6 Meeting October 17th, 2024

Title: Presentability of ∞-categories
Speaker: Fangji Liu

A natural question to ask for the title is.

> **Question 6.1.** What is a **presentable** ∞-categories? Why do we need a presentable ∞-category?

Most of the talk today will be devoted to defining this category. The intuition is that a presentable category should satisfy the notion of:

1. The simplest kind of categories are small categories, but most categories are not small.
2. The idea of a presentable category is - although it is not small, it should be "generated" by some small subcategories.

There are some interests in why we need presentable ∞-categories too! For instance,

- Presentable ∞-categories are more tractable and hence easier to study.
- Another motivation came from the universal characterization of K-theory (by BGT). The construction utilized some additive/localizing invariants in $\mathrm{Cat}_\infty^{ex} \to D$ where we required D to go into some presentable ∞-category
$$\mathrm{Cat}_\infty^{ex} \to D \hookrightarrow \text{presentable } \infty\text{-category}$$
- There is a recent development called **continuous K-theory** which is a functor
$$K : \{\text{dualizable presentable } \infty\text{-categories}\} \to \mathrm{Sp}$$
which extends the standard functor we have
$$K : \mathrm{Cat}_{small} \to \mathrm{Sp}\,.$$
- Adjoint functor theorem.
- There is a correspondence between presentable ∞-categories and combinatorial model categories.

6.1 Cocompletion and Ind-completion

To discuss the construction, we will first talk about cocompletion and ind-completion. For an ordinary category $\mathcal{C}$, it need not be cocomplete (meaning that it admits all small colimits). There is, however, a very natural way to produce a cocompletion of $\mathcal{C}$ (it can be thought of as an analog of free group).

> **Theorem 6.2.** The **free cocompletion of** $\mathcal{C}$ is the presheaf category of $\mathcal{C}$, ie.
> $$\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, Set).$$
> The fully-faithful embedding of $\mathcal{C}$ in $\mathcal{P}(\mathcal{C})$ is given by the Yoneda embedding, ie
> $$i : \mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C}), c \mapsto [-, c]$$
> We call the essential image of $\mathcal{C}$ as the **representables** in $\mathcal{P}(\mathcal{C})$.
>
> In other words, let $\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), D)$ be all the functors that preserve colimits, then there is an equivalence of category given by restriction
> $$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Proof. Let $H \in \mathcal{P}(\mathcal{C})$, we essentially want to show that

$$H = \text{colimit of some representables}\,.$$

There is a very explicit construction of this colimit. We take the category C/H where

- The objects of C/H are objects $x \in H(c)$ for all c.
- The morphisms from $x \in H(c) \to x' \in H(c')$ is a morphism
$$f : c \to c' \text{ such that } H(f) \cdot x' = x.$$
- In other words, C/H is the full-subcategory of $\mathcal{C}$ spanned by the representables of $\mathcal{P}(C)/H$ (slice category).

One can check that

$$H = \operatorname{colim}_{C/H} F$$

Here each functor $F : C/H \to \mathcal{P}(C)$ sends $x \in H(c) \mapsto i(c)$ (recall i is the Yoneda embedding). ■

This is the discussion for 1-category, but the construction generalizes to ∞-categories!

Theorem 6.3. Let $\mathcal{C}$ be an ∞-category, then the free cocompletion of $\mathcal{C}$ is exactly

$$\mathcal{P}(\mathcal{C}) = \operatorname{Fun}(\mathcal{C}^{op}, \operatorname{Spc}).$$

Proof Sketch. The idea is to find an ∞-category analog of a slice category and apply similar arguments. The slice category is given by the homotopy pullback

$$\begin{array}{ccc} C/H & \longrightarrow & * \\ \downarrow & & \downarrow \\ \operatorname{Fun}(\Delta^1, \mathcal{C}) & \xrightarrow{\Delta} & \operatorname{Fun}(\Delta^0, \mathcal{C}) \end{array}$$

In this case, we will have again that $H = \operatorname{colim}_{C/H} i(c)$. ■

On the other hand, Ind completion is given by the concept of filtered colimits.

Definition 6.4 (Filtered Categories)**.** A 1-category $\mathcal{C}$ is a **filtered category** if

- For any finite list of objects $\{c_i\}_{i=1}^n$, there exists $d \in \operatorname{obj}(c)$ with morphisms $c_i \to d$ for all $i = 1$ to n.
- For any finite collection of morphisms $h_i : c \to c'$ for $i = 1$ to n, there exists a morphism $f : c' \to d$ such that
$$f \circ h_i = f \circ h_j \text{ for all } i, j.$$

Definition 6.5. A **filtered colimit** is a colimit whose index diagram is a filtered category.

The presheaf category is the free cocompletion, we want a suitable analog for cocompletion that only contains all filtered colimits.

Definition 6.6. For an ordinary category $\mathcal{C}$, we define $\operatorname{Ind}(\mathcal{C})$ to be the full subcategory of $\mathcal{P}(\mathcal{C})$ consisting of H such that C/H is a filtered category (or equivalently that H is a filtered colimit of C).

Of course, from here, we have the following.

Proposition 6.7. $\mathrm{Ind}(\mathcal{C})$ is the free filtered cocompletion (also called an Ind completion) of $\mathcal{C}$. In other words, we have an equivalence

$$\mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where the LHS is the filtered-colimit preserving functors.

This is the construction for 1-categories, but the catch is that the same construction does not quite work for ∞-categories. Let us however analyze some properties of filtered categories to see if they can motivate a definition.

Proposition 6.8. A 1-category is filtered if and only if for all finite simplicial sets I, for a map $I \to N(\mathcal{C})$, there exists an extension $I^{\triangle} \to N(\mathcal{C})$. Here $I^{\triangle}$ refers to the cocone (this is just saying every map has a cocone).

Definition 6.9. We say that an ∞-category $\mathcal{C}$ is filtered if for all finite simplicial set I, a map $I \to \mathcal{C}$ extends to $I^{\triangle} \to \mathcal{C}$.

6.2 Compactness

Once we have the notion of filtered colimit, there is a notion of a compact object.

Definition 6.10. An object $d \in \mathcal{C}$, where $\mathcal{C}$ is an ordinary category, is called **compact** if the functor

$$[d, -] : \mathcal{C} \to \mathrm{Sets}$$

preserves filtered colimits. Let C^{ω} be the full subcategory spanned by compact objects.

Here are some examples of compact objects.

Category	Compact Objects
Set	Finite Sets
Vect_k	Finite dimensional vector space
Mod_R	Finitely presented modules
Grps	Finitely presented groups
Top	Fintie Sets with discrete topology
$\mathrm{Open}(X)$	compact open sets in X
sSet	Finite simplicial sets

Table 1: Some examples of categories and their compact objects.

Note that the compact objects are Top are not exactly all the compact spaces...

Proposition 6.11. We make two observations for every category $\mathcal{C}$ (with the exception of Top) in Table 1:

1. $\mathcal{C}$ is generated by compact objects (being colimits of compact objects).
2. The subcategory of compact objects in $\mathcal{C}$ is small.

Definition 6.12. A cardinal κ is called regular if for a collection $\{A_i\}_{i \in I}$ where I has cardinal $< \kappa$ and each A_i has cardinal $< \kappa$, the union $\bigcup_{i \in I} A_i$ has cardinal $< \kappa$.

Example 6.13. $0, \omega$, and the continuum are examples of a regular cardinal. Here ω refers to the cardinality of the natural numbers.

Definition 6.14. For any regular cardinal κ, we can define a κ-filtered category whose collection of objects and morphisms in the definition are no longer finite, but of cardinality $< \kappa$ (they are called κ-small). We can also define κ-compact sets similarly, and $\mathrm{Ind}_\kappa(\mathcal{C})$ similarly. These notions extend similarly to ∞-categories.

6.3 Presentable ∞-category

We are finally able to define a presentable ∞-category.

Definition 6.15. An ∞-category $\mathcal{C}$ is called **presentable** if there exists a regular cardinal κ, a small ∞-category C', such that C' admits κ-small colimits, and

$$\mathcal{C} = \mathrm{Ind}_\kappa(C')$$

Definition 6.16. A functor $f : \mathcal{C} \to \mathcal{D}$ is a **localization** if it has a fully faithful right adjoint. A functor $f : \mathcal{C} \to \mathcal{D}$ is **accessible** if there exists regular cardinal κ, $\mathcal{C}, \mathcal{D}$ admits κ-filtered colimits, and f preserves them.

Theorem 6.17. The following are equivalent:

1. $\mathcal{C}$ is presentable.
2. $\mathcal{C}$ is equivalent to $\mathrm{Ind}_\kappa(C^\kappa)$, where C^κ is the full subcategory of κ-compact objects, and C^κ is essentially small (note no κ!), and admits κ-small colimits.
3. $\mathcal{C}$ is equivalent to $\mathrm{Ind}_\kappa(C')$ such that C' is small and $\mathcal{C}$ (no $'$) admits colimits.
4. There exists a small ∞-category C' and an "accessible localization" in the sense there is a localization $\mathcal{P}(C') \to \mathcal{C}$ whose fully faithful right adjoint is accessible.
5. $\mathcal{C}$ is locally small, cocomplete, and there exists a regular cardinal κ, a set S consisting of κ-compact objects, such that S generates C under small colimits.

Remark 6.18. The condition that $\mathcal{C}$ is equivalent to $\mathrm{Ind}_\kappa(C')$ in (3) such that C' is small is called being "**accessible**".

Example 6.19. For a small category $\mathcal{C}$ that is cocomplete. $\mathcal{C}$ is presentable if and only if $\mathcal{C}$ is idempotent complete.

We will end the meeting with a discussion on the adjoint functor theorem.

Theorem 6.20. Presentable ∞-categories are complete and cocomplete.

Theorem 6.21 (Adjoint Functor Theorem)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories.

1. F is a left adjoint if and only if F preserves colimits.
2. F is a right adjoint if and only if F preserves limits and is accessible.

Remark 6.22 (Remark by Nir Gadish)**.** If every object is the colimit of compact objects, then we can compute the hom-set $[x, y]$ as

$$\begin{aligned}[x, y] &= [\operatorname{colim}_I c, \operatorname{colim}_J d] \\ &= \lim_I [c, \operatorname{colim}_J d] \\ &= \lim_I \operatorname{colim}_J [c, d]\end{aligned}$$

Thus, every morphism can also be hit by morphisms in the compact subcategory.