5 Meeting October 10th, 2024

Topic: Stable ∞-categories
Speaker: Colton

5.1 Definition of Stable ∞-categories

Today we will be talking about the stable ∞-categories. Most of what we are talking about is a mix of Maximillen's notes, Gallagher's, and Lurie's higher algebra (with an emphasis on the last source). We will concretely investigate two specific examples of them:

1. The ∞-category of Spectra.
2. Derived categories.

Note that while we could form a derived category for any abelian category, the general construction of a "stable" derived category is very general.

Definition 5.1. An ∞-category $\mathcal{C}$ is pointed if there exists an object 0 that is both initial and final. This just means that

$$\mathrm{Hom}_{\mathcal{C}}(0, X) \simeq * \simeq \mathrm{Hom}_{\mathcal{C}}(X, 0)$$

for all objects $X \in \mathrm{obj}(\mathcal{C})$.

Remark 5.2. We remark that $\mathcal{C}$ is pointed if and only if there exists an initial object $\emptyset$, a final point $*$, and a one-morphism $* \to \emptyset$. These conditions imply the $\emptyset$ agrees with $*$ already.

Definition 5.3. Let $\mathcal{C}$ be a pointed ∞-category, a **triangle** is a square $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

Here 0 is the initial and final object.

1. A triangle is a fiber (resp. cofiber) sequence if it is a pullback (resp. pushout). Note that Maximilien Péroux calls this exact and coexact instead.

2. Let $g : X \to Y$ be a morphism, a kernel/fiber of g is a fiber sequence of the form

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Y \end{array}$$

3. Let $g : X \to Y$ be a morphism, a cokernel/cofiber of g is a cofiber sequence of the form

$$\begin{array}{ccc} X & \xrightarrow[g]{} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & W \end{array}$$

Now we are ready to definition a stable ∞-category.

Definition 5.4. A pointed ∞-category $\mathcal{C}$ is stable if it satisfies the following 2 conditions

1. For every morphism $g : X \to Y$, its fibers and cofibers exist.
2. Every triangle has the property that - it is a fiber sequence if and only if it is a cofiber sequence.

We can regard this definition as a sort of generalization of triangulated categories. The motivation behind why we want to look at stable ∞-categories because triangulated categories requires sort of a choice rather than an intrinsic property that stable ∞-categories offer.

5.2 Spectra

Definition 5.5. A spectrum E is a collection of pointed spaces $(E_n)_{n\geq 0}$ with structure maps

$$\Sigma E_n \to E_{n+1}$$

There is also a morphism of spectra from $E \to E'$ given by $E_n \to E'_n$ for all n that respects structure maps.

Definition 5.6. There is also a notion of Ω-spectrum where we require that the adjoints of the structure maps are weak equivalences.

Example 5.7. Let X be a pointed space, the suspension spectrum $\Sigma^\infty X$ given by $\Sigma^\infty X_n = \Sigma^n X$, and the morphisms of the structure maps are the identity. A specific example of the suspension spectrum is the sphere spectrum $\mathbb{S}$ when we take $X = S^0$.

There is a suitable notion of homotopy groups of a spectrum.

Definition 5.8. Let E be a spectrum, we define

$$\pi_n(E) \coloneqq \mathrm{colim}_k\, \pi_{n+k}(E_k).$$

In the specific case where E is the sphere spectrum $\mathbb{S}$, $\pi_n(\mathbb{S})$ is exactly the n-th stable homotopy group of spheres.

Example 5.9. Here is another example of spectrum. Let G be an abelian group, we can form the Eilenberg-Maclane spectrum HG where $HG_n = K(G, n)$. There is a canonical weak equivalence given by

$$K(G, n) \simeq \Omega K(G, n + 1),$$

which gives the structure map in suspension. Taking the homotopy groups of HG gives the singular homology is coefficient G.

There is a remarkable theorem that relates spectra to cohomology theories.

Theorem 5.10 (Brown Representability)**.** There is a correspondence between Ω-spectra and cohomology theories.

Definition 5.11. A weak equivalence of spectra E and E' is a morphsim $f : E \to E'$ that induces isomorphism on all of their homotopy groups. SH is the localization of (Spectra) by weak equivalence.

5.3 Loop Space and Suspension

We can define a suitable notion of suspension and loop functor in pointed ∞-categories.

Definition 5.12. Let $\mathcal{C}$ be a pointed ∞-category. Let M^Σ (resp. M^Ω) to be the full subcategory of squares that look like the following

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & X' \end{array}$$

such that the square is a pushout (resp. pullback). Here $0, 0'$ are zero objects.

We have the following theorem that is not at all easy.

Theorem 5.13. Assume that fibers and cofibers all exist. Then, there exists a trivial Kan fibration $M^\Sigma \to \mathcal{C}$ with section $s : \mathcal{C} \to M^\Sigma$. Let $e : M^\Sigma \to \mathcal{C}$ return the object X' - the bottomr right corner of the square. From here we define the suspension functor as

$$\Sigma = e \circ s.$$

We can similarly define ΩX. From here, we get the squares:

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

Lemma 5.14 (Loop-Suspension Adjunction)**.** Σ is left adjoint to Ω. Furthermore, when $\mathcal{C}$ is stable, the functors Σ, Ω gives an equivalence.

We have talked about spectra and stable ∞-categories. Now we will try to relate the two.

Definition 5.15. If $c \in \mathrm{obj}(\mathcal{C})$ is some final object, we can define $\mathcal{C}_*$ the ∞-category of pointed objects to be the full subcategory with morphisms of the form $c \to d$.

Definition 5.16 (Stabilization)**.** We define $\mathrm{Sp}(\mathcal{C})$ as the limit of the sequence

$$\mathcal{C}_* \xleftarrow{\Omega} \mathcal{C}_* \xleftarrow{\Omega} \mathcal{C}_* \longleftarrow \ldots$$

In the specific case when $\mathcal{C} = \mathrm{Spc}$, we call $\mathrm{Sp}(\mathrm{Spc})$ the stable ∞-category of spectra.

Proposition 5.17. If $\mathcal{C}$ has finite limits, then $\mathrm{Sp}(\mathcal{C})$ is stable.

5.4 Derived Category

The construction $\mathrm{Sp}(\bullet)$ gives a lot of ways to construct stable ∞-categories. We will look at another major example in the world of derived categories. The general results that motivate this construction is as follows:

Theorem 5.18. Let $\mathcal{C}$ be a stable ∞-category, then its homotopy category $h\mathcal{C}$ has the structure of a triangulated category.

Let us clarify some terminologies first.

Definition 5.19 (Additive Category)**.** An **additive** category $\mathcal{C}$ is a category equipped with the following additional data...

- For $A, B \in \mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(A, B)$ is given the structure of an abelian group.

satisfying...

1. Composition distributes over addition, ie.

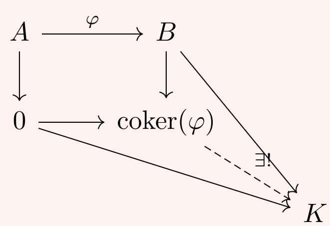
$$(f + g) \circ h = (f \circ h) + (g \circ h) \text{ and } f \circ (g + h) = (f \circ g) + (f \circ h)$$

2. $\mathcal{C}$ has a zero object, meaning that it is both the initial and final object.

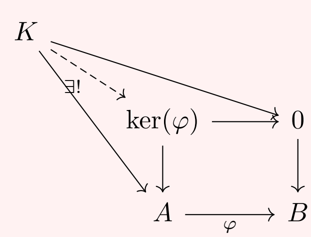
3. $\mathcal{C}$ has finite products.

An additive category is called **abelian** if...

4. kernels and cokernels exist. In the sense that if we have a morphism $\varphi : A \to B$, the cokernel of this morphism $\mathrm{coker}(\varphi)$,



Similarly for kernel, ie. they are pushouts or pullbacks.



5. Every monomorphism is the kernel of its cokernel. In the sense that for a monomorphism $\varphi : A \to B$, consider the map $A \to B \to \mathrm{coker}(\varphi)$, then the kernel of this morphism $B \to \mathrm{coker}(\varphi)$ is (A, φ).

6. Every epimorphism is the cokernel of its kernel.

Definition 5.20. An additive category $\mathcal{C}$ is **triangulated** if we have

1. A morphism $T : X \in \mathrm{obj}(\mathcal{C}) \to X$ given by $X \mapsto X[1]$.

2. A collection of distinguished triangles of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

such that they satisfy some axioms which we omit for this talk.

A sad fact about triangulated categories is that they are generally very hard to work with.

Remark 5.21. For the stable ∞-category $\mathcal{C}$, it has the structure of a triangulated category if we take T to be the suspension functor.

There is a general procedure to produce a derived category of abelian category, which will be examples of triangulated categories.

Definition 5.22. Let A be an abelian category. We say that A "has enough projectives" (or injectives) if every object admits a projective (or injective) resolution.

Remark 5.23. Let A be an abelian category with enough projectives (or injectives). We can produce a category $D^{\pm}(A)$ as a stable ∞-category such that its homotopy category $hD^{\pm}(A)$ is the usual derived category.

Definition 5.24. Let K be a commutative ring with unity, a dg-category $\mathcal{C}$ (roughly speaking) consists of

1. $\mathrm{Obj}(\mathcal{C})$ - object class
2. For all objects X and Y, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are the chain complexes of K-modules with a notion of tensor product and composition.

We can do this over any ring, but in the specific case where $K = \mathbb{Z}$ is the ring of integers, we have the following construction.

Definition 5.25 (dg Nerve)**.** Let $\mathcal{C}$ be a dg category and $n \geq 0$, we can (roughly speaking) define $N_{dg}(\mathcal{C})_n$ to be set of pairs of the form

$$(\{X_i\}_{i=0}^n, \{f_I\})$$

such that

1. X_i is an object in $\mathcal{C}$ for all i.
2. For all $I = \{i_- < i_n < \ldots < i_1 < i_+\} \subseteq [n]$, $f_I \in \mathrm{Hom}(X_{i_-}, X_{i_+})_m$ satisfying

$$df_I = \sum_{i=0}^{m} (-1)^j (\ldots)$$

where ... is some combinatorial arrangement.

Example 5.26. In the dg Nevre construction, the 0-simplex are the objects, and 1-simplex are degree 0 morphisms $f : X \to Y$ with $df = 0$, and so on.

Lemma 5.27. We have the following:

1. $N_{dg}(\mathcal{C})$ is an ∞-category.
2. Let A be an additive category, the category $\mathrm{Ch}(A)$ of chain complexes on A is a dg-category.

Definition 5.28. Let A be an additive category. We define $Ch^{-}(A)$ as the category of chain complexes where $M_n = 0$ for $n << 0$. We similarly define $Ch^{+}(A)$ as the category of chain complexes where $M_n = 0$ for $n >> 0$.

Definition 5.29 (∞-Derived Categories)**.** Let A be an abelian category with enough injectives (resp. projectives), we can define $D^{+}(A)$ (resp. $D^{-}(A)$) as the dg-nerve of $\mathrm{Ch}^{+}(A_{inj})$ (resp. $\mathrm{Ch}^{-}(A_{proj})$).

It turns out that both categories are stable ∞-categories, which follows from the following general fact in Lurie.

Proposition 5.30. Let A be an additive category, then $N_{dg}(Ch(A))$ is stable.