By Hirzebruch signature theorem, $\frac{1}{3}p_1(TS^4) \cdot [S^4] = \sigma(S^4) = 0$. Hence

$$\begin{aligned} 0 &= \frac{1}{3}p_1(TS^4) \cdot [S^4] \\ &= \big(2c_2(TS^4E) - (c_1(TS^4))^2\big) \cdot [S^4]. \end{aligned}$$

Note that c_2 is the top Chern class of TS^4, so it is the Euler class. Evaluating at $[S^4]$, $c_2 \cdot [S^4] = \chi(S^4) = 2$. This implies $(c_1(TS^4))^2 \cdot [S^4] = 4$, which is impossible because $H^2(S^4) = 0$. Contradiction!

We can complexify TM to $TM \otimes \mathbb{C}$ and decompose it into $T^{1,0} \oplus T^{0,1}$, where $T^{1,0}$ is the eigenspace associated with i and $T^{0,1}$ is the eigenspace associated with $-i$. Dually, we can decompose the complexified cotangent bundle into $(T^{1,0})^*$ and $(T^{0,1})^*$.

Given a complex manifold M with an atlas $\{(U, \phi)\}$, we can obtain a canonical almost complex structure through the following: start with a local coordinate $(z_1, \cdots, z_n)$ for some arbitrarily chosen $p \in U \subset M$, where $z_j = x_j + iy_j$. A basis for T_pM can be chosen to be the span of $\{\partial_{x_j}, \partial_{y_j}\}$. Set $J : TM \to TM$ restricting at p to be $J_p(\partial_{x_j}) = \partial_{y_j}$ and $J_p(\partial_{y_j}) = -\partial_{x_j}$. It is an easy exercise to check that J is an almost complex structure.

Definition 1.4. An almost complex structure J on a manifold M is said to be **integrable** if it comes from a complex structure.

We have two methods leading to the same theorem deciding whether an almost complex structure is integrable.

Definition 1.5. Let M be a manifold, and $E \subset TM$ be a subbundle of rank k. Then

- E is **involutive** if the Lie bracket of any two sections of E is again a section;

- E is **integrable** if for any point $p \in M$, there is a neighborhood U of p with a diffeomorphism $\phi_U : U \to \mathbb{R}^{n-k}$ such that $E\mid_U = \ker(d\phi_U)$. That is, any fiber ϕ_U^{-1} is a submanifold of U with tangent space $E \mid \phi_U^{-1}$.

Theorem 1.6 (Frobenius)**.** *Let M be a manifold and $E \subset TM$ be a subbundle of rank k. Then E is involutive iff E is integrable. If M further admits a complex structure with $\dim_{\mathbb{C}} M = n$, then E involutive iff E is holomorphically integrable (i.e. ϕ_U in the definition of integrability is chosen to be holomorphic).*

Definition 1.7. Let M be a manifold with an almost complex structure J. Then (M, J) is called **real analytic** if M has a real analytic atlas, and in each of these local coordinate charts, J is a real analytic family of matrices.

Theorem 1.8 (Newlander-Nirenberg, version 1)**.** *Let (M, J) be real analytic. Then J is integrable iff $T^{0,1}$ is involutive.*

Another way to state the Newlander-Nirenberg theorem is via the Nijenhuis tensor. Let X, Y be vector fields on M. The Nijenhuis tensor of X, Y is

$$N_J(X, Y) \coloneqq [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

One can check that $N_J = 0$ iff $[T^{0,1}, T^{0,1}] \subset T^{0,1}$, i.e. $T^{0,1}$ is involutive. So we have:

Theorem 1.9 (Newlander-Nirenberg, version 2)**.** *J is integrable iff $N_J = 0$.*

In the following chapter, we will see that the vanishing of Nijenhuis tensor can be characterized by other criteria, e.g. $d = \partial + \overline{\partial}$ or $\overline{\partial}^2 = 0$.

2. Complexes

2.1. **de Rham Complexes.** Proofs of this section are omitted. See Chapter 5 of course notes of C3.3 Differentiable Manifolds for details.

Recall that an n-**form** of a manifold M is a section of the Grassmann exterior algebra of cotangent bundle. Namely, $\Omega^m(M) = \mathbb{C}^\infty(M, \bigwedge^n T^*M)$. Here $\bigwedge^\bullet V$ for an arbitrary finite dimensional R-vector space V is a graded associative algebra $\bigwedge^\bullet V = \bigoplus \bigwedge^k V$ together with an injective linear map $\imath : V \to \bigwedge^\bullet V$ with $\bigwedge^0 V = V$ and $\bigwedge^1 V = \imath(V) \cong V$, which is universal.

We have an exterior derivative $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ such that it is universal and linear, with the following properties:

(1) $d^2 = 0$;
(2) for any $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, we have $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k \omega\wedge d\eta$;
(3) for any $F : M \to N$ smooth, $\omega \in \Omega^\bullet(M)$, we have $d(F^*\omega) = F^* d\omega$;

For any $f \in \Omega^0(M)$, $df \in \Omega^1(M)$, for any vector field X, we define

$$df(X) = \mathcal{L}_X f = Xf,$$

where $\mathcal{L}_X\omega$, the Lie derivative of k-form ω, is given by

$$\mathcal{L}_X\omega(p) = \lim_{t\to 0} \frac{(\phi_t^X)^*(\omega(\phi_t^X(p))) - \omega(p)}{t}$$

for $p \in M$, where ϕ_t^X is the flow of X near p. In general, let $V_0, \cdots, V_k$ be vector fields. Then the exterior derivative of a k-form is given by

$$d\omega(V_0, \cdots, V_k) = \sum_{j=0}^{k} (-1)^j \mathcal{L}_{V_j}(\omega(V_0, \cdots, \hat{V}_j, \cdots, V_k)) + \sum_{j<\ell} (-1)^{j+\ell} \omega([V_j, V_k], V_0, \cdots, \hat{V}_j, \cdots, \hat{V}_\ell \cdots, V_k).$$

In particular, when $k = 1$,

$$d\omega(V_0, V_1) = \mathcal{L}_{V_0}(\omega(V_1)) - \mathcal{L}_{V_1}(\omega(V_0)) - \omega([V_0, V_1]).$$

$(\Omega^\bullet(M), d)$ now constitutes a well-defined cochain complex:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots.$$

An k-form ω is called **closed** if $d\omega = 0$, and called **exact** if $d\eta = \omega$ for some $(k-1)$-form η.

Definition 2.1. The **k-th de Rham cohomology group** of M is defined to be

$$H^k_{dR}(M) = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}.$$

Corollary 2.2. *$H^0_{dR}(M) = \{f \in C^\infty(M) \mid df = 0\}$ = locally constant functions on M = $\mathbb{R}^{\pi_0 M}$.*

Lemma 2.3. *de Rham cohomology is a graded commutative algebra with multiplication given by $[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$. Hence, H^*_{dR} : $\mathbf{Manifolds}^{op} \to \mathbf{GradedCommAlg}$ is a well-defined contravariant functor.*

Let $F : M \to N$. We can pull back the k-forms on N via F^*, commuting with the exterior derivative d. In fact, F^* induced a map on de Rham cohomology group $F^* : H^*_{dR}(N) \to H^*_{dR}(M)$, where $[\omega] \in H^*_{dR}(N)$ is sent to $[F^*\omega] \in H^*_{dR}(M)$. This map only depends only on the homotopy class of F.

Lemma 2.4. *Let $H : M \times [0, 1] \to N$ be a smooth homotopy between F_0, F_1, with $F_t = H \mid_{M \times \{t\}}$. Then $F_0^* = F_1^*$.*

Corollary 2.5. *If A is a deformation retract of M, then M and A have the same de Rham cohomology.*

Corollary 2.6 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^n$ be a smoothly contractible subspace. Then $H^k_{dR}(U) = 0$ for all $k > 0$.*

Poincaré lemma also exists in compactly supported de Rham cohomology $H^*_{dR,c}$. Let M be a manifold, consider the projection $\pi : M \times \mathbb{R} \to M$. The push-forward map (**NOT** pullback π^*!) $\pi_* : \Omega^*_c(M \times \mathbb{R}) \to \Omega^{*-1}_c(M)$ defined an induced map of π on compactly supported de Rham complex as follows: note that every compactly supported form on $M \times \mathbb{R}$ is a linear combination of two types of form

(1) $\pi^*\omega \cdot f(x, t)$ for $\omega \in \Omega^*(M \times \mathbb{R})$ and f being a function with compact support. In this case,
$$\pi_*(\pi^*\omega \cdot f(x, t)) = 0;$$
(2) $\pi^*\omega \cdot f(x, t)dt$. In this case,
$$\pi_*(\pi^*\omega \cdot f(x, t)dt) = \omega \int_{-\infty}^{\infty} f(x, t)dt.$$

It is an easy exercise to show that π_* is a cochain map, hence it induces a map on compactly supported de Rham cohomology $\pi_* : H^*_{dR,c}(M \times \mathbb{R}) \to H^{*-1}_{dR,c}(M)$. Now let $e = e(t)dt$ be a compactly supported 1-form on $\mathbb{R}$ integrating to 1. Define $e_* : \Omega^*_c(M) \to \Omega^{*+1}_c(M\times\mathbb{R})$ by sending ω to $\pi^*(\omega)\wedge e$. One can check e_* is a well-defined cochain map (Exercise). So e_* induces a map $e_* : H^*_{dR,c}(M) \to H^{*+1}_{dR,c}(M \times \mathbb{R})$. We have

Lemma 2.7. *e_* and π_* induce a pair of mutual inverses:*

$$H^*_{dR,c}(M \times \mathbb{R}) \underset{e_*}{\overset{\pi_*}{\rightleftharpoons}} H^{*-1}_{dR,c}(M).$$

Corollary 2.8 (Poincaré lemma for compactly supports)**.** *Let $U \subset \mathbb{R}^n$ be a smoothly contractible subspace. Then $H^k_{dR,c}(U) = 0$ for all $0 \le k < n$, and $H^n_{dR,c}(U) \cong \mathbb{R}$. Here the last isomorphism in n-th compactly supported de Rham cohomology is given by applying π_* iteratively.*

Mayer-Vietoris sequences work for both de Rham complexes and compactly supported de Rham complexes. Namely,

Theorem 2.9 (Mayer-Vietoris)**.** *Let $M = U\cup V$ with U, V open. Then the following sequence is exact:*

$$\begin{array}{ccccccc} 0 \to \Omega^*(M) \to & \Omega^*(U) \oplus \Omega^*(V) & \to & \Omega^*(U \cap V) \to 0 \\ & (\omega, \tau) & \mapsto & \tau - \omega & . \end{array}$$

In the compactly supported case,

$$\begin{array}{ccccccc} 0 \leftarrow \Omega^*_c(M) \leftarrow & \Omega^*_c(U) \oplus \Omega^*_c(V) & \leftarrow & \Omega^*_c(U \cap V) \leftarrow 0 \\ & (-\jmath_*\omega, \jmath_*\omega) & \leftarrow\!\shortmid & \omega & , \end{array}$$

where $\jmath_*$ is induced by the inclusion $\jmath$ of open subsets of M to M, extending a form on a open subset by zero to a form on M.

Like in usual cohomology theory, the following results hold for (compactly supported) de Rham cohomology.

Theorem 2.10 (Künneth formula)**.** *Let M, N be manifolds. Then for every $n \ge 0$,*

$$H^n_{dR}(M \times N) = \bigoplus_{p+q=n} H^p_{dR}(M) \otimes H^q_{dR}(N).$$

If further M, N admit finite good covers (i.e. open cover $\{U_j\}$ with all the U_j and all their non-empty finite intersections are contractible), then

$$H^n_{dR,c}(M \times N) = \bigoplus_{p+q=n} H^p_{dR,c}(M) \otimes H^q_{dR,c}(N).$$

Theorem 2.11 (Poincaré duality for orientable manifolds)**.** *If M is an orientable manifold of dimension n admitting a finite good cover, then for any integer $0 \le p \le n$,*

$$H^p_{dR}(M) \cong (H^{n-p}_{dR,c}(M))^*,$$

where the isomorphism is induced by the non-degenerate bilinear form

$$\int_M : H^p_{dR}(M) \otimes_{\mathbb{R}} H^{n-p}_{dR,c}(M) \to \mathbb{R}.$$

If the de Rham cohomology of M is finite-dimensional, then we also have

$$H^p_{dR,c}(M) \cong (H^{n-p}_{dR}(M))^*.$$

Theorem 2.12 (Poincaré duality for non-orientable manifolds)**.** *If M is a manifold of dimension n admitting a finite good cover, then for any integer $0 \le p \le n$, there are non-degenerate bilinear forms*

$$\int_M : H^p_{dR}(M) \otimes_{\mathbb{R}} H^{n-p}_{dR,c}(M, L) \to \mathbb{R},$$

and

$$\int_M : H^p_{dR,c}(M) \otimes_{\mathbb{R}} H^{n-p}_{dR}(M, L) \to \mathbb{R}.$$

where L is the line bundle over M.

Proof of the last three theorems in this section can be found in Chapter I of Bott & Tu's book *Differential Forms in Algebraic Topology*.

2.2. **Dolbeault Complexes.** Let J be an almost complex structure on the manifold M. Recall that J induces a bundle map $J : T^*M \to T^*M$ by $J\omega(V) = \omega(J(V))$, so we can decompose $T^*M \otimes \mathbb{C}$ into $(T^{1,0})^* \oplus (T^{0,1})^*$, where $(T^{1,0})^*$ is the eigenspace associated with i and $(T^{0,1})^*$ is the eigenspace associated with $-i$.

If J is integrable, then it makes sense to use the complex coordinates $(z_1, \cdots, z_n)$ in local coordinate charts with $z_j = x_j + iy_j$ in real coordinate $(x_1, \cdots, x_n, y_1, \cdots, y_n)$. Now $T^{1,0}$ is spanned by $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$, $T^{0,1}$ is spanned by $\partial_{\overline{z_j}} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$. Thus, $(T^{1,0})^*$ is spanned by $dz_j = \frac{1}{2}(d(x_j) - id(y_j))$, $(T^{0,1})^*$ is spanned by

$$d\overline{z_j} = \frac{1}{2}(d(x_j) - id(y_j))$$

. It is straightforward that

$$\begin{array}{cc} dz_j(\partial_{z_k}) = \delta_{jk}, & dz_j(\partial_{\overline{z_k}}) = 0, \\ d\overline{z_j}(\partial_{z_k}) = 0, & d\overline{z_j}(\partial_{\overline{z_k}}) = \delta_{jk}. \end{array}$$

We define

$$\bigwedge^{p,0} T^*M = \left(\bigwedge^{1,0} T^*M\right)^{\wedge p},$$

$$\bigwedge^{0,q} T^*M = \left(\bigwedge^{0,q} T^*M\right)^{\wedge q},$$

and

$$\bigwedge^{p,q} T^*M = \bigwedge^{p,0} T^*M \otimes \bigwedge^{0,q} T^*M.$$

We refer to the sections of $\bigwedge^{p,q} T^*M$ as (p,q)**-form**. In local holomorphic coordinate, a (p,q)-form ω can be written as

$$\omega = \sum_{\substack{|\boldsymbol{\alpha}|=p \\ |\boldsymbol{\beta}|=q}} f dz_{\boldsymbol{\alpha}} \wedge d\overline{z_{\boldsymbol{\beta}}},$$

where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_p)$ and $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_q)$, and $dz_{\boldsymbol{\alpha}} = dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_p}$ and $d\overline{z_{\boldsymbol{\beta}}} = d\overline{z_{\beta_1}} \wedge \cdots \wedge d\overline{z_{\beta_q}}$. Note that

$$d(f dz_{\boldsymbol{\alpha}} \wedge d\overline{z_{\boldsymbol{\beta}}}) = \sum_{j=1}^n \partial_{z_j}(f) dz_j \wedge dz_{\boldsymbol{\alpha}} \wedge d\overline{z_{\boldsymbol{\beta}}} + \sum_{j=1}^n \partial_{\overline{z_j}}(f) d\overline{z_j} \wedge dz_{\boldsymbol{\alpha}} \wedge d\overline{z_{\boldsymbol{\beta}}}.$$

This implies $d(\bigwedge^{p+q}) \subset \bigwedge^{p+1,q} \otimes \bigwedge^{p,q+1}$. Hence we can decompose $d = \partial + \overline{\partial}$, where

$$\begin{aligned}\partial &: \Omega^{p,q}M = \Omega^{p+1,q}M,\\ \overline{\partial} &: \Omega^{p,q}M = \Omega^{p,q+1}M.\end{aligned}$$

Here $\Omega^{p,q}(M) = \mathbb{C}^\infty(M, \bigwedge^{p,q} T^*M)$. It is obvious that $\partial\omega = \overline{\overline{\partial}\omega}$. (Check!) The operator $\partial, \overline{\partial}$ satisfies their own Leibniz's rules. Namely, for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$,

$$\begin{aligned}\partial(\omega \wedge \eta) &= \partial\omega \wedge \eta + (-1)^k \omega \wedge \partial\eta,\\ \overline{\partial}(\omega \wedge \eta) &= \overline{\partial}\omega \wedge \eta + (-1)^k \omega \wedge \overline{\partial}\eta.\end{aligned}$$

It is immediate that $\partial^2 = \overline{\partial}^2 = 0$. From $d^2 = 0$, we know $(\partial + \overline{\partial})^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2 = 0$. So $\partial\overline{\partial} = -\overline{\partial}\partial$.

In general, if J is not necessarily integrable, it does **NOT** make sense to give a basis for $(T^{0,1})^*$ and $(T^{1,0})^*$ via dz_j and $d\overline{z_j}$. In this case, d has four types of components instead of two:

$$d : \Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p,q+1} \oplus \Omega^{p+2,q-1} \oplus \Omega^{p-1,q+2}.$$

Let us give an example to demonstrate this phenomenon. Let $\omega \in \Omega^1(M)$. ω has $(1,0)$-type if for $\pi_{i,j}$ = projection of Ω^k onto $\Omega^{i,j}$ $(i + j = k)$, $\omega(V) = \omega(\pi_{1,0}V) = \pi_{1,0}\omega(V)$, and $\omega(\pi_{0,1}V) = \pi_{0,1}\omega(V) = 0$, where V is an arbitrary vector field. By definition, $d\omega(V_0, V_1) = \mathcal{L}_{V_0}(\omega(V_1)) - \mathcal{L}_{V_1}(\omega(V_0)) - \omega[V_0, V_1]$. Note that

$$\begin{aligned}\pi_{2,0}d\omega(V_0, V_1) &= d\omega(\pi_{1,0}V_0, \pi_{1,0}V_1),\\ \pi_{1,1}d\omega(V_0, V_1) &= d\omega(\pi_{1,0}V_0, \pi_{0,1}V_1) + d\omega(\pi_{0,1}V_0, \pi_{1,0}V_1),\\ \pi_{0,2}d\omega(V_0, V_1) &= d\omega(\pi_{0,1}V_0, \pi_{0,1}V_1).\end{aligned}$$

None of them can be guaranteed to vanish. If J is integrable, then by Newlander-Nirenberg theorem,

$$\pi_{0,2}d\omega(V_0, V_1) = -\omega[\pi_{0,1}V_0, \pi_{0,1}V_1] = -\omega(\pi_{0,1}W) = 0$$

for some $W \in T^{0,1}$. On the other hand, if $\pi_{0,2}d\omega = 0$ for a $(1,0)$-form ω, then

$$\pi_{1,0}\omega[\pi_{0,1}V_0, \pi_{0,1}V_1] = 0$$

for all V_0, V_1. This implies that $[T^{0,1}, T^{0,1}] \subset T^{0,1}$, i.e. $T^{0,1}$ is involutive. Packaging the information we obtain the following result.

Theorem 2.13. *Let (M, J) be an almost complex manifold. TFAE:*

(1) J is integrable;
(2) $T^{0,1}$ is involutive;
(3) $d : \Omega^{1,0} \to \Omega^{2,0} \oplus \Omega^{1,1}$;
(4) $d : \Omega^{0,1} \to \Omega^{0,2} \oplus \Omega^{1,1}$;
(5) $d : \Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p,q+1}$.

We now assume J is integrable in the following paragraphs.

Definition 2.14. Fix p. Since $\overline{\partial}^2 = 0$, the $(p, *)$-forms on a manifold M of dimension n constitute a cochain complex:

$$0 \to \Omega^{p,0}(M) \xrightarrow{\overline{\partial}} \Omega^{p,1}(M) \xrightarrow{\overline{\partial}} \Omega^{p,2}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,n-p}(M) \to 0,$$

called the **Dolbeault complex** of M. The (p, q)**-th Dolbeault cohomology group** of M is then defined to be

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\ker \overline{\partial}}{\operatorname{im} \overline{\partial}}.$$

The complex dimension of $H^{p,q}_{\overline{\partial}}(M)$, denoted $h^{p,q}(M)$, is called the **Hodge numbers** of M.

Similarly, we can give the definition of the conjugate Dolbeault cohomology by

$$H^{p,q}_{\partial}(M) = \frac{\ker \partial}{\operatorname{im} \partial}.$$

From the definition, $H^{p,0}_{\overline{\partial}}(M)$ is the holomorphic sections of $\bigwedge^p (T^*M)^{1,0}$. If M is closed, then $H^{p,0}_{\overline{\partial}}(M)$ is clearly finite dimensional. This holds in general case:

Lemma 2.15. *If M is a closed complex manifold, then $H^{p,q}_{\overline{\partial}}(M)$ is a finite dimensional vector space.*

Theorem 2.16 ($\overline{\partial}$-Poincaré lemma)**.** *For any $\omega \in \Omega^{p,q}(D)$, where $q > 0$ and $D \subset \mathbb{C}^n$ is a polydisc (possibly unbounded), ω is both $\overline{\partial}$-exact and $\overline{\partial}$-closed.*

Given a Dolbeault cohomology, one would wonder its relationship with de Rham cohomology. Unfortunately, there are **NO** natural maps between Dolbeault cohomology groups and de Rham cohomology groups on general complex manifolds. However, we can construct ones through other objects.

Definition 2.17. Note that $\partial\overline{\partial}(\overline{\partial} + \partial) = \partial\overline{\partial}\partial = -\partial\overline{\partial}^2 = 0$, we may get a cochain complex with differentials $\partial\overline{\partial}$ and $\partial + \overline{\partial}$. The (p, q)**-th Bott-Chern cohomology group** is defined to be

$$H^{p,q}_{BC}(M) = \frac{\ker(\partial + \overline{\partial} : \Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p,q+1})}{\operatorname{im}(\partial\overline{\partial} : \Omega^{p-1,q-1} \to \Omega^{p,q})}.$$

Definition 2.18. One can check $(\partial+\overline{\partial})\partial\overline{\partial} = 0$. Similar to the preceding definition, we can define the (p, q)-th **Aeppli cohomology group** to be

$$H^{p,q}_{A}(M) = \frac{\ker(\partial\overline{\partial})}{\operatorname{im}(\partial + \overline{\partial})}.$$

There are natural maps

$$\begin{tikzcd}
 & H^{\bullet,\bullet}_{BC}(M) \arrow[dl] \arrow[d, "\vartheta"] \arrow[dr] & \\
H^{\bullet,\bullet}_{\partial}(M) \arrow[r, Rightarrow] \arrow[dr] & H^{\bullet}_{dR}(M) \arrow[d] & H^{\bullet,\bullet}_{\overline{\partial}}(M) \arrow[l, Rightarrow] \arrow[dl] \\
 & H^{\bullet,\bullet}_{A}(M) &
\end{tikzcd} \tag{2.19}$$

We give a brief illustration on this diagram and encourage the readers to [5] for a detailed discussion. The middle horizontal double arrows are given by the Frölicher spectral sequence that will be introduced in the next section. The maps from Bott-Chern cohomology to (conjugate) Dolbeault cohomology, and then to Aeppli cohomology are induced by the definitions. Explicitly, for example, since a d-closed form is again $\overline{\partial}$-closed, there is a natural map

$$\{\alpha \in \Omega^{p,q} : d\alpha = 0\} \to H^{p,q}_{\overline{\partial}}.$$

It follows from $\overline{\partial}\partial\overline{\partial} = 0$ that we have a canonical map

$$H^{p,q}_{BC} \to H^{p,q}_{\overline{\partial}}.$$

Finally, we have the following theorem about ϑ in the diagram:

Theorem 2.20. *Let the underlying complex manifold be M.*

1. *If ϑ is injective, then all maps are isomorphism.*
2. *If M satisfies $\partial\overline{\partial}$-lemma, i.e. $(\ker\partial \cap \ker\overline{\partial} \cap \operatorname{im} d) \subset \operatorname{im}\partial\overline{\partial}$, then ϑ is injective.*

There are various criteria regarding the $\partial\overline{\partial}$-lemma. The most commonly used one is that if M is Kähler, then M satisfies $\partial\overline{\partial}$-lemma (see Lemma 4.37). In [5], the author gave another characterization: for any $k \geq 0$ and $\dim_{\mathbb{C}} M = n$,

$$\Delta^k := \sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(M) + \dim_{\mathbb{C}} H^{n-q,n-p}_{BC}(M)\right) - 2b_k \geq 0,$$

and the equality holds iff M satisfies $\partial\overline{\partial}$-lemma. Focusing on both Bott-Chern cohomology and Aeppli cohomology, M satisfies $\partial\overline{\partial}$-lemma iff

$$\sum_k \left| \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{BC}(M) - \dim_{\mathbb{C}} H^{p,q}_{A}(M) \right| = 0.$$

2.3. ***Off-topic: An Invitation to the Frölicher Spectral Sequence.** Spectral sequences are very powerful tools in algebraic topology. They can be used to compute various (co)homology and determine some multiplicative structures on them. Spectral sequences are the generalization of long exact sequences which associate to chain complexes with filtrations.

Definition 2.21. A **filtrated R-module** A is an R-module with an increasing sequence of submodules $F_pA \subset F_{p+1}A$ indexed by $p \in \mathbb{Z}$, such that $\bigcup_p F_pA = A$ and $\bigcap_p F_pA = \{0\}$. The filtration is bounded if $F_pA = \{0\}$ for sufficiently small p and $F_{p'}A = A$ for sufficiently large p'. The **associated graded module of** $\{F_pA\}$ is defined by $G_pA = F_pA/F_{p-1}A$.

Similarly, we can define the **cofiltration** on an R-module A to be a decreasing sequence of submodules $F^{p+1}A \subset F^pA$ indexed by $p \in \mathbb{Z}$, with the conditions in analog to the above definition in the way as you would expect. In fact, we can package the information to define a functor F with domain being a poset and codomain being $\mathrm{Ch}(\mathcal{A})$ for $\mathcal{A}$ an abelian category. But this part is of less interest to this topic. Given the cofiltrated R-module A, we can also define the **associated graded module of** $\{F^pA\}$ by $G^pA = F^pA/F^{p+1}A$.

Definition 2.22. A **cofiltrated cochain complex** is a cochain complex $(C^\bullet, d^\bullet)$ together with a cofiltration $\{F^pC^n\}$ of each C^n, such that $F^0(C^\bullet) = C^\bullet$ and $F^{n+1}(C^n) = 0$ for all n, and the differential preserves the cofiltration, namely $d(F^p(C^n)) \subset F^p(C^{n+1})$, where $F^p(C^n) = F^p(C^\bullet) \cap C^n$. This implies that we have an **associated graded cochain complex** $\{G^pC^\bullet\}$.

Let $C^\bullet$ be a cochain complex and $F^\bullet$ be a decreasing filtration on $C^\bullet$ preserved by d. $F^\bullet$ induces a decreasing filtration on cohomology, defined by

$$F^pH^k(C^\bullet) = \{\alpha \in H^k(C^\bullet) : \alpha = [x] \text{ for some } x \in F^pC^k\}.$$

We will use the notion F^pH^k if the underlying cochain complex is clear. If we focus on the cocycles and coboundaries, we can set $Z^k = (\ker d^k) \cap C^k$ and $B^k = (\operatorname{im} d^{k-1}) \cap C^k$, and define

$$\begin{aligned} F^pZ^k &= F^p(C^\bullet) \cap Z^k = F^p(C^k) \cap Z^k, \\ F^pB^k &= F^p(C^\bullet) \cap B^k = F^p(C^k) \cap Z^k. \end{aligned}$$

Note that $F^{p+1}Z^k = F^{p+1}(C^k) \cap Z^k \subset F^p(C^k) \cap Z^k = F^pZ^k$, and so $F^\bullet$ induces a decreasing filtration on $Z^\bullet$, and similarly on $B^\bullet$. Note that $H^k = Z^k/B^k$, we set

$$F^pH^k = \frac{(\ker d^k) \cap F^pC^k}{(\operatorname{im} d^{k-1}) \cap F^pC^k} = \frac{F^pZ^k}{F^pB^k}.$$

It is straightforward to check both definitions for $F^pH^k(C^\bullet)$ are equivalent, with $F^{p+1}H^k \subset F^pH^k$, $F^0H^k = H^k$, and $F^{k+1}H^k = 0$. Note that $F^{p+1}B^k = F^{p+1}Z^k \cap F^pB^k$, we get

$$F^{p+1}H^k = \frac{F^pZ^k}{F^pB^k} = \frac{F^pZ^k}{F^{p+1}Z^k \cap F^pB^k} \cong \frac{F^{p+1}Z^k + F^pB^k}{F^pB^k}$$

by the second isomorphism theorem. The associated graded pieces G^pH^k is then

$$G^pH^k := \frac{F^pH^k}{F^{p+1}H^k} \cong \frac{F^pZ^k}{F^{p+1}Z^k + F^pB^k}. \tag{2.23}$$

The associated graded homology is closely related to the homology by definition. A natural question is to analyze what the associated graded homology looks like for an arbitrary cochain complex. It is usually not easy to know exactly how the right-hand side of equation 2.23 behaves, so we would like to form a sequence of approximations to the associated graded homology from the associated graded cochain complex itself. The idea is that, for each F^pC^k, we take those cochains whose coboundary lives in some higher filtration level F^{p+r}, modulo forms in the next filtration level F^{p+1} with the same property (i.e. coboundary in F^{p+r}), and module coboundaries in F^p of elements in a lower filtration, F^{p-r+1}. If r is large enough, then F^{p-r+1} gives C^* and F^{p+r} gives $\{0\}$. In this case,

$$\frac{\{x \in F^pC^k \mid dx \in F^{p+r}C^{k+1}\}}{\{y \in F^{p+1}C^k \mid dy \in F^{p+r}C^{k+1}\} + d(F^{p-r+1}C^{k-1}) \cap F^pC^k} \cong \frac{F^pZ^k}{F^{p+1}Z^k + F^pB^k}.$$

When $r = 0$, the left-hand side is just $F^pC^k/F^{p+1}C^k$. Now move from one approximating space to another, we define (write $k = p + q$)

$$E_r^{p,q} := \frac{\{x \in F^pC^{p+q} \mid dx \in F^{p+r}C^{p+q+1}\}}{\{y \in F^{p+1}C^{p+q} \mid dy \in F^{p+r+1}C^{p+q+1}\} + d(F^{p-r+1}C^{p+q-1}) \cap F^pC^{p+q}}.$$

It is clear that $E_0^{p,q} = F^pC^{p+q}/F^{p+1}C^{p+q}$. When r is large enough, then we denote

$$E_\infty^{p,q} = \lim_{r\to\infty} E_r^{p,q} := \frac{\{x \in F^pC^{p+q} \mid dx = 0\}}{\{y \in F^{p+1}C^{p+q} \mid dy = 0\} + d(C^{p+q-1}) \cap F^pC^{p+q}} \cong \frac{F^pH^{p+q}}{F^{p+1}H^{p+q}}. \tag{2.24}$$

At $r = 0$, since the differential d is compatible with the filtration, it induces a map d_0 by

$$E_0^{p,q} = \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} \xrightarrow{d_0} \frac{F^p C^{p+q+1}}{F^{p+1} C^{p+q+1}} = E_0^{p,q+1}.$$

Exercise 2.25. Check that d induces a map d_1 by

$$E_1^{p,q} \xrightarrow{d_1} E_1^{p+1,q}.$$

Hint: write down the definition of E_1-page.

Consider

$$\frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} \xrightarrow{d_0} \frac{F^p C^{p+q+1}}{F^{p+1} C^{p+q+1}} \xrightarrow{d_0} \frac{F^p C^{p+q+2}}{F^{p+1} C^{p+q+2}}.$$

$d_0^2 = 0$ since $d^2 = 0$. Taking the cohomology at the middle term gives

$$\begin{aligned} H^{p,q+1}(E_0^{*,*}, d_0) = \frac{\ker d_0}{\operatorname{im} d_0} &= \frac{\frac{\{x \in F^p C^{p+q+1} \mid dx \in F^{p+1} C^{p+q+2}\}}{F^{p+1} C^{p+q+1}}}{\frac{d(F^{p+1} C^{p+q}) + F^{p+1} C^{p+q+1}}{F^{p+1} C^{p+q+1}}} \\ &= \frac{\{x \in F^p C^{p+q+1} \mid dx \in F^{p+1} C^{p+q+2}\}}{d(F^{p+1} C^{p+q}) + F^{p+1} C^{p+q+1}} \\ &= E_1^{p,q+1}. \end{aligned}$$

In fact, this formula holds for each $r \geq 0$. That is, E_{r+1}-page will be the cohomology of (E_r, d_r)-page. We have the following theorem:

Theorem 2.26. *For each $r \geq 0$, the cochain complex $(C^\bullet, d)$ gives rise to $E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}$, and the cohomology of $(E_r^{*,*}, d_r)$ is isomorphic to $E_{r+1}^{*,*}$, where p, q are natural integers.*

For a detailed proof, the readers are referred to [6]. To summarize, we make the following definition.

Definition 2.27. A **(cohomological) spectral sequence** consists of

- An R-module $E_r^{p,q}$ for each natural numbers p, q and each integer $r \geq 0$.
- Differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^2 = 0$ and E_{r+1} is the cohomology of (E_r, d_r).

A spectral sequence **converges** if for every p, q, if r is large enough, then d_r vanishes on $E_r^{p,q}$. In this case, $E_r^{p,q}$ is independent of r and it is actually $E_\infty^{p,q}$ (Check!).

Now we focus on $\Omega^k = \bigoplus_{p+q=k} \Omega^{p,q}$ with underlying manifold M of dimension n. We assume further that M is Kähler (so $d = \partial + \overline{\partial}$, see next section). Define the filtration on Ω^k by

$$F^p \Omega^k = \bigoplus_{\substack{i \geq p \\ i+j=k}} \Omega^{i,j}.$$

Obviously, $F^0 \Omega^k = \Omega^k$ and $F^{n+1} \Omega^k = \{0\}$ for all k. We use the construction in the preceding paragraphs to get

$$E_0^{p,q} = \frac{F^p \Omega^{p+q}}{F^{p+1} \Omega^{p+q}} = \frac{\bigoplus_{\substack{i \geq p \\ i+j=p+q}} \Omega^{i,j}}{\bigoplus_{\substack{i \geq p+1 \\ i+j=p+q}} \Omega^{i,j}} \cong \Omega^{p,q}.$$

One can check that $d_0 = \overline{\partial}$ (Exercise. *Hint: write down the element and use* $d = \partial + \overline{\partial}$*)*. Moreover, $d_r = 0$ for all $r \geq 1$. By Theorem 2.26,

$$E_1^{p,q} = H_{\overline{\partial}}^{p,q}(M).$$

Turn to the E_∞-page. By equation 2.24,

$$E_\infty^{p,q} = \frac{F^p H^k}{F^{p+1} H^k}.$$

Summing up all possible p,

$$\bigoplus_{p+q=k} E_\infty^{p,q} = \bigoplus_{0 \leq p \leq k} \frac{F^p H^k}{F^{p+1} H^k} = H^k(M),$$

which is the de Rham cohomology of M. This is part of the Frölicher spectral sequence. Explicitly,

Theorem 2.28. *Let M be a compact Kähler manifold. Then the Frölicher spectral sequence $(E_r^{*,*}, d_r)$ collapses at E_1-page (i.e. $d_r = 0$ for all $r \geq 1$, and so $E_\infty = E_1$). Furthermore, there is a isomorphism $H^k(M) = \bigoplus_{p+q=k} H_{\overline{\partial}}^{p,q}(M)$.*

So in the diagram 2.19, the horizontal double arrows are exactly the Frölicher spectral sequences given above.

Corollary 2.29. *Let M be a compact complex manifold. Then the Hodge number $h^{p,q}$ and the Betti number b_k satisfy*

$$\sum_{p+q=k} h^{p,q} \geq b_k.$$

In particular, if M is Kähler, then the equality is achieved.

Proof. Note that $\dim E_r^{p,q} \geq \dim E_{r+1}^{p,q}$, since E_{r+1} is the cohomology of (E_r, d_r). By Theorem 2.28, $E_1^{p,q} \cong H_{\overline{\partial}}^{p,q}$, and

$$H^k(M) = \bigoplus_{p+q=k} H_{\overline{\partial}}^{p,q}(M).$$

This implies

$$\sum_{p+q=k} \dim E_1^{p,q} \geq \sum_{p+q=k} \dim E_\infty^{p,q} = b_k.$$

$\square$

Corollary 2.30. *Let M be a compact complex manifold. Then $\sum_{p,q}(-1)^{p+q} h^{p,q} = \chi(M)$.*

3. Kähler Manifolds

Let (M, J) be an almost complex manifold. $J^2 = -\mathrm{id}$. Recall that

Definition 3.1. A **Riemannian metric** g on a smooth manifold M is a smoothly varying family of inner products on the fibers of the tangent bundle. Explicitly, for each $\xi \in M$, a map $g_\xi : T_\xi M \times T_\xi M \to \mathbb{R}$ satisfying

(1) $g(u, v)$ is $\mathbb{R}$-linear in u for all v.
(2) $g(u, v) = g(v, u)$ for any $u, v \in T_\xi M$.
(3) $g(u, u) \geq 0$ and $g(u, u) = 0$ iff $u = 0$.
(4) If $s_1, s_2 \in C^\infty(M, TM)$, then $g(s_1, s_2) \in C^\infty(M)$.

g is a (real) bundle metric. We introduce the Hermitian metric on bundles for future reference.

Definition 3.2. Let $E \to M$ be a complex vector bundle over a smooth manifold M. A **Hermitian metric** h^E on E is a smooth family of Hermitian inner products on the fibers of E. That is, for each $\xi \in M$, $h^E_\xi : E_\xi \times E_\xi \to \mathbb{C}$ satisfies

(1) $h^E(u, v)$ is $\mathbb{C}$-linear in u for each $v \in E_\xi$.
(2) $h^E(u, v) = \overline{h^E(v, u)}$.
(3) $h^E(u, u) \geq 0$ and $h^E(u, u) = 0$ iff $u = 0$.
(4) If $s_1, s_2 \in C^\infty(M, E)$, then $h^E(s_1, s_2) \in C^\infty(M)$.

Actually, h^E is equivalent to a $\mathbb{C}$-anti-linear bundle isomorphism $h^\flat : E \to E^*$ with

$$h^\flat(u)(v) = h^E(v, u).$$

We say g **is compatible with** J if $g(u, v) = g(Ju, Jv)$. Let (M, ω) be a symplectic manifold. Say ω **is compatible with** J if $\omega(u, v) = \omega(Ju, Jv)$.

Theorem 3.3. *Let (g, J, ω) be a compatible triple. Then any two determines the third.*

We refer the proof to the notes by Dekun. In particular, given (J, ω), we can define the Riemannian metric $g(u, v) = \omega(u, Jv)$. Given (g, J), we can define $\omega(u, v) = g(Ju, v)$, which is a non-degenerate 2-form (**NOT** necessarily closed).

Definition 3.4. A **Kähler manifold** (M, g, J, ω) is a complex manifold with a compatible triple (g, J, ω), where J is integrable, such that $d\omega = 0$. This ω is sometimes called the **Kähler form**.

One can get a Kähler structure from a Hermitian metric. Let (M, J) be an almost complex manifold and $h = h^{TM}$ be a Hermitian metric on TM (as a $\mathbb{C}$-vector bundle). Separating h into real and imaginary parts gives $h(u, v) = g(u, v) + i\omega(u, v)$, then one can check that g is a Riemannian metric on M and ω is a 2-form, i.e. $\omega \in \Omega^2(M)$. Since

$$h(J(u), J(v)) = i \cdot (-i) \cdot h(u, v) = h(u, v),$$

we have

$$\begin{aligned} g(J(u), J(v)) &= g(u, v), \\ \omega(J(u), J(v)) &= \omega(u, v). \end{aligned}$$

Similarly, $h(J(u), v) = ih(u, v)$ implies $g(J(u), v) = \omega(u, v)$ and $\omega(J(u), v) = -g(u, v)$. We sometimes refer to the compatible triple (g, J, ω) as a **Hermitian structure**. In local holomorphic coordinates $\{z_j\}$, let $H \in \mathrm{GL}_n(\mathbb{C})$ be the matrix with entries $h_{jk} = h(\partial_{z_j}, \partial_{z_k})$, then $H = H^*$ and H is thus positive definite. Recall that there is a natural $\mathbb{C}$-vector bundle isomorphism

$$\begin{aligned} (TM, J) &\xrightarrow{\phi} T^{1,0}M \\ v &\mapsto \frac{1}{2}(v - iJ(v)). \end{aligned}$$

To find the Riemannian metric, write $z_j = x_j + iy_j$. Note that $\phi(\partial_{x_j}) = \partial_{z_j}$ and $\phi(\partial_{y_j}) = \phi(J(\partial_{x_j})) = i\partial_{z_j}$. Thus we have, for instance,

$$\begin{aligned} g(\partial_{x_j}, \partial_{x_k}) &= \Re h(\partial_{z_j}, \partial_{z_k}) = \Re h_{jk}, \\ g(\partial_{x_j}, \partial_{y_k}) &= \Re h(\partial_{z_j}, i\partial_{z_k}) = \Re(-ih(\partial_{z_j}, \partial_{z_k})) = \Im h_{jk}. \end{aligned}$$

So in the basis $\partial_{x_1}, \cdots, \partial_{x_n}, \partial_{y_1}, \cdots, \partial_{y_n}$, g is the $2n \times 2n$-matrix

$$G = \begin{bmatrix} \Re H & \Im H \\ -\Im H & \Re H \end{bmatrix}.$$

Next consider the 2-form ω. It is not hard to find

$$\begin{aligned} \omega(\partial_{x_j}, \partial_{x_k}) &= -\Im h(\partial_{z_j}, \partial_{z_k}) = -\Im h_{jk}, \\ \omega(\partial_{x_j}, \partial_{y_k}) &= -\Im h(\partial_{z_j}, i\partial_{z_k}) = \Re h_{jk}, \\ \omega(\partial_{y_j}, \partial_{y_k}) &= -\Im h(i\partial_{z_j}, i\partial_{z_k}) = -\Im h_{jk}. \end{aligned}$$

Extend ω bilinearly to the complexified tangent spaces $TM \otimes \mathbb{C}$. We want to express ω in terms of dz_j and $d\overline{z_k}$. Note that

$$\begin{aligned} \omega(\partial_{z_j}, \partial_{\overline{z_k}}) &= \omega(\partial_{x_j} - i\partial_{y_j}, \partial_{x_k} + i\partial_{y_k}) \\ &= \omega(\partial_{x_j}, \partial_{x_k}) - i\omega(\partial_{y_j}, \partial_{x_k}) + i\omega(\partial_{x_j}, \partial_{y_k}) + \omega(\partial_{y_j}, \partial_{y_k}) \\ &= -\Im h_{jk} + i\Re h_{jk} + i\Re h_{jk} - \Im h_{jk} \\ &= 2ih_{jk}. \end{aligned}$$

Similar computations show that $\omega(\partial_{z_j}, \partial_{z_k}) = 0$, $\omega(\partial_{\overline{z_j}}, \partial_{\overline{z_k}}) = 0$. This yields

$$\omega = \frac{i}{2} \sum h_{jk} dz_j \wedge d\overline{z_k}.$$

In particular, ω is of type $(1,1)$.

Example 3.5. Equip $\mathbb{C}$ with the standard metric such that $\partial_{z_1}, \cdots \partial_{z_n}$ is a unitary basis. Then $H = \mathrm{id}_n$, $G = \left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right)$ is the standard metric on $\mathbb{R}^{2n}$, and

$$\omega = \frac{i}{2} \sum dz_j \wedge d\overline{z_j} = \sum dx_j \wedge dy_j$$

is the standard symplectic form on $\mathbb{R}^{2n}$. Note $d\omega = 0$, so it is Kähler.

Example 3.6. $\mathbb{CP}^n$ admits a $U(n+1)$-invariant Kähler structure. Let $z_1, \cdots, z_{n+1}$ be the standard coordinates on $\mathbb{C}^{n+1}$ and $\rho = \|z\|^2 = \sum_j z_j^2$. Set

$$\begin{aligned} \tilde{\omega} &= \frac{i}{2\pi} \partial\overline{\partial} \log \rho = \frac{i}{2\pi} \left[\frac{\partial\overline{\partial}\rho}{\rho} - \frac{\partial\rho \wedge \overline{\partial}\rho}{\rho^2} \right] \\ &= \frac{i}{2\pi} \left[\frac{\|z\|^2 \sum dz_j \wedge d\overline{z_j} - (\sum \overline{z_j} dz_j) \wedge (\sum z_j d\overline{z_j})}{\|z\|^4} \right]. \end{aligned}$$

It is $U(n+1)$-invariant since it only depends on ρ. It is also $\mathbb{C}^\times$-invariant since the numerator and denominator are homogeneous of degree 4. Hence $\tilde{\omega}$ pushes forward to a 2-form ω on $\mathbb{CP}^n$. To see that the resulting 2-form ω is positive definite (i.e. $\omega(J(\cdot), \cdot) > 0$), we evaluate it at the point $(1 : 0 : 0 : \cdots : 0)$. This is clearly positive. Appealing to $U(n+1)$-invariance to see that ω is positive definite at all points. This is known as the **Fubini-Study form**.

Proposition 3.7. *(M, g, J, ω) is a Kähler manifold iff for each $\xi \in M$, there are local holomorphic coordinates $z_1, \cdots, z_n$ centered at ξ such that the Hermitian metric satisfies*

$$h = \mathrm{id}_n + O\left(\sum |z_i|^2\right).$$

Theorem 3.8. *(M, g, J, ω) is Kähler manifold iff for each $\xi \in M$, there exists a neighborhood of ξ and $f : U \to \mathbb{R}$ smooth, such that*

$$\omega = i\partial\overline{\partial} f \qquad \text{on } U.$$

*Here f is called a **local Kähler potential**.*

Proof. ($\Leftarrow$) is trivial by $d\omega = (\partial + \overline{\partial})(i\partial\overline{\partial} f) = 0$. For ($\Rightarrow$), let U be a coordinate chart identified with a polydisc. By Poincaré's Lemma, we know that $d\omega = 0$, and $\omega = d\eta$ for some η on U. Extend ω and η to $\mathbb{C}$-vector fields and let η be real, i.e. $\eta^{1,0} = \overline{\eta^{0,1}}$. ω is of type $(1,1)$, yielding

$$d\eta = \overline{\partial}\eta^{1,0} + \partial\eta^{0,1}.$$

So $\partial\eta^{1,0} = 0 = \overline{\partial}\eta^{0,1}$. Applying the $\overline{\partial}$-Poincaré's Lemma, we know that there exists φ on U with $\partial\varphi = \eta^{1,0}$ and $\overline{\partial}\overline{\varphi} = \eta^{0,1}$. Let $f = 2\Im\varphi = i(\varphi - \overline{\varphi})$, then

$$i\partial\overline{\partial} f = -\partial\overline{\partial}\varphi + \partial\overline{\partial}\overline{\varphi} = \overline{\partial}\eta^{1,0} + \partial\eta^{0,1} = d\eta = \omega.$$

□

4. Hodge Theory

4.1. Elliptic Operators.

Let M be a manifold.

Definition 4.1. A **linear differential operator of order** k is a $\mathbb{F}$-linear map $L : C^\infty(M, \mathbb{F}) \to C^\infty(M, \mathbb{F})$ that for any choice of local coordinates at $\xi \in M$, it takes the form

$$Lf(\xi) = \sum_{|\alpha| \le k} a_\alpha(\xi) D^\alpha f = \sum_{\alpha_1 + \alpha_2 + \cdots + \alpha_n \le k} a_{\alpha_1, \cdots, \alpha_n}(\xi) \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n} f.$$

So L is a polynomial in vector fields.

Remark 4.2. In another approach by Grothendieck, a linear differential operator of order k is defined inductively with respect to k. Namely, When $k = 0$, $\mathrm{Diff}^k(M)$ is just the multiplication by a smooth function. When $k > 0$, $L \in \mathrm{Diff}^k(M)$ iff $[L, f] \in \mathrm{Diff}^{k-1}(M)$ for any $f \in C^\infty(M)$.

Let $E \to M$, $F \to M$ be vector bundles over M. We can define a linear differential operator of order k in $\mathrm{Diff}^k(M; E, F)$ to be $L : C^\infty(M, E) \to C^\infty(M, F)$, which has the same form as above, with $a_\alpha(\xi) \in \hom(E_\xi, F_\xi)$. The explicit expression for L in local coordinates depends strongly on the choice of coordinates, but the highest order part can be defined invariantly. This part is called the principal symbol of L.

Definition 4.3. Let $L : C^\infty(M, E) \to C^\infty(M, F)$ and $\xi \in M$. Its **principle symbol** $\sigma_k(L)$ at ξ is

$$\sigma_k(L) = \sum_{|\alpha| = k} a_\alpha(\xi)(i\xi)^\alpha,$$

obtained from the highest order derivatives by replacing $\partial_{x_j}^{\alpha_j}$ with $i\xi_j^{\alpha_j}$.

The principle symbol of $L \in \mathrm{Diff}^k(M; E, F)$ can also be related to cotangent bundles as follows. For any $v \in T_\xi^*(M)$, choose $f \in C^\infty(M)$ with $df(\xi) = v$. Claim that

$$\sigma_k(L)(v) = \lim_{t \to \infty} \frac{e^{-it} L(e^{itf})}{t^k}.$$

Indeed,

$$\partial_{x_j} e^{itf} = (it\partial_{x_j} f)e^{itf},$$
$$\partial_{x_j}^{\alpha_j} e^{itf} = (it\partial_{x_j} f)^{\alpha_j} e^{itf} + \psi(t)e^{itf},$$

where $\psi(t)$ is the lower order terms in t. This implies

$$e^{-it}L(e^{itf}) = t^k\sigma_k(L) + \psi(t).$$

If $f \in C^\infty(M)$, then for any $v \in T^*_\xi(M)$,

$$\sigma_k(fL)(v) = f(\xi)\sigma_k(L)(v).$$

This implies that $\sigma_k(L) \in C^\infty(T^*M, \pi^* \hom(E, F))$ for $\pi : T^*M \to M$.

Example 4.4. If $k = 1$, then $\sigma_1(L)(v) = i[L, f](\xi)$ for any smooth function f such that $df(\xi) = v$. If $k = 2$, then similarly

$$\sigma_2(L)(v) = -\frac{1}{2}\left[[L, f], f\right](\xi).$$

Definition 4.5. An operator L is called **elliptic at** $\xi \in M$ if $\sigma_k(L)$ is nowhere vanishing for any $v \in T^*_\xi(M)\backslash\{0\}$. L is **elliptic** if it is elliptic at all $\xi \in M$.

Example 4.6. Let $\Delta = -\sum \partial^2_{x_j}$, the negative of the Laplacian in $\mathbb{R}^n$. The principle symbol of it is simply

$$\sigma_2(\Delta)(\xi) = -\sum(i\xi_j)^2 = \sum(\xi_j)^2 = |\xi|^2.$$

So Δ is an elliptic operator.

4.2. **Formal Adjunctions.** We assume that (M, g) is a Riemannian manifold in this section. There is an L^2-pairing on $C^\infty(M)$:

$$C_c^\infty(M) \times C_c^\infty(M) \xrightarrow{(\cdot,\cdot)_M} \mathbb{F}$$
$$(f_1, f_2) \mapsto \int_M f_1 \cdot \overline{f_2} dV_g,$$

and the norm

$$\|f\|^2_{L^2} = (f, f)_{L^2(M)} = \int_M |f|^2 dV_g,$$

where V_g is the volume form. Define $L^2(M)$ to be the completion of $C_c^\infty(M)$ with respect to $\|\cdot\|_{L^2}$. If E is an $\mathbb{F}$-vector bundle over M equipped with an $\mathbb{F}$-bundle metric h^E, then there is an L^2-pairing on $C_c^\infty(M, E)$:

$$C_c^\infty(M, E) \times C_c^\infty(M, E) \xrightarrow{(\cdot,\cdot)_E} \mathbb{F}$$
$$(s_1, s_2) \mapsto \int_M h^E(s_1, s_2) dV_g,$$

which yields the completion $L^2(M, E)$.

Definition 4.7. Let $L \in \mathrm{Diff}^k(M; E, F)$, and $E \to M$, $F \to M$ be vector bundles over M with bundle metric $h^E = (-,-)_E, h^F = (-,-)_F$, respectively. The **formal adjoint** of L is the operator $L^* \in \mathrm{Diff}^k(M; F, E)$ such that

$$(Ls, \tilde{s})_F = (s, L^*\tilde{s})_E,$$

where $s \in C_c^\infty(M^\circ, E)$, $s \in C_c^\infty(M^\circ, F)$, and M° is the interior of M (in case M is not closed).

Proposition 4.8. *The formal adjoint of an elliptic operator is again elliptic.*

Proof. Let $L \in \mathrm{Diff}^k(M; E, F)$ be an elliptic operator. The principle symbol of L^* satisfies

$$(\sigma_k(L)(x)u, v)_F = (u, \sigma_k(L^*)(x)v)_E$$

for all $x \in T^*_\xi M$, $u \in E_\xi$, $v \in F_\xi$. That is, $\sigma_k(L^*) = (\sigma_k(L))^*$. □

Remark 4.9. The principal symbol is actually a homomorphism:

$$\sigma_{k+\ell}(L \circ L') = \sigma_k(L) \circ \sigma_\ell(L'),$$

where L, L' are linear differential operators of order k, ℓ, respectively.

Theorem 4.10. *Let M be a closed smooth manifold, $E \to M$ and $F \to M$ be two vector bundles, and $L \in \mathrm{Diff}^k(M; E, F)$. If L is elliptic, then*

(1) $\ker L = \ker_{C^\infty} L = \{u \in C^\infty(M, E) : Lu = 0\}$ *is finite dimensional.*
(2) $\mathrm{im}\, L = L(C^\infty(M, E))$ *is a closed subspace of* $C^\infty(M, F)$*.*
(3) $\mathrm{coker}\, L = C^\infty(M, F)\big/L(C^\infty(M, E)) \cong \ker L^*$ *is finite dimensional.*

Corollary 4.11.

$$C^\infty(M, E) \cong \ker L \oplus \mathrm{im}\, L^*,$$
$$C^\infty(M, F) \cong \ker L^* \oplus \mathrm{im}\, L.$$

This is also true if we replace all instances of C^∞ with L^2-spaces. Moreover,

$$\ker_{C^\infty} L = \ker_{L^2} L.$$

This is called the ***elliptic regularity****. In this case, $L : C^\infty(M, E) \to C^\infty(M, F)$ and $L : L^2(M, E) \to L^2(M, F)$ are called* ***Fredholm operators****.*

A general version of the elliptic regularity uses the theory of Sobolev spaces. We will not discuss this due to space limitations.

4.3. **Formal Adjoint of d.** In de Rham complexes, the differential

$$d \in \mathrm{Diff}^1(M; \bigwedge^k T^*M, \bigwedge^{k+1} T^*M)$$

is a linear differential operator of order 1 for each $k \geq 0$. A natural question to ask is what is the principle symbol of d.

Lemma 4.12. $\sigma_1(d)(v) = iv \wedge -$, *for all* $v \in T^*_\xi M$.

Proof. Since $\sigma_1(d)(v) = i[d, f](\xi) : \bigwedge^k T^*_\xi M \to \bigwedge^{k+1} T^*_\xi M$, where $df(\xi) = v$, for every $\omega \in \bigwedge^k T^*_\xi M$,

$$\begin{aligned}
\sigma_1(d)(v)(\omega) &= i[d, f](\xi)(\omega) \\
&= i(d(f\omega) - f d\omega) \\
&= i(df \wedge \omega + f d\omega - f d\omega) \\
&= i df \wedge \omega \\
&= iv \wedge \omega.
\end{aligned}$$

□

Let (M, g) be Riemannian manifold of dimension n. Then $\bigwedge^{k+1} T^*M$ has a bundle metric given by g for all k. It makes sense to define the formal adjoint of d, denoted by δ. Before we discuss the property of δ, we need the following notion:

Definition 4.13. If V is a vector field and $\omega \in \Omega^k(M)$, then the **interior product** of V and ω is

$$\imath(V)(\omega) = V \lrcorner\, \omega \in \Omega^{k-1}(M),$$

where

$$(V \lrcorner\, \omega)(V_1, \cdots, V_{k-1}) = \omega(V, V_1, \cdots, V_{k-1}).$$

g gives rise to a bundle isomorphism $g^\flat : TM \to T^*M$ by sending $v \in TM$ to $g(v, -)$. Write $g^\sharp : T^*M \to TM$ for its inverse. Then $g^\flat$ takes the basis of TM to its dual basis. For each nonzero $\eta \in T^*_\xi M$ and $\omega \in \bigwedge^k T^*_\xi M$, ω can be uniquely decomposed as $\eta \wedge \omega' + \omega''$. Note

$$\imath(g^\sharp \eta)(\omega) = |\eta|^2 \omega'$$

by writing $\eta = \sum \alpha_j dx_j$ and looking at the value of $(\eta \wedge \omega')(\sum \alpha_j \partial_{x_j}, V_1, \cdots, V_k)$. Hence,

$$\begin{aligned} g(\eta \wedge \alpha, \beta) &= g(\eta \wedge \alpha, \eta \wedge \beta' + \beta'') \\ &= g(\eta \wedge \alpha, \eta \wedge \beta') = g(\eta, \eta) g(\alpha, \beta') \\ &= |\eta|^2 g(\alpha, \beta') \\ &= g(\alpha, \imath(g^\sharp \eta)\beta), \end{aligned}$$

i.e. the adjoint of $\mathrm{ext}(\eta)$ is $\imath(g^\sharp \eta)$. Moreover, it is easy to see that

$$\begin{aligned} \mathrm{ext}(\eta)\imath(g^\sharp \eta)(\omega) &= |\eta|^2 \eta \wedge \omega', \\ \imath(g^\sharp \eta)\mathrm{ext}(\eta)(\omega) &= |\eta|^2 \omega''. \end{aligned}$$

Thus

$$(\mathrm{ext}(\eta)\imath(g^\sharp \eta) + \imath(g^\sharp \eta)\mathrm{ext}(\eta))(\omega) = |\eta|^2 \omega.$$

Lemma 4.14. *Let $\delta = d^* : \Omega^k(M) \to \Omega^{k-1}(M)$. Then*

$$\sigma_1(\delta)(v) = -i \cdot \imath(g^\sharp v)$$

for all $v \in T^*_\xi M$, where (M, g) is a Riemannian manifold with g compatible with the action of i.

Proof. Write $\mathrm{ext}(\eta) \coloneqq \eta \wedge -$. By Proposition 4.8, it suffices to find the formal adjoint of $\sigma_1(d)(v) = i\, \mathrm{ext}(v)$. By preceding discussion, the formal adjoint of $\sigma_1(d)(v)$ is just $-i \cdot \imath(g^\sharp v)$. Note that a negative sign is needed since g is invariant under multiplication of i, i.e. $g(u, v) = g(iu, iv)$. $\square$

Definition 4.15. The **Hodge Laplacian**, also known as the **Laplace–de Rham operator**, of k-forms on a Riemannian manifold is the differential operator

$$\Delta_k : \Omega^k(M) \to \Omega^k(M),$$

defined as

$$\Delta_k = d\delta + \delta d = (d + \delta)^2 \mid_{\Omega^k}.$$

Remark 4.16. In the definition above, we implicitly use the fact that $\delta^2 = 0$. This fact will be proved in the next section.

By Remark 4.9 and preceding discussion, the principle symbol of Δ_k is

$$\begin{aligned}\sigma_2(\Delta_k)(v) &= (\sigma_1(d)\sigma_1(\delta) + \sigma_1(\delta)\sigma_1(d))(v) \\ &= \mathrm{ext}(v)\imath(g^\sharp v) + \imath(g^\sharp v)\mathrm{ext}(v) = |v|^2.\end{aligned}$$

So Δ_k is elliptic. Furthermore, Δ_k is self-adjoint. That is, $\Delta_k^* = \Delta_k$, since $(d\delta)^* = \delta^* d^* = d\delta$ and $(\delta d)^* = d^*\delta^* = \delta d$.

Theorem 4.17 (Maximum principle)**.** *The only functions f satisfying $\Delta f = 0$ (called the **harmonic functions**) on a closed, connect and oriented Riemannian manifold are the constant functions.*

Proof. Let $f \in \ker \Delta$. Then

$$0 = g(\Delta f, f) = g(d\delta f, f) + g(\delta df, f) = g(\delta f, \delta f) + g(df, df) = \|\delta f\|^2 + \|df\|^2.$$

This implies f is constant. In fact, we proved $\ker \Delta_k = \ker \delta \cap \ker d$ ("$\supset$" is obvious). ☐

Theorem 4.18 (Hodge's theorem for the de Rham complex)**.** *Let M be a closed Riemannian manifold. For each k, we have*

$$\Omega^k(M) = \ker \Delta_k \oplus \operatorname{im} \Delta_k = \ker \Delta_k \oplus \operatorname{im} d \oplus \operatorname{im} \delta.$$

In particular,

$$H^k_{\mathrm{Hod}}(M) = \ker \Delta_k \cong H^k_{\mathrm{dR}}(M) = \frac{\ker d}{\operatorname{im} d} = \frac{\ker \Delta_k \oplus \operatorname{im} d}{\operatorname{im} d}$$

*is finite dimensional. Here H^k_{Hod} is called the **Hodge cohomology**.*

We need to justify $\operatorname{im} \Delta_k = \operatorname{im} d \oplus \operatorname{im} \delta$. "$\subset$" is clear by definition. Observe that

$$d\Delta_k = d\delta d = \Delta_{k+1} d, \quad \delta\Delta_k = \delta d\delta = \Delta_{k-1}\delta.$$

From $\Omega^k(M) = \ker \Delta_k \oplus \operatorname{im} \Delta_k$, we see that for $u \in \Omega^k(M)$,

$$du = d(u_0 + \Delta_k u'),$$

where u_0 is the part of u lying in $\ker \Delta_k$. By $\ker \Delta_k = \ker d \cap \ker \delta$, $du_0 = 0$. This yields

$$du = d\Delta_k u' = d\delta d u'.$$

Hence $\operatorname{im} d \subset \operatorname{im}(d\delta)$ and $\operatorname{im} \delta \subset \operatorname{im}(\delta d)$. This implies

$$\operatorname{im} \Delta_k = \operatorname{im}(d\delta) \oplus \operatorname{im}(\delta d) = \operatorname{im} d \oplus \operatorname{im} \delta.$$

Proof of Theorem 4.18 amounts to the fact that Δ_k is elliptic. The readers are referred to [2] for a detailed proof.

4.4. Hodge Star Operator.

Definition 4.19. Let (M, g) be a closed and orientable Riemannian manifold with $\dim M = n$. For any $\alpha, \beta \in \Omega^k(M)$, $0 \le k \le n$, we define the **Hodge star** $\star : \Omega^k(M) \xrightarrow{\simeq} \Omega^{n-k}(M)$ by

$$\alpha \wedge \star\beta = g(\alpha, \beta) dV_g,$$

where V_g is the volume form. In general coordinate chart,

$$dV_g = \sqrt{|\det g|}\, dx_1 \wedge \cdots \wedge dx_n.$$

Example 4.20. Consider $(\mathbb{R}^3, g_{\mathbb{R}^3})$, where $g_{\mathbb{R}^3}$ is the standard metric, with the volume form $dV_g = dx_1 \wedge dx_2 \wedge dx_3$. Then

$$\begin{aligned}
&\star 1 = dx_1 \wedge dx_2 \wedge dx_3,\\
&\star dx_1 = dx_2 \wedge dx_3, \quad \star dx_2 = -dx_1 \wedge dx_3, \quad \star dx_3 = dx_1 \wedge dx_2,\\
&\star (dx_1 \wedge dx_2) = dx_3, \quad \star(dx_2 \wedge dx_3) = dx_1, \quad \star(dx_1 \wedge dx_3) = -dx_2,\\
&\star (dx_1 \wedge dx_2 \wedge dx_3) = 1.
\end{aligned}$$

Proposition 4.21. $\star^2 |_{\Omega^k} = \pm \mathrm{id}$. *In fact,*

$$\star^2 |_{\Omega^k} = (-1)^{k(n-k)}.$$

Proof. It suffices to check on a basis element $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $i_1 < \cdots < i_k$. Let $J = (j_1, \cdots, j_{n-k})$ be the complementary increasing multi-index. We want to find out the multiplication of signs of the permutations: $\mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau)$, where σ takes IJ to $(1, \cdots, n)$ and τ takes JI to $(1, \cdots, n)$. Denote $\overline{J}$ be the reverse of J (i.e. a decreasing multi-index). Note that the sign is the same if we replace $JI \to (1, \cdots, n)$ with $\overline{IJ} \to (n, \cdots, 1)$. This yields

$$\begin{aligned}
\mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau) &= \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\overline{IJ} \to (n, \cdots, 1))\\
&= (\mathrm{sgn}(\sigma))^2 \cdot \mathrm{sgn}(\overline{I} \to I) \cdot \mathrm{sgn}(\overline{J} \to J) \cdot \mathrm{sgn}((1, \cdots, n) \to (n, \cdots, 1))\\
&= \mathrm{sgn}(\overline{I} \to I) \cdot \mathrm{sgn}(\overline{J} \to J) \cdot \mathrm{sgn}((1, \cdots, n) \to (n, \cdots, 1))\\
&= (-1)^{\ell},
\end{aligned}$$

where

$$\ell = \frac{(k-1)k + (n-k-1)(n-k) + n(n-1)}{2} = k^2 + n^2 - n - kn \equiv k(n-k) \mod 2.$$

$\square$

Remark 4.22. The L^2-pairing on $\Omega^k(M)$ is

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta,$$

which is an inner product. On can use this to express $\delta = d^*$ through $\star$ and d. Let $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$, and M be a closed manifold, then

$$\begin{aligned}
\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle &= \int_M d\alpha \wedge \star\beta\\
&= \int_M d(\alpha \wedge \star\beta) - (-1)^k \int_M \alpha \wedge d(\star\beta)\\
\text{(Stokes' Theorem)} \quad &= (-1)^{k+1} \int_M \alpha \wedge d(\star\beta)\\
\text{(Proposition 4.21)} \quad &= (-1)^{k+1} \cdot (-1)^{k(n-k)} \int_M \alpha \wedge \star\star\, d(\star\beta)\\
&= (-1)^{kn+1+k(1-k)} \int_M \alpha \wedge \star(\star d \star \beta)\\
&= (-1)^{kn+1} \langle \alpha, \star d \star \beta \rangle .
\end{aligned}$$

Thus,

$$\delta\beta = d^*\beta = (-1)^{kn+1}(\star d \star)\beta.$$

Corollary 4.23. *For any* $\alpha, \beta \in \Omega^k(M)$,

$$\langle \alpha, \beta \rangle = \langle \star\alpha, \star\beta \rangle .$$

Proof. By direct computation,

$$\begin{aligned}
\langle \star\alpha, \star\beta \rangle &= \int \star\alpha \wedge \star\star\beta \\
&= (-1)^{k(n-k)} \int \star\alpha \wedge \beta \\
&= (-1)^{2k(n-k)} \int \beta \wedge \star\alpha \\
&= \int g(\beta, \alpha) V_g = \int g(\alpha, \beta) V_g \\
&= \langle \alpha, \beta \rangle .
\end{aligned}$$

□

Corollary 4.24. $\delta = (-1)^{kn+1} \star d \star$. *It is then easy to see that* $\delta^2 = 0$. *Moreover,* $\star\Delta_k = \Delta_{n-k}\star$.

Theorem 4.25 (Poincaré duality)**.** *Let* M *be a closed and orientable manifold and* $\dim M = n$. *Then its de Rham cohomology satisfies*

$$H^k_{\mathrm{dR}}(M) \cong H^{n-k}_{\mathrm{dR}}(M),$$

for any $0 \le k \le n$.

Proof. This follows immediately from Corollary 4.24 by taking the kernels of both sides of $\star\Delta_k = \Delta_{n-k}\star$. □

Now we assume (M, g, J, ω) is a Kähler manifold with complex dimension m. Extend J to differential forms, i.e. $\omega \in \Omega^k(M)$ giving $J\omega \in \Omega^k(M)$,

$$(J\omega)(V_1, \cdots, V_k) = \omega(JV_1, \cdots, JV_k).$$

So J acts on $\Omega^{p,q}(M)$ by multiplication by i^{p-q} (because for $V = V^{1,0} \oplus V^{0,1}$, J acts on $V^{1,0}$ by multiplication by i, and by $-i$ on $V^{0,1}$). Define $d^c = J^{-1} \circ d \circ J : \Omega^k(M) \to \Omega^{k+1}(M)$. Since $d = \partial + \overline{\partial}$, we can write

$$\begin{aligned}
d^c\omega &= J^{-1}(\partial + \overline{\partial})J\omega = J^{-1}(\partial + \overline{\partial})i^{p-q}\omega \\
&= i^{p-q}(J^{-1}\partial\omega + J^{-1}\overline{\partial}\omega) = i^{p-q}\left(\frac{1}{i^{p+1-q}}\partial\omega + \frac{1}{i^{p-(q+1)}}\overline{\partial}\omega\right) \\
&= \frac{1}{i}\partial\omega + i\overline{\partial}\omega = i(\overline{\partial} - \partial)\omega.
\end{aligned}$$

It is easy to check that $dd^c = -d^c d$ and $(d^c)^2 = 0$. We can then define a cochain complex of k-forms with differentials d^c. Denote the cohomology of this new complex by $H^\bullet_{d^c}(M)$.

Proposition 4.26. *The formal adjoint of* d^c *is* $-\star d^c \star$.

Proof. Basically copy of Remark 4.22. □

The Hodge star operator can also be extended to complexified k-forms. Namely, we require that

$$\alpha \wedge \star\beta = h(\alpha,\beta) dV_g,$$

where h is the Hermitian metric on M. If $\alpha = \sum u_{I,J} dz_I \wedge d\overline{z}_J$ and $\beta = \sum v_{I,J} dz_I \wedge d\overline{z}_J$ both have type (p,q), then

$$h(\alpha,\beta) = \sum u_{I,J}\overline{v_{I,J}},$$

and so (under appropriately chosen orthogonal frame)

$$\alpha \wedge \star\overline{\beta} = h(\alpha,\beta) dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z_1} \wedge \cdots \wedge d\overline{z_n}.$$

In particular, since dV_g has type (n,n), $\star$ is a $\mathbb{C}$-linear isometry $\Omega^{p,q}(M) \to \Omega^{n-q,n-p}(M)$. Like d^c, we can define the formal adjoint of ∂ and $\overline{\partial}$ to be

$$\partial^* : \Omega^{p,q}(M) \to \Omega^{p-1,q}(M), \quad \overline{\partial}^* : \Omega^{p,q}(M) \to \Omega^{p,q-1}(M).$$

Proposition 4.27. *$\partial^* = -\star\overline{\partial}\star$, and $\overline{\partial}^* = -\star\partial\star$.*

Proof. For any $\omega \in \Omega^{p-1,q}(M)$ and $\eta \in \Omega^{p,q}(M)$,

$$\begin{aligned}
\langle \partial\omega, \eta\rangle_{\Omega^{p,q}} &= \langle \omega, \partial^*\eta\rangle_{\Omega^{p-1,q}} \\
&= \int \partial\omega \wedge \star\overline{\eta} = \int \partial(\omega\wedge\star\overline{\eta}) - (-1)^{p-1}\int \omega\wedge\partial(\star\overline{\eta}) \\
&= \int d(\omega\wedge\star\overline{\eta}) - (-1)^{p-1}\int \omega\wedge\partial(\star\overline{\eta}) \\
\text{(Stokes' Theorem)} \qquad &= (-1)^p \int \omega\wedge\partial(\star\overline{\eta}) \\
\text{(Proposition 4.21)} \qquad &= (-1)^p\cdot(-1)^{(p-1)(m-p+1)}\int \omega\wedge\star\star\,\partial(\star\overline{\eta}) \\
&= (-1)^{p(1+m-p)-m-1}\int \omega\wedge\star\overline{\left(\star\overline{\partial}(\star\eta)\right)}.
\end{aligned}$$

When p is even, $p(1+m-p)$ is even since m is even, and $p(1+m-p)-m-1$ is odd. When p is odd, $p(1+m-p)$ is even, $p(1+m-p)-m-1$ is also odd. In particular, $(-1)^{p(1+m-p)-m-1} = -1$. So $\partial^* = -\star\overline{\partial}\star$. Proof of the other equality is the similar. $\square$

For operators $p \in \{d, d^c, \partial, \overline{\partial}\}$, we define the **complexified Hodge-Laplacian** to be $\Delta_p = pp^* + p^*p$. By Example 4.4, it is not hard to find

$$\begin{aligned}
\sigma_2(\Delta_{d^c})(v) &= |v|^2, \\
\sigma_2(\Delta_{\overline{\partial}})(v) &= \frac{1}{2}|v|^2, \\
\sigma_2(\Delta_{\overline{\partial}})(v) &= \frac{1}{2}|v|^2.
\end{aligned}$$

Since $\sigma_2(\Delta_d) = \sigma_2(\Delta_{d^c}) = 2\sigma_2(\Delta_{\overline{\partial}}) = 2\sigma_2(\Delta_{\overline{\partial}})$, all of $\Delta_d, \Delta_{d^c}, \Delta_{\overline{\partial}}, \sigma_2(\Delta_{\overline{\partial}}$ are elliptic. So we have Hodge theories for them as follows.

Theorem 4.28 (Hodge's theorem for the Dolbeault complex)**.** *Let M be a closed Kähler manifold, whose $\mathbb{C}$-dimension is $m = 2n$, where $2n$ is its corresponding $\mathbb{R}$-dimension. For each k, the cohomology groups*

(1) $H^{p,q}_{\mathrm{Hod},\partial}(M) = \ker \Delta_\partial|_{\Omega^{p,q}} \cong H^{p,q}_\partial(M)$

(2) $H^{p,q}_{\mathrm{Hod},\overline{\partial}}(M) = \ker \Delta_{\overline{\partial}} \mid_{\Omega^{p,q}} \cong H^{p,q}_{\overline{\partial}}(M)$
(3) $H^{k}_{\mathrm{Hod},d^c}(M) = \ker \Delta_{d^c} \mid_{\Omega^{k}} \cong H^{k}_{d^c}(M)$

are finite dimensional. Moreover,

$$\Omega^{p,q}(M) = \ker \overline{\partial} \cap \ker \overline{\partial}^* \oplus \overline{\partial}\left(\Omega^{p,q-1}(M)\right) \oplus \overline{\partial}^*\left(\Omega^{p,q+1}(M)\right),$$

and the Hodge star $\star$ *induces the Poincaré duality:*

$$\star : H^{p,q}_{\mathrm{Hod},\partial}(M) \xrightarrow{\cong} H^{n-p,n-q}_{\mathrm{Hod},\partial}(M).$$

Similar results hold for $\overline{\partial}, d^c$.

4.5. **Lefschetz Operator and Kähler Identities.** Let (M, g, J, ω) be a Kähler manifold.

Definition 4.29. The **Lefschetz operator** L on (M, g, J, ω) is

$$L : \Omega^k(M) \to \Omega^{k+2}(M)$$
$$\alpha \mapsto \omega \wedge \alpha$$

After complexifying, it restricts to

$$L : \Omega^{p,q}(M) \to \Omega^{p+1,q+1}(M).$$

Lemma 4.30. *The formal adjoint of* L*, denoted by* $\Lambda : \Omega^k(M) \to \Omega^{k-2}(M)$*, is given by* $\Lambda = (-1)^k \star L \star$.

Proof. Indeed, for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+2}(M)$,

$$\begin{aligned}
\langle L\alpha, \beta\rangle &= \langle \alpha, \Lambda\beta\rangle \\
&= \int L\alpha \wedge \star\beta = \int \omega \wedge \alpha \wedge \star\beta \\
&= (-1)^{2k} \int \alpha \wedge \omega \wedge \star\beta \\
&= (-1)^{(k+2)(m-k-2)} \int \alpha \wedge \star\star(\omega \wedge \star\beta) \\
&= \left\langle \alpha, (-1)^{(k+2)(m-k-2)} (\star L \star)\beta \right\rangle
\end{aligned}$$

Note $(k+2)(m-k-2) = k(m-k) + 2(m-2k-2)$. When k is odd, $k(m-k)$ is odd; when k is even, $k(m-k)$ is even. In particular, $(-1)^{(k+2)(m-k-2)} = (-1)^k$, which yields the lemma. □

The famous Kähler identities are the core of this section.

Theorem 4.31 (Kähler Identities)**.** *Let* (M, g, J, ω) *be a Kähler manifold, then*

(1) $[\overline{\partial}^*, L] = i\partial$.
(2) $[\partial, L] = -i\overline{\partial}$, $[\Lambda, \overline{\partial}] = -i\partial^*$, $[\Lambda, \partial] = i\overline{\partial}^*$.
(3) $\partial\overline{\partial}^* = -\overline{\partial}^*\partial$ *and* $\overline{\partial}\partial^* = -\partial^*\overline{\partial}$.
(4) $\Delta_\partial = \Delta_{\overline{\partial}} = \Delta_d/2 = \Delta_{d^c}/2$.
(5) $d^c d^* = -d^* d^c$, $d(d^c)^* = -d^{c*} d$, $[\Lambda, d] = -(d^c)^*$.

The proof of these identities is fruitful. However, I will not present them in this paper. One can see Theorem 34 in Chapter 6.4 of my note on Complex Geometry for details.

Corollary 4.32. $[\Delta_\partial, L] = 0$, $[\Delta_d, L] = 0$.

Corollary 4.33. *Let (M, g, J, ω) be a Kähler manifold, then*

$$d\omega = \partial\omega = \overline{\partial}\omega = d^c\omega = \delta\omega = \partial^*\omega = \overline{\partial}^*\omega = (d^c)^*\omega = 0.$$

Definition 4.34. Let $p \in \{d, d^c, \partial, \overline{\partial}\}$. A form α is called **p-harmonic** if $p\alpha = 0$.

Corollary 4.35 (Hodge decomposition)**.** *If (M, g, J, ω) is a Kähler manifold, then*

$$H^k_{\mathrm{dR}}(M) \cong H^k_{d^c}(M) \cong \bigoplus_{p+q=k} H^{p,q}_{\partial}(M) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M).$$

Moreover, conjugation induces an isomorphism $\overline{H^{p,q}_{\partial}(M)} \cong H^{q,p}_{\partial}(M)$, and $\star$ operator induces an isomorphism

$$\star : H^{p,q}_{\overline{\partial}}(M) \to H^{n-q,n-p}_{\overline{\partial}}(M).$$

Proof. Since $\Delta_d = \Delta_{d^c} = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$, we can deduce

$$\ker \Delta_d \mid_{\Omega^k} = \ker \Delta_{d^c} \mid_{\Omega^k} = \ker \Delta_\partial \mid_{\bigoplus_{p+q=k} \Omega^{p,q}} = \ker \Delta_{\overline{\partial}} \mid_{\bigoplus_{p+q=k} \Omega^{p,q}} .$$

Also, for any harmonic form α of type (p, q), $\overline{\alpha}$ has type (q, p), and $\overline{\Delta_\partial \overline{\alpha}} = \Delta_{\overline{\partial}}\alpha = 0$. So $\overline{\alpha}$ is also harmonic. □

Corollary 4.36 (dd^c-Lemma)**.** *Let α be a form such that $d\alpha = 0$, $d^c\alpha = 0$, and $\alpha = d\gamma$ for some γ, then $\alpha = dd^c\beta$ for some β.*

Proof. Write $\gamma = \gamma_0 + d^c\gamma_1 + (d^c)^*\gamma_2$ using $\Omega^{k-1}(M) = \ker \Delta_{d^c} \oplus \mathrm{im}\ (d^c) \oplus \mathrm{im}\ (d^c)^*$. So

$$\alpha = d\gamma = d\gamma_0 + dd^c\gamma_1 + d\,(d^c)^*\gamma_2.$$

Since $\ker \Delta_{d^c} = \ker \Delta_d = \ker d \cap \ker d^*$, $d\gamma_0 = 0$. On the other hand,

$$0 = d^c\alpha = d^c dd^c\gamma_1 + d^c d\,(d^c)^*\gamma_2 = -(d^c)^2 d\gamma_1 - d^c\,(d^c)^* d\gamma_2 = -d^c\,(d^c)^* d\gamma_2.$$

Thus, $-(d^c)^* d\gamma_2 = d\,(d^c)^*\gamma_2 \in \ker d^c \cap \mathrm{im}\ (d^c)^* = \{0\}$. So $\alpha = dd^c\gamma_1$. Write $\beta = \gamma_1$ and we are done. □

If

$$\ker d \cap \ker d^c \cap \mathrm{im}\, d = \mathrm{im}\, dd^c$$

holds over $\mathbb{R}$, then it continues to hold over $\mathbb{C}$. Since on $\mathbb{C}$, $dd^c = i(\partial + \overline{\partial})(\overline{\partial} - \partial) = 2i\partial\overline{\partial}$, dd^c-Lemma in $\mathbb{C}$ is equivalent to the following lemma:

Lemma 4.37 ($\partial\overline{\partial}$-Lemma)**.** *Let (M, g, J, ω) be a Kähler manifold. If $\alpha \in \Omega^{p,q}(M)$ is d-closed and either ∂ or $\overline{\partial}$-exact, then there exists $\beta \in \Omega^{p-1,q-1}(M)$ such that $\alpha = \partial\overline{\partial}\beta$.*

Proof. Basically copy of Proof of Lemma 4.36. □

Corollary 4.38. *Let M be a closed Kähler manifold and $m = \dim M$ be its $\mathbb{C}$-dimension. Let $b_k = \dim H^k_{\mathrm{dR}}(M)$, $h^{p,q} = \dim H^{p,q}_{\overline{\partial}}(M)$. The following holds:*

(1) $b_k = \sum_{p+q=k} h^{p,q}$.
(2) $h^{p,q} = h^{q,p} = h^{m-q,m-p} = h^{m-p,m-q}$.
(3) $h^{p,p} \neq 0$ *for any* $p \in \{1, \cdots, n\}$.
(4) b_k *is even if* k *is odd.*

Proof. $1 \sim 3$ are trivial. 4 follows from 1 and 2. □

4.6. **Lefschetz Decomposition.** Assume that we are working in a closed Kähler manifold (M, g, J, ω) with $\dim_{\mathbb{C}} M = m = 2n$.

Definition 4.39. A form α is called **primitive** if it is not in the image of L, i.e. $\alpha \neq \omega \wedge \tilde{\alpha}$ for any $\tilde{\alpha}$. We are interested in those who are not primitive because they come from lower-degree forms.

By Corollary 4.11, we can decompose Ω^k into $\ker L^* \oplus \operatorname{im} L = \ker \Lambda \oplus \operatorname{im} L$. So α is primitive if $\Lambda\alpha = 0$. Before we continue, we need the following tools.

Lemma 4.40 (Lefschetz identity). *$[L, \Lambda] : \Omega^k(M) \to \Omega^k(M)$ satisfies*

$$[L, \Lambda]\alpha = (k - m)\alpha,$$

for any $\alpha \in \Omega^k(M)$.

The proof of the lemma is tedious. The reader is referred to Lemma 13 in Chapter 6.4 of my note on Complex Geometry. With this in hand, we can easily prove a general version of the preceding lemma.

Corollary 4.41 (Generalized Lefschetz identity). *For $0 \leq k \leq m$ and $0 \leq r \leq m - k$, we have*

$$[L^r, \Lambda]\alpha = (r(k - m) + r(r - 1))\, L^{r-1}\alpha,$$

for any $\alpha \in \Omega^k(M)$, where M has $\mathbb{C}$-dimension m.

Proof. We do it by induction. The base case is just Lemma 4.40. Note that $[L^r, \Lambda] = L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1}$. By inductive hypothesis, we obtain

$$\begin{aligned}
[L^r, \Lambda]\alpha &= L[L^{r-1}, \Lambda]\alpha + [L, \Lambda]L^{r-1}\alpha \\
&= L\left((r-1)(k-m) + (r-2)(r-1)\right) L^{r-2}\alpha + (2r - 2 + k - m)L^{r-1}\alpha \\
&= \Big((r-1)(k-m) + r^2 - 3r + 2 + (k-m) + 2r - 2\Big)\, L^{r-1}\alpha \\
&= (r(k-m) + r(r-1))\, L^{r-1}\alpha.
\end{aligned}$$

□

Remark 4.42. The generalized Lefschetz identity induces an isomorphism $L^{m-k} : \Omega^k(M) \to \Omega^{2m-k}(M)$ for $k \leq m$.

Lemma 4.43. *For $k \leq m$, $\alpha \in \Omega^k(M)$ is primitive iff $L^{m-k+1}\alpha = 0$.*

Proof. $[L^{m-k+1}, \Lambda]\alpha = 0$ by generalized Lefschetz identity. So $L^{m-k+1}\Lambda\alpha = \Lambda L^{m-k+1}\alpha$. Now $\Lambda\alpha \in \Omega^{k-2}(M)$, and $L^{m-(k-2)}$ is an isomorphism on $\Omega^{k-2}(M)$. L^{m-k+1} is then injective on $\Omega^{k-2}(M)$. This implies $L^{m-k+1}\Lambda\alpha = 0$ is equivalent to $\Lambda\alpha = 0$. On the other hand, $L^{m-k+1}\alpha \in \Omega^{2m-k+2}(M)$, and $\Lambda^{m-k+2} : \Omega^{2m-k+2}(M) \to \Omega^{k-2}(M)$ is an isomorphism. So Λ is injective on $\Omega^{2m-k+2}(M)$. $\Lambda L^{m-(k-1)}\alpha = 0$ is equivalent to $L^{m-(k-1)}\alpha = 0$. Hence, α is primitive iff $\Lambda\alpha = 0$, iff $L^{m-k+1}\alpha = 0$. □

Theorem 4.44 (Lefschetz decomposition of differential forms). *Every $\alpha \in \Omega^k(M)$ admits a unique decomposition of the form $\alpha = \sum L^r\alpha_r$, where α_r is of degree $k - 2r \leq \min(2m - k, k)$ and primitive.*

Proof. WLOG, we assume $k \leq m$. Start with uniqueness. Suppose $\sum_{r\geq 0} L^r\alpha_r = 0$. We want to show $\alpha_r = 0$. If $\alpha_0 = 0$, then $L\left(\sum L^{r-1}\alpha_r\right) = 0$ implies $\sum L^{r-1}\alpha_r = 0$

and we are done by induction. Now suppose $\alpha_0 \neq 0$. Since $\alpha_0 \in \Omega^k(M)$ and it is primitive, we know $L^{m-k+1}\alpha_0 = 0$. From

$$L^{m-k+1}\left(\sum L^r \alpha_r\right) = 0 = L^{m-k+2}\underbrace{\left(\sum_{r>0} L^{r-1}\alpha_r\right)}_{\text{degree } k-2},$$

and the fact that L^{m-k+2} is an isomorphism on Ω^{k-2}, we know $\sum_{r>0} L^{r-1}\alpha_r = 0$. Induction on k, we get $\alpha_r = 0$ for all $r > 0$, implying $\alpha_0 = 0$. Combining the previous result yields the desired result.

To prove the existence, first note

$$L^{m-k+1}\alpha \in \Omega^{2m-k+2}(M) = L^{m-k+2}\left(\Omega^{k-2}(M)\right).$$

Thus, there exists $\beta \in \Omega^{k-2}(M)$ such that $L^{m-k+1}\alpha = L^{m-k+2}\beta$. So $\alpha_0 = \alpha - L\beta$ is primitive and $\alpha = \alpha_0 + L\beta$. Induction on degrees, we can assume that β has a Lefschetz decomposition and so does α. ☐

Theorem 4.45 (Hard Lefschetz theorem)**.** *For all $k \leq m$, L^{m-k} induces an isomorphism $H^k_{dR}(M) \to H^{2m-k}_{dR}(M)$.*

Proof. Denote $\mathcal{H}^k(M) = \ker \Delta_d |_{\Omega^k(M)} \cong H^k_{\mathrm{dR}}(M)$. Corollary 4.32 tells us $[\Delta_d, L] = 0$, so $L^{m-k} : \mathcal{H}^k(M) \to \mathcal{H}^{2m-k}(M)$ is injective. On the other hand, $\dim \mathcal{H}^k(M) = \dim \mathcal{H}^{2m-k}(M)$ since $\star$ is an isomorphism. Thus, L^{m-k} is also surjective. ☐

Corollary 4.46 (Lefschetz decomposition of cohomology)**.** *Write $H^k(M)_{\mathrm{prim}} = \ker L^{m-k+1} \subset H^k(M)$ for $k \leq m$. Then for any k, $H^k_{dR}(M) = \bigoplus_r L^r H^{k-2r}(M)_{\mathrm{prim}}$.*

Remark 4.47. If $k \leq m$, then $b_k \leq b_{k+2}$ ($h^{p,q} \leq h^{p+1,q+1}$, respectively); and if $k \geq m$, then $b_k \geq b_{k+2}$ ($h^{p,q} \geq h^{p+1,q+1}$, respectively). Thus $\dim H^k(M)_{\mathrm{prim}} = b_k - b_{k-2}$.

In terms of forms, we might observe that L^{m-k} and $\star$ play similar roles in decomposition and duality. Naturally, one would ask if there is any relationship between these two operators. This is answered by the following proposition.

Proposition 4.48. *If $\alpha \in \Omega^k(M)$ is primitive, then*

$$\star\frac{L^j\alpha}{j!} = (-1)^{\frac{k(k+1)}{2}}\frac{L^{m-k-j}J(\alpha)}{(m-k-j)!}.$$

The proof of this proposition is done by brute force calculation and is therefore omitted.

4.7. **Hodge Index Theorem.**

Definition 4.49. Let Q be a bilinear form on $\Omega^*(M)$, satisfying

(1) $Q(\alpha, \beta) = 0$, if $|\alpha| \neq |\beta|$.
(2) If $\alpha, \beta \in \Omega^k(M)$, then

$$Q(\alpha,\beta) = (-1)^{\frac{k(k+1)}{2}}\int_M L^{m-k}(\alpha\wedge\beta) = (-1)^{\frac{k(k+1)}{2}}\int_M \omega^{m-k}\wedge\alpha\wedge\beta.$$

We call Q an **intersection form** on $\Omega^*(M)$.

It is easy to check that Q satisfies the following properties: (choose $\alpha, \beta \in \Omega^k$)

(1) $Q(\alpha, \beta) = Q(\beta, \alpha)$ if k is even, and $Q(\alpha, \beta) = -Q(\beta, \alpha)$ if k is odd;
(2) $Q(L\alpha, L\beta) = -Q(\alpha, \beta)$.

Theorem 4.50 (Hodge-Riemann bilinear relation)**.** *The following holds:*

(1) $H^{p,q}(M)_{\text{prim}}$ and $H^{r,s}(M)_{\text{prim}}$ are orthogonal with respect to Q, except for $(p, q) = (r, s)$. That is, $Q(\alpha, \beta) = 0$ for all $\alpha \in H^{p,q}(M)_{\text{prim}}$ and $\beta \in H^{r,s}(M)_{\text{prim}}$ with $(p, q) \neq (r, s)$.
(2) The Lefschetz decomposition $H^k_{dR}(M) = \bigoplus L^r H^{k-2r}(M)_{\text{prim}}$ is orthogonal for Q.
(3) If $\alpha \in H^{p,q}(M)_{\text{prim}}$ is nonzero, then

$$i^{p-q} Q(\alpha, \overline{\alpha}) > 0.$$

In particular, Q is non-degenerate.

Proof. Note that Q descends to cohomology since, by Stokes Theorem, if α and β are closed and either of them is exact, then

$$\int_M L^{m-k}(\alpha \wedge \beta) = 0.$$

(1) If $\alpha \in \Omega^{p,q}(M)$ and $\beta \in \Omega^{r,s}(M)$, then $L^{m-k}(\alpha \wedge \beta)$ has type $(n - k + p + r, n - k + q + s)$. Since the volume form has type (n, n), the integral vanishes except when $-k + p + r = 0 = -k + q + s$, i.e. $p + r = k = q + s$. But $k = p + q$, yielding $r = q$, $s = p$.
(2) Suppose $\alpha = L^r \alpha_0$, $\beta = L^s \beta_0$, where α_0, β_0 are primitive and $r < s$. Since $\alpha_0 \in \Omega^{k-2s}(M)_{\text{prim}}$, $L^{m-k+2r+1}\alpha_0 = 0$. We see

$$\begin{aligned} Q(\alpha, \beta) = Q(L^r \alpha_0, L^s \beta_0) &= (-1)^r Q(\alpha_0, L^{s-r}\beta_0) \\ &= \pm \int L^{m-k+2r}(\alpha_0 \wedge L^{s-r}\beta_0) \\ &= \pm \int L^{m-k+2r+1}\alpha_0 \wedge L^{s-r-1}\beta_0 = 0. \end{aligned}$$

(3) Let $\alpha \in \Omega^{p,q}(M)$ be primitive. $\Lambda\overline{\alpha} = \overline{\Lambda\alpha} = 0$. So $\overline{\alpha} \in \Omega^{q,p}(M)_{\text{prim}}$. By Proposition 4.48 $\star\overline{\alpha} = (-1)^{\frac{k(k+1)}{2}} i^{p-q} \frac{L^{m-k}\overline{\alpha}}{(m-k)!}$, we compute

$$\begin{aligned} i^{p-q} Q(\alpha, \overline{\alpha}) &= (-1)^{\frac{k(k+1)}{2}} i^{p-q} \int L^{m-k}(\alpha \wedge \overline{\alpha}) \\ &= (-1)^{\frac{k(k+1)}{2}} i^{p-q} \int \alpha \wedge L^{m-k}\overline{\alpha} \\ &= (m-k)! \, \|\alpha\|^2 \geq 0. \end{aligned}$$

This yields the desired result. In particular, Q is non-degenerate.

□

(3) of Theorem 4.50 says that $i^{p-q}Q$ is positive definite on $H^{p,q}(M)_{\text{prim}}$. The Hodge index theorem is an immediate corollary describing the index (or the signature) of the intersection form Q on $H^m_{dR}(M)$ for a closed Kähler manifold M with complex dimension m. Recall that

Definition 4.51. The **index** (or **signature**) of Q is the number of positive eigenvalues minus the number of negative eigenvalues.

Note that on $H^m_{dR}(M)$, $Q(\alpha,\beta) = \pm\int_M \alpha\wedge\beta$. Define

$$\widetilde{Q}(\alpha,\beta) = \int_M \alpha\wedge\beta.$$

If M is orientable, $\widetilde{Q}$ is non-degenerate since $\widetilde{Q}(\alpha,\star\alpha) = \int_M \alpha\wedge\star\alpha = \|\alpha\|^2 \geq 0$.

Definition 4.52. The **index** (or **signature**) of M is the signature of $\widetilde{Q}$, denoted by $\sigma(M)$.

Theorem 4.53 (Hodge index theorem)**.** *Let (M,g,J,ω) be a closed Kähler manifold with complex dimension m. We have*

$$\sigma(M) = \sum_{p,q}(-1)^p h^{p,q}.$$

Proof. Extend $\widetilde{Q}$ to a Hermitian form on $H^m_{dR}(M,\mathbb{C})$ through $\widetilde{Q}(\alpha,\beta) = \int \alpha\wedge\overline{\beta}$. Lefschetz orthogonal decomposition gives $H^m_{dR}(M,\mathbb{C}) = \bigoplus L^r H^{p,q}(M)_{\text{prim}}$. From the Hodge-Riemann bilinear relations, $(-1)^p\widetilde{Q}$ is positive definite on $L^r H^{p,q}(M)_{\text{prim}}$. Thus

$$\begin{aligned}
\sigma(M) &= \sum_{p+q=m-2r} (-1)^p \dim H^{p,q}(M)_{\text{prim}} \\
&= \sum_{p+q=m-2r} (-1)^p \left(h^{p,q} - h^{p-1,q-1}\right) \\
&= \sum_{p+q=m-2r} (-1)^p h^{p,q} + (-1)^{p-1} h^{p-1,q-1} \\
&= \sum_{p+q=m} (-1)^p h^{p,q} + 2\sum_{\substack{p+q=m-2r\\ r\neq 0}} (-1)^p h^{p,q} \\
&= \sum_{p+q=m} (-1)^p h^{p,q} + \sum_{\substack{p+q \text{ even}\\ p+q\neq m}} (-1)^p h^{p,q} \\
&= \sum_{p+q \text{ even}} (-1)^p h^{p,q}.
\end{aligned}$$

On the other hand, by applying complex conjugation,

$$\sum_{p+q \text{ odd}} (-1)^p h^{p,q} = \sum_{p+q \text{ odd}} (-1)^p h^{q,p} = -\sum_{p+q \text{ odd}} (-1)^q h^{q,p} = 0.$$

Hence,

$$\sigma(M) = \sum_{p,q}(-1)^p h^{p,q}.$$

□

4.8. ***Off-topic: Cohomology with Holomorphic Coefficients and Serre Duality.** In this section, we allow the cohomology to have coefficients in holomorphic vector bundle, instead of $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition 4.54. Let $E \to M$ be a $\mathbb{C}$-vector bundle over a Kähler manifold (M,g,J,ω) of complex dimension m, and h^E be a Hermitian metric on E inducing

$h^\flat : E \to E^*$, which is a $\mathbb{C}$-anti-linear bundle isomorphism. Define for each $\xi \in M$,

$$\overline{\star}_E : \bigwedge^{p,q} T^*_\xi M \otimes E_\xi \to \bigwedge^{m-p,m-q} T^*_\xi M \otimes E^*_\xi$$

by requiring $\overline{\star}_E(\alpha \otimes s) = \overline{\star \alpha} \otimes h^\flat(s)$.

Remark 4.55. $\overline{\star}_E$ is a $\mathbb{C}$-anti-linear isomorphism such that for any $\alpha, \beta \in \Omega^{p,q}(M, E)$

$$\alpha \wedge \overline{\star}_E \beta = h^E(\alpha, \beta) dV_g.$$

Also, $\overline{\star}_E \star \overline{\star}_E = (-1)^{p+q}$ on $\bigwedge^{p,q} T^*_\xi M \otimes E$.

If $(E, h^E) \to M$ is holomorphic, then we can define

$$\overline{\partial}_E : \Omega^{p,q}(M, E) \to \Omega^{p,q+1}(M, E),$$
$$\overline{\partial}^*_E : \Omega^{p,q}(M, E) \to \Omega^{p,q-1}(M, E).$$

Proposition 4.56. *The Laplacian $\Delta_{\overline{\partial}_E} = \overline{\partial}_E \overline{\partial}^*_E + \overline{\partial}^*_E \overline{\partial}_E$ is elliptic. Hence, there is a Hodge decomposition:*

$$\Omega^{p,q}(M, E) = \underbrace{\ker \overline{\partial}_E \cap \ker \overline{\partial}^*_E}_{\ker \Delta_{\overline{\partial}_E}} \oplus \operatorname{im} \overline{\partial}_E \oplus \operatorname{im} \overline{\partial}^*_E,$$

and

$$H^{p,q}(M, E) \cong \ker \Delta_{\overline{\partial}_E} |_{\Omega^{p,q}(M,E)} .$$

It is not hard to find $\overline{\partial}^*_E = -\overline{\star}_{E^*} \overline{\partial}_{E^*} \overline{\star}_E$, which generalizes the equality $\overline{\partial}^* = - \star \partial \star$, since

$$-\overline{\star} \left(\overline{\partial} \overline{\star} \alpha \right) = -\overline{\star} \left(\overline{\partial} (\overline{\star \alpha}) \right) = -\overline{\star} \left(\overline{\partial \star \alpha} \right) = - \star \partial \star \alpha.$$

Exercise 4.57. Show that $\overline{\star}_E \Delta_{\overline{\partial}_E} = \Delta_{\overline{\partial}_{E^*}} \overline{\star}_{E^*}$. So we have a $\mathbb{C}$-anti-linear isomorphism

$$H^{p,q}(M, E) \xrightarrow{\overline{\star}_E} H^{m-p,m-q}(M, E^*).$$

One can think about the natural pairing

$$\begin{array}{ccc} H^{p,q}(M, E) \otimes H^{m-p,m-q}(M, E^*) & \to & \mathbb{C} \\ (\alpha, \beta) & \mapsto & \int_M \alpha \wedge \beta \end{array}$$

It is non-degenerate since $(\alpha, \overline{\star}_E \alpha) \mapsto \int_M h_E(\alpha, \alpha) dV_g = \|\alpha\|^2_{h^E}$. Thus, we have a $\mathbb{C}$-linear isomorphism

$$\left(H^{m-p,m-q}(M, E^*) \right)^* \cong H^{p,q}(M, E).$$

In this context, it is known as the **Serre duality**.

Remark 4.58. By Corollary 4.35, the duality becomes

$$H^q(M, \Omega^p \otimes E) \cong \left(H^{m-q}(M, \Omega^{m-p} \otimes E^*) \right)^* .$$

The sheaf Ω^m_M is known as the **structure sheaf** of M, denoted by K_M. It satisfies

$$H^q(M, E) \cong \left(H^{m-q}(M, K_M \otimes E^*) \right)^* .$$

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