10 Meeting November 14th, 2024

Speaker: Kartik Tandon
Title: Monadicity in ∞-Categories.

For reference, what we are talking about in the lecture (at least when it gets to higher algebra) is adapted from Section 4.7 of Lurie's Higher Algebra.

10.1 The Classical Setting of Monads

There are a few fundamental questions we can ask that are all connected by the idea of Mondas.

Question 10.1.
1. When are two rings R and S Morita equivalent? In other words, there is an (additive) equivalence of category between Mod_R and Mod_S.
2. Let R be a ring, when is there an equivalence between $D(R)$ and $\mathrm{Mod}(HR)$ (in terms of infinity category theory).
3. Descent Theorems
4. Koszul duality, Serre's Affineness criterion

Here is a classical proposition in Morita equivalences:

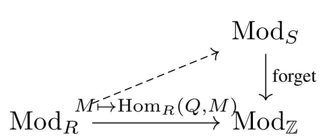
Proposition 10.2. Let $R = k$ be a field, then $M_n(k)$ (the $n \times n$ matrix ring over k) is Morita equivalent to k.

More generally, we have that

Theorem 10.3. Let R be a ring, then R is Morita equivalent to $\mathrm{End}_{Mod_R}(Q^{op})$ if Q satisfies:

- Q is finitely presented (ie. $\mathrm{Map}(Q, -)$ commutes with filtered colimits. This should be thought of as a compactness condition - recall finitely presented R-modules are the compact objects in Mod_R).
- Q is projective (ie. $\mathrm{Map}(Q, -)$ preserves split coequalizers and additivity).
- Q is a generator (ie. $\mathrm{Map}(Q, -)$ is faithful. As a remark, this actually implies that $\mathrm{Ext}^n(Q, -) = 0$, for all $n > 0$).

Let $S = \mathrm{End}_{Mod_R}(Q^{op})$. After rewriting the three conditions in languages closer to our seminar, we see that the proof of Morita equivalence is obtained - if we have a lift:



and if the three conditions given by the theorem guarantees this lift is an equivalence.

This is where we introduce the proof of monads. To give the definition of monad, we have the following:

Definition 10.4. A **monad** $T \in \mathcal{C}$ is a monoid in the category of endofunctors $\mathrm{End}(\mathcal{C})$. Specifically, T is the data $(T : \mathcal{C} \to \mathcal{C}, \mu : T^2 \implies T, \eta : 1_{\mathcal{C}} \implies T)$ and we want the following two diagrams to commute

$$
\begin{array}{ccc}
T \circ T \circ T & \xrightarrow{\mu T} & T \circ T \\
{\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
T \circ T & \xrightarrow[\mu]{} & T
\end{array}
\qquad\qquad
\begin{array}{ccc}
T & \xrightarrow{\eta T} & T \circ T \\
{\scriptstyle T\eta}\downarrow & \searrow^{id} & \downarrow{\scriptstyle \mu} \\
T \circ T & \xrightarrow[\mu]{} & T
\end{array}
$$

Remark 10.5. "A monad is a monoid in the category of endofunctors" is a long running joke in the world of Functional Programming.

Definition 10.6. An algebra A over a monad is the pair $(A \in \mathcal{C}, \alpha : TA \to A)$ such that it is a T-module in the $\mathrm{End}(\mathcal{C})$-tensored category $\mathcal{C}$. That is, we want the following two diagrams to commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & TA \\
 & {\scriptstyle id_A}\searrow & \downarrow{\scriptstyle \alpha} \\
 & & A
\end{array}
\qquad\qquad
\begin{array}{ccc}
T \circ T(A) & \xrightarrow{T\alpha} & TA \\
{\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \alpha} \\
TA & \xrightarrow[\alpha]{} & A
\end{array}
$$

A morphism $f : (A, \alpha) \to (B, \beta)$ between two T-algebras is a map $f : A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\
A & \xrightarrow[f]{} & B
\end{array}
$$

Composition and identities are the same as in $\mathcal{C}$.

Together, this forms a category of T-algebras denoted $\mathrm{Alg}_T(\mathcal{C})$. This is also called the **Eilenberg-Moore Category**.

Remark 10.7. Monads arise from adjunctions. Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint and $G : \mathcal{D} \to \mathcal{C}$ be right adjoint, then $G \circ F$ is a monad! In this talk, F will always be left and G will always be right.

Proposition 10.8. Let (T, μ, η) be a monad in $\mathcal{C}$, then there is an adjunction between $\mathcal{C}$ and $\mathrm{Alg}_T(\mathcal{C})$ given by

$$F^T : \mathcal{C} \rightleftarrows \mathrm{Alg}_T(\mathcal{C}) : U^T$$

Here U^T is the forgetful functor and F^T (called the **free T-algebra functor**) is given by

$$F^T(A) = (TA, \mu_A : T^2 A \to TA) \text{ and } F^T f = Tf.$$

Fruthermore, the adjunction $U^T \circ F^T$ recovers the monad (T, μ, η).

Motivated by the proposition, we also give the following definition

Definition 10.9. A free T-algebra $X \in \mathcal{C}$ is an object in the image of

$$X \mapsto (T(X), \mu_X : T \circ T(X) \to T(X))$$

Definition 10.10. An adjunction $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ is **monadic** if the monad $T = G \circ F$ induces an equivalence $\tilde{G} : \mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$ in the following sense:

$$\begin{array}{ccc} & & \mathrm{Alg}_T(\mathcal{C}) \\ & \overset{\tilde{G}}{\nearrow} & \big\downarrow \scriptstyle{\text{Forget}} \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

Example 10.11. Here are some examples:

1. There exists a monad T on Set such that $\mathrm{Alg}_T(Set) \cong \mathrm{Grp}$ (so the algebra of T over sets are groups). T comes from the Free and Forgetful functor in the adjunction. The free T-algebras are the free groups.
2. There exists a monad T_R on abelian groups such that $\mathrm{Alg}_{T_R}(Ab) \cong \mathrm{RMod}$ (so the algebra of T_R over Ab are R-modules). T_R comes from the Tensor and Hom functors in adjunction. The free T-algebras are the free R-modules.

In this case, both adjunctions are monadic.

Given a pair of adjoint functors, we would like a sufficient criterion to determine if they would be monadic.

Theorem 10.12 (Barr-Beck Monadicity Theorem)**.** Let F, G be adjoints as before. Suppose $\mathcal{D}$ admits split coequalizers, then if

- i) G is conservative (if. $G \circ f$ is an equivalence, then f is an equivalence).
- ii) G preserves split coequalizers.

Then $F \dashv G$ is monadic.

Remark 10.13. Here even though we say the word "equivalence", they are isomorphisms as morphisms in the 1-category. It is in higher algebra, we end up calling isomorphisms as "equivalences".

Proof Idea. We have a lift of the following for F and G - Here is a picture for G:

$$\begin{array}{ccc} & & \mathrm{Alg}_T(\mathcal{C}) \\ & \overset{\tilde{G}}{\nearrow} & \big\downarrow \scriptstyle{\text{Forget}} \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

The proof may be decomposed in a few steps:

1. Step 1 - Showing $\tilde{F} \dashv \tilde{G}$: We know that we have the following coequalizer in $\mathrm{Alg}_T(\mathcal{C})$:

$$TTA \underset{\mu_A}{\overset{T\alpha}{\rightrightarrows}} TA \xrightarrow{\alpha} A$$

We also have that $\tilde{F}(T(A)) = F(A)$ and $\tilde{F}$ preserves colimits.

In this case, we have the following coequalizer:

$$F(T(A)) \rightrightarrows F(A) \longrightarrow \tilde{F}(A)$$

2. Step 2: The unit map $id_{Alg_T(\mathcal{C})} \to \tilde{G} \circ \tilde{F}$ is an equivalence.

   The idea for showing the equivalence is that to recall that G preserves reflexive coequalizers. In this case the following is a coequalizer

$$GF(GF(A)) \rightrightarrows GF(A) \xrightarrow{G\theta} G\tilde{F}(A)\ ,$$

   and we also obtain a coequalizer of the form

$$GF(GF(A)) \rightrightarrows GF(A) \xrightarrow{\alpha} A$$

   This gives us the following commutative diagram:

$$\begin{array}{ccc} GFA & \xrightarrow{h\theta} & G\tilde{F}(A,\alpha) \\ & \searrow^{\alpha} & \downarrow \\ & & A \end{array}$$

3. Step 3: The co-unit map is also an equivalence. The proof is similar in this case.

■

Let us come back to prove Theorem 10.3.

Proof of Theorem 10.3. Consider this diagram

$$\begin{array}{ccc} & & \mathrm{Mod}_S \\ & \nearrow & \downarrow \text{forget} \\ \mathrm{Mod}_R & \xrightarrow{M \mapsto \mathrm{Hom}_R(Q,M)} & \mathrm{Mod}_{\mathbb{Z}} \end{array}$$

Here the horizontal map (call it G) is adjoint in the tensor-hom adjunction. Since Q is a generator, G is conservative. The other two conditions shows that it preserves relfexive coequalizers. Thus, this adjunction is monadic, so we have an equivalence between Mod_R and Mod_S. Here Mod_S is equivalent to $\mathrm{Alg}_{T_R}(Ab)$ (recall the example given earlier). ■

10.2 The Story of Monads in Higher Algebra

In the setting of ∞-categories, we would like to do the following modifications. First we will give a quick introduction to the theory of (co)Cartesian fibrations.

Definition 10.14. Let S and T be a simplicial set, with a morphism $F : S \to T$ of simplicial sets. Let $f : x \to y$ be an edge in S.

We say f is a F-**Carteisan edge** if the following lifting problem has a solution

$$\begin{array}{ccc} \Lambda^n_n & \xrightarrow{\sigma_0} & X \\ \downarrow & \nearrow & \downarrow F \\ \Delta^n & \xrightarrow[\sigma]{} & Y \end{array}$$

when $n \geq 2$ and the following composite map corresponds to the edge $f : X \to Y$

$$\Delta^1 \simeq N_\bullet(\{n-1 < n\}) \hookrightarrow \Lambda^n_n \to_{\sigma_0} X.$$

We say f is a F**-coCarteisan edge** if the following lifting problem has a solution

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{\sigma_0} & X \\ \downarrow & \nearrow & \downarrow F \\ \Delta^n & \xrightarrow[\sigma]{} & Y \end{array}$$

when $n \geq 2$ and the following composite map corresponds to the edge $f : X \to Y$

$$\Delta^1 \simeq N_\bullet(\{0 < 1\}) \hookrightarrow \Lambda_0^n \to_{\sigma_0} X.$$

Definition 10.15. Let S and T be simplicial sets. A morphism $F : S \to T$ of simplicial sets is a **Cartesian fibration** if it is an inner fibration (recall Definition 3.4) and for an edge $f : x \to y$ in T and every $y' \in S$ such that $F(y') = y$, there exists an F**-Cartesian edge** $f' : x' \to y'$ such that $F(x') = x$.

We say F is a **Cartesian co-fibration** if it is an inner fibration and for an edge $f : x \to y$ in T and every $x' \in S$ such that $F(x') = x$, there exists an F**-co-Cartesian edge** $f' : x' \to y'$ such that $F(y') = y$.

Example 10.16. Let S be a simplicial set an consider the map $F : S \to \Delta^0$ to the zero simplex. Then, S is an ∞-category if and only if F is a Cartesian fibration, if and only if, F is a co-Cartesian fibration.

Definition 10.17 (Monoidal ∞-Category)**.** The idea of monoidal ∞-categories is to look at co-Cartesian fibrations and insist the **Segal condition**. A **monoidal** ∞**-category** $(\mathcal{C}, \otimes)$ is composed of a simplicial set $\mathcal{C}^{\otimes}$ and a co-Cartesian fibration $\rho_{\otimes} : C^{\otimes} \to N(\Delta)^{op}$. For each $n \in \mathbb{N}$, there is a sequence of induced map $C^{\otimes}_{[n]} \to C^{\otimes}_{i,i+1}$ for all $i = 0, ..., n-1$. We also require that the universal property of products gives an equivalence of the following ∞-categories for each n:

$$C^{\otimes}_n \to C^{\otimes}_{0,1} \times ... \times C^{\otimes}_{n-1,n} \simeq (C^{\otimes}_{[1]})^n.$$

Definition 10.18 (Algebraic Objects)**.** Let $(\mathcal{C}, \otimes)$ be a monoidal ∞-category. This has a co-Cartesian fibration $\rho_{\otimes} : \mathcal{C}^{\otimes} \to N(\Delta)^{op}$. An **algebra** of $(\mathcal{C}, \otimes)$ is, roughly speaking, a section $s : N(\Delta)^{op} \to \mathcal{C}^{\otimes}$.

We also want to obtain an analog of endofunctors.

Definition 10.19. Let $\mathcal{C}$ be an ∞-category and consider the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. Observe that the composition and evaluation maps:

$$\mathrm{Fun}(\mathcal{C}, \mathcal{C}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \text{ and } \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \times \mathcal{C} \to \mathcal{C}$$

gives $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ the structure of a simplicial monoid with a left action on $\mathcal{C}$. Thus, we can regard $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ as a monoidal ∞-category.

Definition 10.20. A monad of an ∞-category $\mathcal{C}$ is an algebraic object of the monoidal ∞-category $\mathrm{End}(\mathcal{C})$. Informally, this should be thought of as the classical monad with endofunctor $T : \mathcal{C} \to \mathcal{C}$ and maps $\mu : T \circ T \to T$, $\eta : 1_C \to T$ satisfying the same diagrams up to coherent homotopy.

We use $\mathrm{Alg}_T(\mathcal{C})$ to denote the ∞-category of (left) T-modules in $\mathcal{C}$. Informally, this is an analog of the algebra over monads in the classical setting for ∞-categories.

Remark 10.21. In Lurie, $\mathrm{Alg}_T(\mathcal{C})$ is denoted as $\mathrm{LMod}_T(\mathcal{C})$. We used the former notation to be consistent with the previous section in the classical setting.

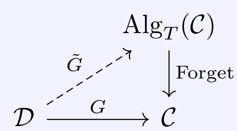
In this case, we still have a way to obtain monads from adjunctions. While we have given one definition of adjunction before, there are many ways to define an adjunction (that are equivalent), the most convenient definition to see how monads arise from adjunctions is the following definition:

Definition 10.22. An adjunction between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $M \to [1] = \{0 \to 1\}$ that is both a coCartesian fibration and a Cartesian fibration. Here we identify the fiber M_0 as $\mathcal{C}$ and M_1 as $\mathcal{D}$ in the usual set-up of an adjunction.

Theorem 10.23 (Barr-Beck-Lurie Monadicity Theorem)**.** Let F, G be adjoints on ∞-categories $\mathcal{C}$ and $\mathcal{D}$. Suppose $\mathcal{D}$ admits **geometric realization** of simplicial objects, then if

- i) G is conservative (if. $G \circ f$ is a equivalence, then f is an equivalence).
- ii) G preserves geometric realizations.

Then $F \dashv G$ is monadic in the sense of ∞-categories. That is, there exists a monad T on $\mathcal{C}$ and an equivalence given in the lift of:



(Here the diagram is up to homotopy)

Why are we doing all the labor to generalize this to the setting of ∞-categories? There are some incredible applications.

Theorem 10.24 (Schwede-Shipley)**.** Let $\mathcal{C}$ be a presentable stable ∞-category with $Q \in \mathcal{C}$ a compact generator, then $\mathcal{C} \cong \mathrm{Alg}_T(\mathrm{Sp}) \cong \mathrm{Mod}(\mathrm{End}(Q^{op}))$. Here T is an appropriately chosen monad coming from the ∞-category analog of tensor-hom adjunction.

Remark 10.25. The original theorem of Schwede-Shipley was done over model categories with a fairly lengthy proof. This proof is shortened and generalized in the language of ∞-categories in Lurie's Higher Algebra.

From this theorem, we obtain the following corollary:

Corollary 10.26. Let R be a ring, then the derived ∞-category $D(R)$ is equivalent to $\mathrm{Mod}(HR)$.

Proof. In the case where $\mathcal{C} = D(R)$, we observe that $\mathrm{End}_{D(R)}(R)$ is concentrated in degree 0 and its degree 0 term is R. Since spectra is determined by maps inducing isomorphisms of homotopy groups, this is exactly HR. Thus, we have that $D(R)$ is equivalent to $\mathrm{Mod}(HR)$. ∎