4 Meeting September 26th, 2024

Speaker: Elle P.

Today we will be talking about limits, colimits, and adjunctions in the setting of ∞-categories. Let us first establish some terminologies for this lecture:

1. $\mathcal{C}$ will denote an ∞-catgeory.
2. For $a, b \in obj(\mathcal{C})$, we denote the mapping space $\mathrm{Map}_{\mathcal{C}}(a, b)$
3. We use $I \in \mathrm{sSet}$ to denote an indexing diagram - this will be used later to index limits (For the purposes of this talk, we say sSet is a small category).
4. $\mathcal{C}^I = \mathrm{Fun}(I \to \mathcal{C})$ is the functor category from I to $\mathcal{C}$.

4.1 Limits

Definition 4.1. A cone of a functor F consists of a pair (y, η), where $\eta : c_y \to F$ is a natural transformation from c_y to F. Here c_y descends as the map

$$c_y : I \to \Delta^0 \to_y \mathcal{C}$$

and is the constant functor. From here we can define a map $Map_{\mathcal{C}}(x, y) \to Map_{\mathcal{C}^I}(c_x, F)$ as the composition of

$$\mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{\quad c \quad} \mathrm{Map}_{\mathcal{C}^I}(c_x, c_y) \xrightarrow{\quad \eta_* \quad} \mathrm{Map}_{\mathcal{C}^I}(c_x, F)$$

Definition 4.2. The cone (y, η) is a **limit cone** if the map $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{C}^I}(c_x, F)$ specified previously is a homotopy equivalence. In this case, we call y the **limit** of F. We write this notationally as

$$y = \lim_I F = \lim_{i \in I} F(i).$$

Proposition 4.3. If we plug in $N(I)$ and $N(\mathcal{C})$ (nerves of 1-categories), then the previous definition are recovers the ordinary 1-limits.

Example 4.4. Here are some common examples for limits:

1. In the specific case where $I = \emptyset$, then $\mathrm{Fun}(I, \mathcal{C}) = \mathrm{pt}$. In this case, y is a limit if for all x

   $$\mathrm{Map}_{\mathcal{C}}(x, y) \simeq \star$$

   In this case, we call y the **terminal object**.

2. Suppose I is a discrete set, ie. I is the disjoint union of some collection of points. Write the elements of $F(I)$ as $F(i)$ for each $i \in I$. Then, we observe that

   $$\mathrm{Fun}(I, \mathcal{C}) = \prod_I \mathcal{C},$$

   where the right hand side are taken as products in sSet. $y \in \mathcal{C}$ is a limit to this diagram if there is a homotopy equivalence

   $$\mathrm{Map}_{\mathcal{C}}(x, y) \to \prod_{i \in I} \mathrm{Map}_{\mathcal{C}}(x, F(i)).$$

Proposition 4.5. One can check that

$$\mathrm{Map}_{\mathcal{C}}(x, \prod_{i \in I} F(i)) \simeq \prod_{i \in I} \mathrm{Map}_{\mathcal{C}}(x, F(i)).$$

Hence $y \in \mathcal{C}$ is a limit to the discrete diagram is we have the equivalence

$$\mathrm{Map}_{\mathcal{C}}(x, y) \simeq \mathrm{Map}_{\mathcal{C}}(x, \prod_{i \in I} F(i))$$

From here we can conclude that $y \simeq \prod_{i \in I} F(i)$.

The definition of limit in ∞-categories, as expected, are also unique.

Proposition 4.6. If y, y' are limits to the same F, then they are equivalent.

Remark 4.7 (Quick Divergence by Nir Gadish)**.** We have that

- $\mathrm{Maps}(\Delta^2, \mathcal{C})$ is the space of all composites in $\mathcal{C}$. This is intuitively the idea of a homotopy:

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "g \circ f"'] \arrow[r, phantom, "h" near end, shift left=4ex] & & \bullet \end{tikzcd}$$

- $\mathrm{Maps}(\Delta^1, \mathcal{C}) \times \mathrm{Maps}(\Delta^1, \mathcal{C})$ is the space of composable morphisms.
- There is a canonica homomotopy equivalence

$$\mathrm{Maps}(\Delta^2, \mathcal{C}) \to \mathrm{Maps}(\Delta^1, \mathcal{C}) \times \mathrm{Maps}(\Delta^1, \mathcal{C})$$

- This is in fact a homotopy equivalence of Kan complexes, so we can get a section s back

$$\mathrm{Maps}(\Delta^2, \mathcal{C}) \leftarrow \mathrm{Maps}(\Delta^1, \mathcal{C}) \times \mathrm{Maps}(\Delta^1, \mathcal{C})$$

Theorem 4.8. The full subcategory spanned by limits in the functor category (this is sometiems called the sSet of limits) is either empty or contractible (ie. trivial).

The last example of a limit we want to talk about is the **pullback**. This actually acts differently than how they are typically in 1-categories.

Example 4.9. Suppose we have the diagram I being

$$\begin{tikzcd} & 0 \arrow[d] \\ 0' \arrow[r] & 1 \end{tikzcd}$$

In this case $\mathrm{Fun}(I, C)$ are diagrams of the form

$$\begin{tikzcd} & b \arrow[d, "h"] \\ c \arrow[r, "k"'] & d \end{tikzcd}$$

Let $b \times_d c$ denote the limit of this diagram. This is called the pullback.

Lemma 4.10. A map $a \in \mathrm{obj}(\mathcal{C}) \mapsto b \times_d c$ is equivalent to the data of

$$i : a \to b, j : a \to k$$

so that we have the homotopy equivalence $h \circ i \simeq k \circ j$ (diagram commutes up to homotopy).

Theorem 4.11. Taking limits commute with mapping spaces, ie. there is a homotopy equivalence of the form

$$\mathrm{Map}_{\mathcal{C}}(x, \lim_i F(i)) \simeq \lim_i \mathrm{Map}_{\mathcal{C}}(x, F(i)).$$

Proof. This is a corollary of a deeper theorem (that we will not prove) that there is an equivalence between

$$(y, \eta) \iff (y^c(y), y^c(\eta)).$$

■

4.2 Colimits

The discussions for the colimits are a lot shorter.

Definition 4.12. For any ∞-category $\mathcal{C}$, there is a canonical notion of an opposite ∞-category $\mathcal{C}^{op}$. The colimit of $\mathcal{C}$ is the limit in the $\mathcal{C}^{op}$. There is a correspondence

$$\{\text{colimits}\, F : I \to \mathcal{C}\} = \{\lim F^{op} : I^{op} \to \mathcal{C}^{op}\}.$$

Dually to the notion of a terminal object, we can define an initial object as the colimit over the empty set. Pushouts are dual to the pullbacks.

4.3 Adjunction

Let $f : \mathcal{C} \to \mathcal{D}, g : \mathcal{D} \to \mathcal{C}$ be functors respectively. They have an adjunction if there is a pair of natural transformations (sometimes aptly called unit maps)

$$\eta : id_{\mathcal{C}} \to gf, \mathcal{E} : fg \to \mathrm{id}_{\mathcal{D}}$$

such that the following diagrams hold up to homotopy

$$\begin{tikzcd} f \arrow[r, "id"] \arrow[d, "\eta"] & f \\ fgf \arrow[ur, "\varepsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, "id"] \arrow[dr, "\eta"'] & g \\ & gfg \arrow[u, "\varepsilon"'] \end{tikzcd}$$

$\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ $\qquad$ $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$

Theorem 4.13. An adjunction gives rise to a homotopy equivalence of spaces

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{D}}(f(c), d) \arrow[r, "g"] & \mathcal{M}\dashv\sqrt{}_{\mathcal{C}}(gf(c), g(d)) \arrow[dl, "\eta^*"] \\ \mathrm{Map}_{\mathcal{C}}(c, g(d)) & \end{tikzcd}$$

for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$.

Example 4.14. We have the following two examples of adjunctions.

$$\begin{tikzcd} \mathrm{Kan} \arrow[r, bend left, "\mathrm{inc}"] \arrow[r, bend right, "(-)^{\simeq}"'] & \mathrm{QCat} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{spc} \arrow[r, bend left, "\mathrm{inc}"] & \mathrm{Cat}_{\infty} \arrow[l, bend left, "(-)^{\simeq}"] \end{tikzcd}$$

Example 4.15. Consider the category $\mathcal{C}$ to $\mathrm{Fun}(BG, C)$. The left adjoints are the homotopy orbits, and the right adjoints are the homotopy fixed points.

Theorem 4.16 (Fundamental Theorem of Adjoint Functors)**.** Left adjoints preserve colimits, and right adjoints preserve limits.