2 Meeting September 12th, 2024

Speaker: Riley

2.1 Simplicial Sets

Definition 2.1. The simplex category Δ has

- The objects are the finite ordinals $[n] = \{0 < ... < n\}$ are the totally ordered notes on $n + 1$ elements.
- The morphisms are order-preserving set functions.

Δ is presented by "co-face maps" of the form

$$d^i : [n] \to [n+1],$$

where d^i is the unique order preserving map from $[n] \to [n+1]$ that misses i. Furthermore, s^i are the "co-degeneracy maps" of the form

$$s^i : [n] \to [n-1]$$

as the unique order preseriving surjective map that repeeats the index i twice. In particular, these maps satisfy certain identities that we will write out for simplicial sets later.

Definition 2.2. A simplicial set is a presheaf of sets on the category Δ. In other words, it is a contravariant functor from Δ to Set. The category sSet is $\mathrm{Set}^{\Delta^{op}}$ where the objects are the functors and the morphisms are the natural transformations.

If we unpack this definition, this means that - for each n, we have a set X_n of "n-simplicies". From the functoriality, we have maps

- $d_i : X_n \to X_{n-1}$ are the face maps that corresponds to d^i earlier.
- $s_i : X_n \to X_{n+1}$ are the degeneracy maps that corresponds to s^i earlier.

In particular they satisfy the identities.

- $d_i d_j = d_{j-1} d_i$ for $i < j$.
- $d_i s_j = s_{j-1} d_i$ for $i < j$.
- If $i = j$ or $i = j + 1$, $d_i s_j = id$.
- $d_i s_j = s_j d_{i-1}$ if $i > j + 1$.
- $s_i s_j = s_{j+1} s_i$ if $i \leq j$.

Definition 2.3. Consider the functor $\Delta(-, [n]) : \Delta \to Set$ (HOM functor into $[n]$). THis is a valid contravariant functor. This is called the **standard n-simplex** and is denoted Δ^n. In this perspective, we note that

$$\mathrm{Hom}(\Delta^n, X) \cong X_n.$$

We in fact have a Kan extension of the form

$$\begin{tikzcd} sSet \arrow[r, "\mathrm{Lan}_y F"] & E \\ \Delta \arrow[u, hook] \arrow[ur, "F"'] & \end{tikzcd}$$

that commutes up the natural isomorphism. If we want this to genuinely commute, F needs to be co-continuous. Here the map $y : \Delta \to sSet$ sends S to the functor $\mathrm{Delta}(-, S)$.

There is also a canonical functor from $\Delta \to \mathrm{Top}$ by sending an ordinal to the standard simplex. This gives a functor

$$|\bullet| : sSet \to Top$$

that is the geometric realization of a simplicial set. This turns out to be left adjoint to another functor

$$\mathrm{Sing} : Top \to sSet$$

which is the singular complex on a topological space given by $\mathrm{Top}(|\Delta^n|, X)$.

Definition 2.4. We can also consider a contravariant functor to Cat, the category of all small categories

$$\Delta \to \mathrm{Cat}$$

Here $[n]$ is mapped to the index category

$$0 \longrightarrow \ldots \longrightarrow n$$

Here when we consider $\mathrm{Cat}([n], C)$, they correspond exactly to the strings of n composable words.

This is the notion of the nerve of a category given by

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & z \end{tikzcd}$$

♣♣♣ Mattie: [tbd]

There is also a notion of **boundary map** given by

$$\partial \Delta^n = \bigcup_i d_i(\Delta^n).$$

Note that we have not really clarified the notion of a union, but we will be somewhat un-rigorous and only consier the geometric intuition.

When we say the word **simplicial horn**, we consider

$$\Lambda^n_k = \bigcup_{i \neq k} d_i(\Delta^n).$$

A horn is inner if $0 < k < n$. Here is an example for Λ^2_1.

$$\begin{tikzcd} & 1 \arrow[dr, dash] & \\ 0 \arrow[ur, dash] & & 2 \end{tikzcd}$$

2.2 ∞-Category

Notation: Here are some notations in our construction:

- An **object** is a 0-simplex.
- A **morphism** is a 1-simplex.
- An n-cell (if $n = 2$, this is a natural transformation, etc.) is an n-simplex.
- A **composable pair** is an inner 2-horm $\Lambda_1^2 \to X$.
- A **composite** is a filler of a composable pairs (ex. for Λ_1^2, we can fill it to a 2-simplex by a 2-cell and a 1-cell).

Definition 2.5. An ∞-category is a simplicial set in which all inner horns have fillers. Meaning if we have an inner horm $\Lambda_k^n \to X$, we have a lift

$$\begin{tikzcd} \Delta^n \arrow[r, dashed] & X \\ \Lambda_k^n \arrow[u] \arrow[ur] & \end{tikzcd}$$

Definition 2.6. Two maps $f : x \to y$ and $g : x \to y$ are **homotopic** if there is a 2-cell of the form

$$\begin{tikzcd} & y \arrow[d, "1_y = s_0(y)"] \\ x \arrow[ur, "f"] \arrow[r, "g"'] & y \end{tikzcd}$$

Here s_0 is the previous degeneracy map.

♣♣♣ Mattie: [come back to this - 3.1.11 of higher algebra user guide]

Proposition 2.7. If $h \simeq g \circ f$ (so this means that we have the diagram:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

) and $h' \simeq g \circ f$ (so this means that , then $h \simeq h'$.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ and $h : X$ ∎

2.3 Nerve of a Category and Homotopy Category

We saw previously that there is a nerve

$$N : \text{Cat} \to \text{sSet}$$

The nerve of a category is an infinite category.

Proposition 2.8. 1. Nerve is a fully faithful functor.
2. The essential image of N is simplicial sets where every inner horn has a unique filler.

Proof. The proof sketch of (2) is as follows:
Looking at the **spine** of the simplicial set (a sequence 0-cell $\to$ 1-cell $\to$ ... $\to$ n-cell). ∎

Definition 2.9. The **homotopy category** $h(\mathcal{C})$ of an ∞-category $\mathcal{C}$ has

1. objects - the 0-cells from $\mathcal{C}$.
2. morphisms - homotopy equivalence classes of morphisms in $\mathcal{C}$.

This is also sometimes called a 1-truncation of $\mathcal{C}$ and denoted $\tau_1(\mathcal{C})$.

Theorem 2.10. There is a canonical isomorphism of 1-categories

$$h(N(C)) \cong \mathcal{C}.$$

Here, $\mathcal{C}$ is a 1-category.

Definition 2.11. A map $f : x \to y$ in an ∞-category is an isomorphism if there is a map $g : y \to x$ such that $1_x \simeq g \circ f$ and $1_y \simeq f \circ g$.

Proposition 2.12. f is an isomorphism if and only if $[f]$ is an isomorphism in $h(C)$.

2.4 ∞-groupoid

Definition 2.13. An ∞-groupoid is an ∞-category in which all morphisms are isomorphisms.

Remark 2.14. All cells above dimension 1 are automatically reversible, by equivalence of homotopy.

The following is a very deep theorem.

Theorem 2.15. The following are equivalent:

1. $\mathcal{C}$ is an ∞-groupoid.
2. $\mathcal{C}$ is a Kan complex, meaning that all horns have fillers.

Corollary 2.16. The singular simplicial complex $\mathrm{Sing}(X)$ of a topological space X is an ∞-groupoid.

Proof. It suffices for us to check that it is a Kan complex. So we want to solve

$$\begin{tikzcd}
\Delta^n \arrow[r, dashed] & \mathrm{Sing}(X) \\
\Lambda^n_k \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

It suffices for us to take this to the geometric realization land (because $|\bullet|$ is adjoint to Sing), we are looking at

$$\begin{tikzcd} {|\Delta^n|} \arrow[r, dashed] & X \\ {|\Delta^n_1|} \arrow[u, hook] \arrow[ur] & \end{tikzcd} .$$

This is solvable in the world of topology, because a horn is an obvious geometric retract of a geometric n-simplex. ■

2.5 Cardinalities in ∞-Category Theory

Let us first discuss this in the world of 1-categories.

Question 2.17. What is the category Set?

It is certainly not a set. The foundations for interpreting this is given by what is called the **NBG (Von Neumann - Bernays- Godel) set theory**.

Definition 2.18. The idea is that - a class is a formula with free variables where the quantifiers range only over sets, with 2 extra axioms.

It turns out this has two very nice properties:

1. Conservative extension of the ZF system.
2. Finitely axiomatizable.

Saunder MacLane thought this was enough to do 1-category theory and claimed everything in his book could be done using this set-up as long as you add the word "small" in front of your category. However, when you want to study larger categories, it becomes theoretically more challenging. An example of this question is looking at

$$y : \mathrm{Top} \to \hat{\mathrm{Top}}$$

where $\hat{\mathrm{Top}}$ is the presheaf category on Top.

This is where the idea of MK (or MT) set theory, which roughly speaking.

Definition 2.19. MK (or MT) set theory is NBG set theory with quantifiers ranging over classes.

Remark 2.20. It is impossible to prove the consistency of this theory from ZF, because this theory implies the consistency of ZF.

This is fine with 1-categories. This does not, however, work well with ∞-categories. This is where the idea of **Grothendieck universe** comes in.

Definition 2.21. A **Grothendieck universe** is a set U closed under:

- Membership, meaning $x \in y \in U \implies x \in U$.
- Pairing $(\{x, y\} \in U)$
- Unions indexed by U.
- Power sets.

A U-set is a set that is in U. We can put an ∞-category of spaces in the realm of a Grothendieck universe which is an appropriate arena to work under.

However, it turns out that a Grothendieck universe is equivalent to what is called an **inaccessible cardinal**.

Definition 2.22. An inaccessible cardinal is a cardinal (set) that cannot be reached by unions and power-sets.

Example 2.23. The cardinality $|\mathbb{N}|$ is inaccessible relative to the cardinality of finite sets (You might ask - why can't I just take an infinite union of finite sets, but to do that, you see to use $|\mathbb{N}|$, whch is not allowed). The cardinality of $|\emptyset|$ is vacuously inaccessible, because, well, it has no sets.

Now, a even more generalization is what is so called the **Tarski-Grothendieck set theory**, which is built on the notion of "every x is contained in a universe". Now a logician will make a statement that everyone else will find confusing:

- It is enough to have a countable number of universes $U_0 \in U_1 \in U_2 \ldots$. This is enough to do all of category theory.
- BUT, to prove this, you need to construct a cardinal $|\mathbb{N}|$, which requires you to use more than a coutable number of universes.

But, there is an argument to be made that, perhaps type theory is more suitable for category theory than this.

In the proposal of Tarski-Grothendieck set theory, Lurie thought everything he claimed in the book could be done using MK. TG set theory was just a more convenient proof. In this sense, the process of finding out what are the minimal set of axioms required to prove something is called **Reverse Mathematics**.

Example 2.24. McLacty in 2011 proved that FOA (fintie order arithmetic) is enough to prove all of EGA and SGA by Grothendieck.

Theorem 2.25 (Levy Reflection Theorem)**.** If ZFC $\vDash \phi(x_1, ..., x_n)$, then ZFC $\vDash (\exists V, V| = \phi(x_1, ..., x_n))$.

This theorem, essentially, implies that if we only prove a theorem with a finite number of axioms, we can find a smaller universe that can also prove this theorem.

Example 2.26. Maclntyre in the 2000s sketched a proof that the Peano Axioms imply Fermat's Last Theorems. No formal proof was every written down.