# 3 Meeting September 19th, 2024

**The $\infty$-category of spaces**

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Today we will be looking at the $\infty$-category of spaces. We also want to talk about functors between $\infty$-categories, mapping spaces, how to identify constructions as examples of $\infty$-categories. Ultimately, it will hopefully give us a tower of abstractions to climb.
**Motivation:** The $\infty$-category of spaces is an analogus construction to the category of sets in the world of 1-categories.

Let us recall some properties of the category $\mathrm{Set}$:

1. They have free cocompletion of a singleton.
2. The morphism of any two objects in a locally small 1-category take value in $\mathrm{Set}$.
3. There is a standard Yoneda embedding for functors from a locally small category $\mathcal{C}$ into $\mathrm{Set}$.

## 3.1 Functors of $\infty$-Category

We first need to make sense a notion of functor between $\infty$-categories.

**Definition 3.1.** Let $\mathcal{C}, \mathcal{D}$ be $\infty$-categories, a functor of $\infty$-categories is a morphism of simplicial sets (ie. it is a natural transformation between the two functors compatible with the face maps $d_i$ and the degeneracy maps $s_i$).

**Definition 3.2.** Let $K$ be a simplicial set and $\mathcal{C}$ an $\infty$-category. We define a new simplicial set $\mathrm{Fun}(K, \mathcal{C})$ concretely as follows:

- $\mathrm{Fun}(K, C)_n = \mathrm{Hom}_{sSet}(K \times \Delta^n, \mathcal{C})$.
- The face and degeneracy maps are induced by

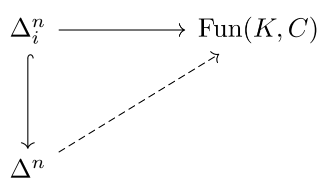
$$d^i : \Delta^{n-1} \to \Delta^n \text{ and } s^i : \Delta^n \to \Delta^{n+1}.$$

Note that this is an internal hom adjunction in sSet.

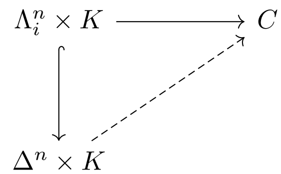
$\mathrm{Fun}(K, \mathcal{C})$ is called the <u>$\infty$-category of functors from $K$ to $\mathcal{C}$</u>.

**Theorem 3.3.** Let $K$ be a simplicial set and $\mathcal{C}$ an $\infty$-category, the simplicial set $\mathrm{Fun}(K, \mathcal{C})$ is an $\infty$-category.

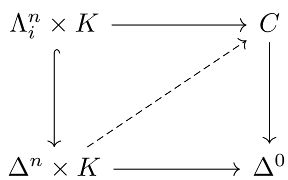
*Proof.* We will use the lifting property of maps of simplicial sets. One wants to show that there is a solution to the following lifting problem.



The usual way one approach these kind of lifting problems is to apply adjunctions in a smart way. Using the internal hom adjunction, this is equivalent indeed to



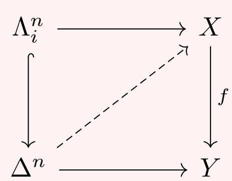

We can augment this to a diagram of the form




where we note the morphism $C \to \Delta^0$ is necessarily unique.

We pause the proof to introduce a definition in the middle.

**Definition 3.4.** A map $f : X \to Y$ is called an **inner fibration** if it satisfies the right lifting property with respect to all hom inclusions. In other words, we have a lift of the form




Hence, we see that the proof amounts to showing that the map $C \to \Delta^0$ is an inner fibration. The proof of this is in fact not categorical at all but is rather an extremely combinatorial proof. The proof follows from the fact that $\mathrm{Fun}(-, K)$ preserves inner fibrations if and only if the claim of maps having the left lifting property (LLP) wr.t. the inner fibrations are closed under $\times K$. ■

**Remark 3.5.** It turns out that $C$ is an $\infty$-category if and only if $C \to \Delta^0$ is an inner fibration.

## 3.2 Mapping Spaces

**Motivation:** When $\mathcal{C}$ is a 1-category and $d, c \in \mathrm{Obj}(\mathcal{C})$. We can consider the pull back of the form

morphisms in $\mathcal{C}$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(c,d) & \longrightarrow & \mathrm{Fun}([1],\mathcal{C}) \\ \Big\downarrow & \lrcorner & \Big\downarrow {\scriptstyle (0,1)\text{,return source and target}} \\ \Lambda^0 & \xrightarrow{(c,d)} & \mathcal{C} \times \mathcal{C} \end{array}$$

terminal 1-category

This guides the definition of simplicial set of morphisms for $\infty$-categories.

Let us now recall the following proposition.

**Proposition 3.6.** For any simplicial set $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ is a Kan complex.
2. $\mathcal{C}$ is an $\infty$-groupoid.

We will obtain that the simplicial set of morphisms will be an $(\infty, 0)$-category (ie. a $\infty$-groupoid).

**Definition 3.7.** Let $\mathcal{C}$ be an $\infty$-catgeory, and $c, d \in \mathrm{Obj}(\mathcal{C})$. We define the **mapping space** as the pullback

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(c,d) & \longrightarrow & \mathrm{Fun}(\Delta^1,c) \\ \Big\downarrow & \lrcorner & \Big\downarrow \\ \Delta^0 & \xrightarrow{(c,d)} & \mathrm{Fun}(\partial\Delta^1,c) \end{array}$$

Note that $\partial\Delta^n$ has two properties:

1. $\mathrm{Fun}(\partial\Delta^1, c)$ are pair of objects.
2. $\mathrm{Fun}(\partial\Delta^1, c) = c \times c$.

Here $\Delta^0$ is a single point and $\Delta^0 \to \mathrm{Fun}(\partial\Delta^n, c)$ goes to the pair $(c, d)$.

$\mathrm{Map}_{\mathcal{C}}(c, d)$ ocnsists of morphisms from $c$ to $d$ that restrict to $c$ and $d$ at the boundary of $\Delta^1$.

**Theorem 3.8.** For $\mathcal{C}$ an $\infty$-category and $c, d \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Map}_{\mathcal{C}}(c, d)$ is a Kan complex, and hence an $\infty$-groupoid by the proposition.

*Proof.* One can show that

$$\mathrm{Fun}(\Delta^1, c) \longrightarrow \mathrm{Fun}(\partial\Delta^1, c)$$

is an inner fibration that is stable under pullback. From here, we can deduce that the map $\mathrm{Map}_{\mathcal{C}}(c, d) \to \Delta^0$ is a "conservative" inner fibration. In particular, this implies that it satisfies the filling condition, and it implies that it is a Kan complex. ∎

## 3.3 The $\infty$-category of Spaces

**Construction:** Take some integer $0 \leq i \leq j$ where $i, j \in \mathbb{N}_0$. From here we define

$$P_{i,j} = \{I \subseteq \{i, ..., j\} \mid \min(I) = i \text{ and } \max(I) = j\}.$$

$P_{i,j}$ itself has a partial order given by inclusion.

**Definition 3.9.** From here we define the simplicial category $C[\Delta^n]$ given by

1. Objects are the numbers $0, 1, ..., n$
2. The morphisms are simplicial Hom Sets of the form

$$\mathrm{Hom}_{C[\Delta^k]}(i, j) = N(P_{i,j})$$

   Here, $N(P_{i,j})$ is the nerve of the poset $P_{i,j}$.

3. Compositions are given by the union.

**Example 3.10.** When $n = 0$, $C[\Delta^0]$ is the terminimal simplicial category that has one simple object $0$ and singleton simplicial mapping space.

When $n = 1$, $C[\Delta^1]$ has objects $0, 1$ and all the simplicial mapping spaces are again trivial (empty or singletons).

The interesting case occurs when $n = 2$. In this case, $C[\Delta^2]$ has three objects $0, 1, 2$. The non-trivial simplicial hom-set is $N(P_{0,2})$. There are two elements in $P_{0,2}$ here, $\{0, 2\} \subseteq \{0, 1, 2\}$.

It turns out this construction is in fact functorial in $n$. Hence, we can define a **simplicial set of spaces** $\mathrm{Spc}$ as the following.

**Definition 3.11.** A simplicial set of spaces $\mathrm{Spc}$ is given by the simplicial set

- $\mathrm{Spc}_n = \mathrm{hom}_{sCat}(C[\Delta^n], \mathrm{Kan})$ where $\mathrm{Kan}$ stand for the category of Kan complexes with simplicial hom sets.
- Note that this can be equivalently written as $\mathrm{Hom}_{sSet}(\Delta^n, \mathrm{Spc})$.

**Remark 3.12.** The construction of $\mathrm{Spc}$ is an example of what is called a **homotopy coherent nerve construction**, which is a construction given to any simplicial category $\mathcal{C}$.

**Example 3.13.** We note that the $0$-simplicies of $\mathrm{Spc}$ are exactly the objects of $\mathrm{Kan}$. The $1$-simplicies are the maps of Kan complexes. The $2$-simplicies are in bijection with maps of Kan complexes $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$ together with a homotopy $g \circ f \simeq h$.