1 Meeting September 5th, 2024

Speaker: Kartik Tandon
The introductory talk is mainly on the motivation of "**why ∞-categories**"?

1. Why homotopy and category?
2. Why in particular ∞-categories?

To be more technical, while ∞-categories meant something broadly before, it now specifically refers to the "quasi-categories" of $(\infty, 1)$-categories developed by Boardman, Vought, Joyal, and Lurie. This is a bit far away for now, and we will focus on more concrete notions for now.

Recall that in category theory, there is the notion of natural transformations.

$$\begin{array}{c} \text{Natural Transformations} \\ \downarrow \\ \text{Functor} \\ \downarrow \\ \text{Objects and Morphisms} \end{array}$$

On the other hand, there are also the notions of universal properties, which relate to limits and colimits. There is in fact a notion of homotopy limits and colimits.

Example 1.1.

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & * \simeq CX \\ \downarrow & & \downarrow \\ * \simeq CX & \longrightarrow & \Sigma X \end{array}$$

colimits homotopy colimits

1.1 Homotopy and Homotopical Category

Here are some examples of homotopy categories:

1. hTop - This is the category obtained from `Top` by modding out the equivalence relation of homotopy equivalences.
2. $Ch[q{-}iso]^{-1}$ - This is the category obtained from $\mathrm{Ch}^*(Ab)$ by moddig out what are called "quasi-isomorphisms".

The notions of ordinary limits and colimits do not typically exist in these categories! Even if they exist, they typically do not agree with limits and colimits.

Example 1.2. Take the **Triangulated categories**:

- Cone is not functorial in this category. This can be partially remedied by the Octahedral Axiom.
- Colimits do not need to be functorial.

Attempts to fix these issues came into the idea of **homotopical categories**. This originally came from **Gabriel-Zismen** in 1967.

- They are of the form $(\mathcal{C}, W)$ where $W \subseteq \mathrm{Mor}(C)$.

They are some issues with this specific framework:

- (i) Formal constructions are technical.
- (ii) Too general of a framework, includes too many things.

1.2 Quillen's Development

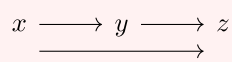
This is where Quillen came in. In 1967, he introduced the idea of **model categories**, which are still prominently used today.

Definition 1.3. Model categories are complete and cocomplete categories of the data

$$(\mathcal{C}, W, \mathrm{Cof}, \mathrm{Fib}),$$

where $W, \mathrm{Cof}, \mathrm{Fib} \subseteq \mathrm{Mor}(\mathcal{C})$. W is typically called **weak equivalences**, Cof cofibrations, and Fib fibrations. They satisfy the following axioms such that

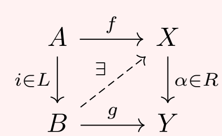
- (i) W has all isomorphisms and is closed under 2 out of 3 in the sense of



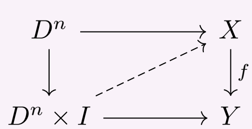
- (ii) $(W \cap \mathrm{Cof}, \mathrm{Fib})$ and $(\mathrm{Cof}, W \cap \mathrm{Fib})$ are both weak factorization systems.

Remark 1.4. $W \cap \mathrm{Cof}$ is called acyclic cofibrations. $W \cap \mathrm{Fib}$ is called acylic fibrations.

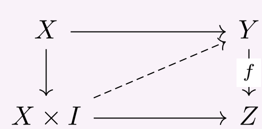
Definition 1.5. A **weak factorization system** $(\mathcal{L}, R)$ is a pair such that the lifting property holds. In other words, for all f, g, the following diagram commutes



Example 1.6. In the category of Top, you can take (W, Fib) where W are the weak equivalences and Fib are the actual fibrations. They are two popular examples of fibrations:



Serre fibrations



Hurewicz fibrations

Exercise 1.7. What are the cofibrations in the example above.

Example 1.8. Now if we take the category of chain complexex over R-modules, we can take

1. W as quasi-isomorphisms.
2. Projective: cofibrations are level-wise monic with projective cokernels.
3. Injective: fibrations are level-wise epic with injective kernels.

In general, the **strategy** is to

- Perform a fibrant/cofibrant replacements with $\{$projective resolutions$\}$.
- Do normal category theory.

1.3 Infinity Categories

So far, we still have not gotten into ∞-categories yet. The analogy between model categories and ∞-categoeis are thought of as follows:

1. Model categories are to picking a basis.
2. ∞-categories are coordinate-free.
3. "The space of choices is contractible".

We can consider the ∞-Category of Spectra,the reason why is outlined as follows

1. There are lots of model categories on Spectra Sp.
2. Lewis's Theorem in 1971 asserted that there does not exist a **convenient** category of spectra (meaning it fails 5 nice properties we hope it to have).
3. We do have a twist map $\tau : A \wedge B \to B \wedge A$. But it might not be homotopic to the identity.
4. Even though the introduction is somewhat technical, the main punchline is that there are some things normal category theory is lacking. In fact, Lewis's Theorem does not hold on ∞-categories (they do really satisfy the 5 nice properties).

Remark 1.9. *Introduction to stable homotopy theory* by Denis Nardin is a great reference - ∞-categories from the start.

Question 1.10. Model categories are not good enough for some things, but what about enriched categories?

There is also a notion of **enriched** categories, perhaps more rooted in physics. They appear in the forms of

1. Top, sSet (simplicial sets).
2. A_∞-categories.
3. dg-categories, etc.

It turns out in fact that A_∞-categories and dg-categories are notions of what's called k-linear stable ∞-categories! Unlike settings outside of enriched category, where you could have statements like "$x \otimes -$ is flat", enriched categories want to start by showing the entire category C is flat, but you would still run into issues.

1.4 Crash Course on Simplicial Sets

Definition 1.11. We define Δ as the category where

1. Objects are $[n] = \{0, ..., n\}$.
2. Morphisms are **order-preserving**.

From here, the category of simplicial sets are the contravariant functors out of Δ into Set. In other words, $sSet = Set^{\Delta^{op}}$. There is a geometric realization of a simplicial set given as a functor

$$|-| : sSet \to Top, [n] \mapsto \Delta^n.$$

A more concrete interpretation of the definition above is as follows.

Definition 1.12. A simplicial set is a graded set over the natural number $\mathbb{N}$, with maps

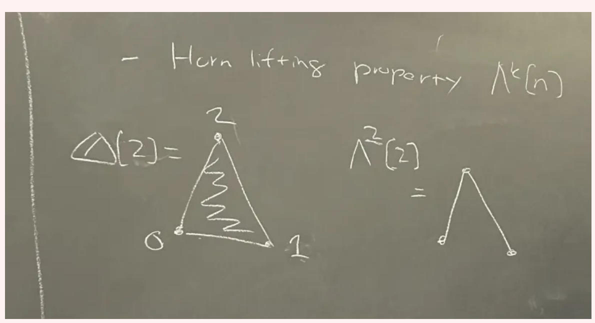
1. Face Maps: $d_m : X_m \to X_{m-1}$
2. Degeneracy Maps; $s_m : X_m \to X_{m+1}$

(If we want to be technical, they should have upper indices). satisfying the conditions

- $d_i d_j = d_{j-1} d_i$ for $i < j$.
- $d_i s_j = s_{j-1} d_i$ for $i < j$.
- If $i = j$ or $i = j + 1$, $d_i s_j = id$.
- $d_i s_j = s_j d_{i-1}$ if $i > j + 1$.
- $s_i s_j = s_{j+1} s_i$ if $i \leq j$.

Definition 1.13 (Kan Complexes). The **Kan complexes** are special cases of simplicial sets that satisfies:

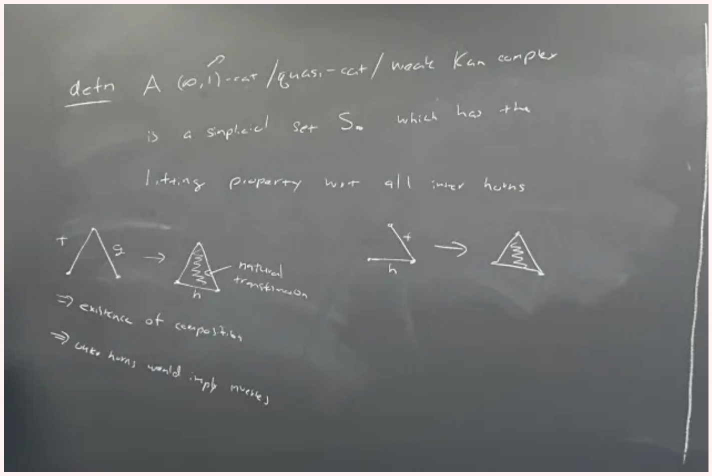
1. The **horn lifting property**. We give a pictorial definition $\Lambda^k[n]$ as, for example when $n = 2$,



   In this case, we note that

2. $\text{Hom}(-, [n]) \in Set^{\Delta^{op}}$. One should think of $\text{Hom}(-, [n])$ as "standard n-simplex". THis also turns out is not a Kan complex.
3. Let $\eta : CAT \to sSet$ be the **nerve functor** of the form $\text{Hom}([n], -)$. $n(C)$ are Kan complexes, fortunately!

Definition 1.14. A $(\infty, 1)$-category/quasicategory/weak Kan complex is a simplical set $S_\bullet$ which has the lifting property with respect to all inner horns. Here is a pictorial representation of what is going on:



1.5 Unique and Interesting Applications of ∞-Categories!!!

Here are some interesting applications:

1. Spectra Sp gives the notion of stable ∞-category enriched in Sp.
   - Cones are all functorial now.
2. Limits and colimits are nice enough. The "ordinary" notion of (co)limits now coincide with the "homotopical" notion of (co)limits.
3. There are some classical constructiosn that are now realizable as new colimit/limit interpretations. (ex. Thom space corresponds to Thom spectra. The classical construction is very technical, but it is very simplified in this new framework . More specifically,
$$M_+X := \operatorname{colim}(M^+ \to \operatorname{Pic}(R) \to \operatorname{Mod}_R).$$
4. Universal Properties in (∞)-category.
   - Thom Spectra.
   - K-theory - BGT (2013) showed that

     Theorem 1.15. There is a functor $\mathbb{K} : Cat_\infty^{Set} \to \mathrm{Sp}$ such that K-theory is the universal additive invariant. In other words,
     $$A \to B \to C \implies K(A) \oplus K(C) \cong K(B).$$

   - Here is another interesting related theorem

     Theorem 1.16 (Beilinson). There is a semi-orthogonal decomposition $\mathcal{D}(\mathbb{P}^1_R) = \langle \mathcal{O}_X, \mathcal{O}_X(-1)\rangle$.

     As a corollary, it shows that

Corollary 1.17. $K(\mathbb{P}^1_R) \cong K(R) \oplus K(R)$.

- Descent - recall for the projective line in algebraic geometry, we can glue it as following pushout

$$\begin{array}{ccc} \operatorname{Spec} \mathbb{Z}[x, x^{-1}] & \longrightarrow & \operatorname{Spec} \mathbb{Z}[u] \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{Z}[t] & \longrightarrow & \mathbb{P}^1 \end{array}$$

The question is, if we look at their derived categories, we have the following:

$$\begin{array}{ccc} \mathcal{D}(\mathbb{P}^2) & \longrightarrow & \mathcal{D}(\mathbb{Z}[u]) \\ \downarrow & & \downarrow \\ \mathcal{D}(\mathbb{Z}[t]) & \longrightarrow & \mathcal{D}(\mathbb{Z}[x, x^{-1}]) \end{array}$$

Is this a pullback?

The answer turns out to be NO for very complicated reasons. There is, however, a remedy in ∞-category, we have that

Theorem 1.18 (Barr-Beck-Lurie)**.** $\mathcal{D}(-)$ is a Cat_∞^{St}-valued sheaf in Sch_{Zar}.