7 Meeting October 24th, 2024

Title: Homotopy theory of ∞-categories
Speaker: Saul Hilsenrath

Today we will be talking about the homotopy theory of ∞-categories.

7.1 Setup

Let us recall a few constructions from earlier talks.

- We have the nerve functor $N : \mathrm{Cat} \to \mathrm{sSet}$ that is full and faithful.
- We also have the 1-truncation functor $\tau : \mathrm{sSet} \to \mathrm{Cat}$ that extends the inclusion of the simplex category $\Delta \to \mathrm{Cat}$.
- From the previous talk, we now know that $\Delta \to \mathrm{Cat}$ can be extended by taking colimits.
- The one-truncation τ is left-adjoint to the nerve functor N.

We also recall a categorical lemma.

Lemma 7.1. Suppose F is left adjoint to a functor U, and U is full and faithful. Then there is a natural isomorphism given by the co-unit

$$F \circ U \simeq id$$

As a corollary of this categorical lemma, we have that

Corollary 7.2. We have a natural isomorphism of the form $\tau \circ N \simeq 1_{Cat}$.

As the first instance of using this construction, we have the following lemma.

Lemma 7.3. τ preserves binary products.

Proof Sketch. Recall that the nerve $N([n]) = \Delta^n$, so

$$\begin{aligned}
\tau(\Delta^m \times \Delta^n) &\cong \tau(N([m]) \times N([n])) \\
&\cong \tau(N([m] \times [n])) && \text{Nerve is right-adjoint and hence preserves products} \\
&\cong [m] \times [n] && \text{By the preceding Corollary} \\
&\cong \tau(\Delta^m) \times \tau(\Delta^n).
\end{aligned}$$

The proof then concludes by extending this using colimits. ■

7.2 Concrete Homotopy Theory (on Simplicial Sets)

Everything we discuss in this section applies to all of sSet. Recall in topology, a homotopy is of the form $H : I \times X \to Y$. We want an analog of the interval. In this section, we first establish some items of terminology (note these are not canonical):

1. For a presheaf X over A, let X_a be the image of $a \in \mathrm{Obj}(A)$ by X. We call X_a the **fiber over** a.
2. For a well-ordered, non-empty, finite poset category E, we use Δ^E to denote the nerve of E, $N(E)$.

Definition 7.4. We use J to denote the nerve of the category $0 \iff 1$ (this is a category with two objects 0 and 1, one isomorphism between 0 and 1, and no additional morphisms besides the identities). In this case, we define $\partial J := \partial\Delta^1$. Note that we can write

$$\partial J = \Delta^{\{0\}} \cup \Delta^{\{1\}}.$$

Here the union is taken fiber-wise, and $\Delta^{\{0\}}, \Delta^{\{1\}}$ are both isomorphic to Δ^0.

Remark 7.5. Δ^0 is the terminal object in sSet. As a result, the product $\Delta^0 \times X$ is isomorphic to X.

Lemma 7.6. For $j = 0, 1$, let $i_j : X \to \partial J \times X$ be the embedding of X as $\Delta^{\{j\}} \times X$. Then

$$(\partial J \times X, i_0, i_1) \text{ is the coproduct } X + X.$$

Proof. The idea is just that

$$\partial J \times X = (\Delta^{\{0\}} \times X) \sqcup (\Delta^{\{1\}} \times X).$$

■

Definition 7.7. In the category of simplicial sets,

1. A J-homotopy is a morphism $J \times X \to Y$.
2. We say that $f, g : X \to Y$ are J-homotopic, written as $f \sim_J g$, if there exists a lift to the problem

$$\begin{array}{ccc} \partial J \times X & \xrightarrow{[f,g]} & Y \\ \big\downarrow & \nearrow_{\exists h} & \\ J \times X & & \end{array}$$

   Note that equivalently, this is saying that

$$\begin{array}{ccc} X + X & \xrightarrow{[f,g]} & Y \\ \big\downarrow & \nearrow_{\exists h} & \\ J \times X & & \end{array}$$

3. We say that $f : X \to Y$ is a J-homotopy equivalence if there exists $g : Y \to X$ such that

$$g \circ f \sim_J 1_X, f \circ g \sim_J 1_Y.$$

We state the following lemma whose verification is left to the reader.

Lemma 7.8. $\sim_J$ is reflexive and symmetric. It is generally not transitive, though.

Proposition 7.9. Let $f, g : X \to Y$. If $f \sim_J g$, then $\tau(f)$ is naturally isomorphic to $\tau(g)$ (here $\tau(f), \tau(g)$ are functors in Cat).

Proof. Recall that τ preserves coproducts and we showed that it preserves binary products. Thus, we have the following diagram

$$\begin{array}{ccc} \tau(X)+\tau(X) & \xrightarrow{[\tau(f),\tau(g)]} & \tau(Y) \\ \downarrow & \nearrow_{\tau(h)} & \\ \tau(J)\times\tau(X) & & \end{array}$$

Here h is given by the $\sim_J$ definition. In particular, by how J is defined, $\tau(J) \cong (0 \iff 1)$. Thus, $\tau(J)$ is going to look like a category of the form $(a_0 \iff a_1)$.

Let $f : a_0 \to a_1$ be the only morphism from a_0 to a_1 in that category. We can consider

$$\eta = (\tau(h)(f \times 1_x))_{x \in \mathrm{Obj}(\tau(X))}$$

which will give the desired natural isomorphism. ■

Corollary 7.10. As a corollary, if $f : X \to Y$ is a J-homotopy equivalence, then $\tau(f)$ is an equivalence of categories.

Proof. From the previous proposition, if $g : Y \to X$ is a J-homotopy inverse, then

$$\tau(g)\tau(f) \sim 1_{\tau(X)} \text{ and } \tau(f)\tau(g) \sim 1_{\tau(Y)}.$$

■

Here we introduce another notation: We have that i (resp. p) has the left (resp. right) lifting property with respect to p (resp. i) if for every commutative square of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

there is a lift in the diagonal of the square, as seen above. We denote this situations as i LLP p or p RLP i.

Definition 7.11. Given a collection I of morphisms, we write $r(I), \ell(I)$ for

$$r(I) = \{p \mid p \text{ RLP } i,\ \forall i \in I\}$$

$$\ell(I) = \{i \mid i \text{ LLP } p,\ \forall p \in I\}.$$

Definition 7.12. We use Cof (cofibrations) to denote the collection of monomorphisms in sSet. We use tFib (called trivial fibrations) to denote $r(\mathrm{Cof})$.

Proposition 7.13. Let $f : X \to Y$ be a trivial fibration. Then f is a J-homotopy equivalence.

Proof. We use $\pi : J \times X \to X$ to denote the projection. Let $\emptyset$ denote the empty presheaf (i.e., the presheaf whose fibers are all empty). Then we have a lift of the form

$$\begin{array}{ccc} \emptyset & \hookrightarrow & X \\ \downarrow & \overset{\exists s}{\nearrow} & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[=]{} & Y \end{array}$$

and a lift of the form

$$\begin{array}{ccc} \partial J \times X & \xrightarrow{[s\circ f, 1_X]} & X \\ \big\downarrow & \overset{\exists h}{\nearrow} & \big\downarrow f \\ J \times X & \xrightarrow[f\circ\pi]{} & Y \end{array}$$

(In both cases, we have lifts because the left sides are monomorphisms.) In particular, the first diagram tells us that $f \circ s = 1_Y$, and the second diagram tells us that $s \circ f \sim_J 1_X$. ■

We establish yet another notation: For simplicial sets A, X, we write $X^A := \mathrm{Fun}(A, X) = \mathrm{Hom}_{\mathrm{sSet}}(h(-) \times A, X)$. Here $h(-)$ is given by the Yoneda embedding $\Delta \hookrightarrow \mathrm{sSet}$.

Thus, a map $f : A \to B$ induces a morphism $X^f : X^B \to X^A$. In particular, the n-th component of X^f is the natural transformation

$$(X^f)_n = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times f, X),$$

given by pre-composition by $1_{\Delta^n} \times f$.

Here we state a proposition that we won't prove for the sake of time.

Proposition 7.14. Let $f : A \to B$ be a J-homotopy equivalence. Then X^f is also a J-homotopy equivalence. In particular, this implies that $\tau(X^f)$ is an equivalence of categories.

7.3 Abstract Homotopy Theory (on ∞-Categories)

Definition 7.15. Let A, B be simplicial sets. We say that $f : A \to B$ is a **categorical weak equivalence** if for all ∞-categories X, $\tau(X^f)$ is an equivalence of categories. We use W to denote the class of categorical weak equivalences.

Example 7.16.

1. J-homotopy equivalences are categorical weak equivalences.
2. As a special case, trivial fibrations are categorical weak equivalences.

Lemma 7.17. W has the 2-out-of-3 property. In other words, if $h = g \circ f$, and two out of f, g, h are in W, then the third morphism is in W.

Proof Sketch. The class of equivalences of categories in Cat has the 2-out-of-3 property. Now use the functoriality of $\tau \circ X^{(-)}$. ■

Definition 7.18. The class of trivial cofibrations tCof is given by $\mathrm{Cof} \cap W$. Further, we write the class of categorical fibrations as

$$\mathrm{Fib} = r(\mathrm{tCof}).$$

Lemma 7.19. Let $I_1 = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$ (note that this is a set). Then

$$\text{Cof} = \ell(r(I_1)).$$

Definition 7.20. Let κ be any cardinal. A simplicial set has size $< \kappa$ if $|\operatorname{Mor}(\Delta/A)| < \kappa$.

Lemma 7.21. There exists a cardinal κ such that if I_2 is the set of trivial cofibrations between simplicial sets of size $< \kappa$, then

$$\text{tCof} = \ell(r(I_2)).$$

Remark 7.22. By definition, the class of trivial fibrations is $r(\text{Cof})$, so we have that

$$\text{tFib} = r(\text{Cof}) = r(\ell(r(I_1))) = r(I_1).$$

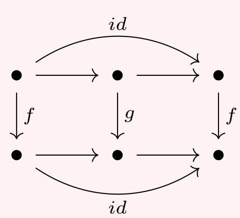
By the same argument

$$\text{Fib} = r(I_2).$$

Finally, $\text{tFib} = r(\text{Cof}) \subset r(\text{tCof}) = \text{Fib}$. Thus, the trivial fibrations form a sub-class of the categorical fibrations.

Definition 7.23. Let $\mathcal{C}$ be a 1-category. A **weak factorization system** (WFS) is a pair (A, B) such that

1. $A, B \subset \operatorname{Mor}(\mathcal{C})$.
2. A, B are closed under retracts. Here, if we have a commutative diagram



   we say that "f is a retract of g".
3. $A \subset \ell(B)$.
4. For any $f \in \operatorname{Mor}(\mathcal{C})$, there exist $i \in A, p \in B$ such that $f = p \circ i$.

Remark 7.24. In the very concrete world of compactly generated weakly Hausdorff spaces, the mapping cylinder would give a weak factorization system.

Lemma 7.25 (Small Object Argument)**.** Let $\mathcal{C}$ be a 1-category and $I \subset \operatorname{Mor}(\mathcal{C})$ be a **set** (note the importance of this being a set) such that for all $i \in I$, $c = \operatorname{dom}(i)$ is small, in the sense that there exists a cardinal κ such that

$$\operatorname{Hom}_{\mathcal{C}}(c, -) \text{ preserves colimits of } \kappa\text{-filtered well-ordered sets.}$$

Then, $(\ell(r(I)), r(I))$ is a weak factorization system.

Lemma 7.26. If A is a small category, then all objects of the presheaf category $\mathrm{Set}^{A^{\mathrm{op}}}$ are small. Here by small, we mean the definition of small in the small object argument.

In particular, this lemma implies that we don't have to worry about the smallness criterion of the small object argument. In particular, we obtain as a corollary

Proposition 7.27. $(\mathrm{Cof}, \mathrm{tFib})$ and $(\mathrm{tCof}, \mathrm{Fib})$ are weak factorization systems.

We know that $\mathrm{tCof} = \mathrm{Cof} \cap W$. We would like a similar result for fibrations.

Lemma 7.28. $\mathrm{tFib} = \mathrm{Fib} \cap W$.

Proof. We know from an earlier remark that $\mathrm{tFib} \subset \mathrm{Fib}$, and we know that every trivial fibration is a categorical weak equivalence. Thus, $\mathrm{tFib} \subset \mathrm{Fib} \cap W$. For the other direction, let $f \in \mathrm{Fib} \cap W$. Since $(\mathrm{Cof}, \mathrm{tFib})$ is a WFS, there exist $i \in \mathrm{Cof}, p \in \mathrm{tFib}$ such that

$$f = p \circ i.$$

Now, $f, p \in W$, so the 2-out-of-3 property gives us that $i \in W$. Now, this implies that $i \in \mathrm{tCof}$.

Thus, f has the RLP with respect to i. A standard category theory argument shows that f is a retract of p, but since WFS is closed under retracts, we have that $f \in \mathrm{tFib}$. ■

With all the setup we have built earlier, we may give a model category structure.

Definition 7.29. A **model category** is a 1-category $\mathcal{C}$ with $W, \mathrm{Fib}, \mathrm{Cof} \subset \mathrm{Mor}(\mathcal{C})$ such that

- $\mathcal{C}$ has all finite limits and colimits.
- W has the 2-out-of-3 property.
- $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ and $(\mathrm{Cof} \cap W, \mathrm{Fib})$ are weak factorization systems.

Theorem 7.30. sSets with $W, \mathrm{Fib}, \mathrm{Cof}$ defined previously is a model category. This model category is called the **Joyal model category** and denoted sSet_J.

7.4 Fibrant Objects

Notation: We use iFib to denote the class of inner fibrations (defined in Mats' talk). We also recall from the same talk that X is an ∞-category if and only if $(!_X : X \to \Delta^0) \in \mathrm{iFib}$.

Lemma 7.31. $\mathrm{Fib} \subset \mathrm{iFib}$.

Proof Idea. One can show that the morphisms in $\ell(\mathrm{iFib})$ (a.k.a., the inner anodyne morphisms) are all trivial cofibrations. ■

For $f \in W$, we know that for all ∞-categories X, $\tau(X^f)$ is an equivalence of categories, so it is essentially surjective. For $f \in tCof$, we have a stronger result.

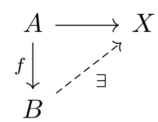
Lemma 7.32. If $f \in \mathrm{tCof}$, then $\tau(X^f)$ is surjective on objects for all ∞-categories X.

Definition 7.33. In a model category $\mathcal{C}$, an object c is **fibrant** if the unique morphism $!_c : c \to 1$ to the terminal object is a fibration.

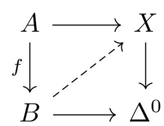
Theorem 7.34. The fibrant objects in sSet_J are exactly the ∞-categories.

Proof. Suppose X is a fibrant object in sSet_J. Then $!_x \in \mathrm{Fib} \subset \mathrm{iFib}$, which we know from Mats' talk implies that X is an ∞-category.

Conversely, suppose X is an ∞-category, and let $f : A \to B$ be a trivial cofibration. Then we know that $\tau(X^f)$ is surjective on objects. In particular, recall that on ∞-categories, τ is the homotopy category construction, so the surjectivity condition is the exact same thing as saying that $\mathrm{Hom}_{\mathrm{sSet}}(f, X)$ (pre-composition by f) is surjective. In particular, all triangles of the following form have a lift $B \to X$.



This implies that when we add the terminal object Δ^0 (which does not affect the rest of the diagram), we have a lift.



In particular, this implies that $!_X$ RLP f. Hence, we have that $!_X \in r(\mathrm{tCof}) = \mathrm{Fib}$. Thus, X is a fibrant object! ■

7.5 Combinatorial Model Categories and Localization

Given a model 1-category, we can define an associated homotopy category by "localizing the weak equivalences", but how can we do that for ∞-categories? This is the goal of this section. We first establish some setup.

Definition 7.35.

1. A 1-category $\mathcal{C}$ is **presentable** if its nerve is presentable.
2. A model category $\mathcal{C}$ is **cofibrantly generated** if it is constructed using the small object argument, i.e., there are sets $I, J \subset \mathrm{Mor}(\mathcal{C})$ (I contains generating cofibrations, J contains generating trivial cofibrations) such that $\mathrm{Cof} = \ell(r(I))$ and $\mathrm{Fib} = r(J)$.
3. A **combinatorial model category** is a cofibrantly generated model category whose underlying category is also presentable.

Example 7.36.

1. The category Set is presentable.
2. For any small category A, the presheaf category $\mathrm{Set}^{A^{\mathrm{op}}}$ is presentable. In particular, this means that sSet is presentable.

3. The Joyal model category sSet_J is a combinatorial model category.

Definition 7.37. Let C be an ∞-category and $i : \tau(C)' \to \tau(C)$ be a subcategory of its one-truncation. We can define a simplicial subset C' as the pullback:

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ N(\tau(C)') & \longrightarrow & N(\tau(C)) \end{array}$$

Here, C' is an ∞-category, and it is called the **subcategory of C spanned by** $\tau(C)'$. If $\tau(C)'$ is full, then C' is the **full subcategory** of C spanned by $\tau(C)'$.

Now we are prepared to define the notion of localization.

Definition 7.38. Let $W \subset A$ be simplicial sets. For each ∞-category X, $\mathrm{Fun}_W(A, X) \subset \mathrm{Fun}(A, X)$ is the full subcategory of functors $A \to X$ taking W to invertible morphisms.

A **localization** of A by W is the data $(L(A), \gamma : A \to L(A))$ such that

1. $L(A)$ is an ∞-category.
2. $\gamma : A \to L(A)$ takes W to invertible isomorphisms.
3. For every ∞-category X, X^γ induces an equivalence of ∞-categories $\mathrm{Fun}(L(A), X) \to \mathrm{Fun}_W(A, X)$.

When $\mathcal{C}$ is a model category, we let $L(C)$ denote the localization of $N(\mathcal{C})$ by the subcategory generated by the weak equivalences W.

Theorem 7.39. Let $\mathcal{C}$ be a combinatorial model category. Then $L(\mathcal{C})$ is presentable.

Corollary 7.40. The ∞-category of small ∞-categories $\infty\text{-Cat} = L(\mathrm{sSet}_J)$ is presentable.

Remark 7.41. Why is ∞-Cat "the same as" $L(\mathrm{sSet}_J)$? This is actually the definition of ∞-Cat in Cisinki's book, but we justify the intuition below.

In ordinary categories, it is a standard theorem in model category theory that given a model category $\mathcal{C}$, its associated homotopy category $\mathrm{Ho}(C)$ is categorically equivalent to $C_{cf}/\sim$, the category whose objects are the objects in C that are both fibrant and cofibrant and whose morphisms are suitable "homotopy classes of maps" in C. (These turn out to just be J-homotopy classes in the case of sSet_J.)

We proved in the previous section that the ∞-categories are all fibrant objects. What about cofibrant objects? Well, we observe that the simplicial set with empty fibers is the initial object and the unique map from the initial object to any other simplicial set is thus a monomorphism. This means that every object in sSet_J is cofibrant! Thus, in light of this standard theorem in model category theory, $\mathrm{Ho}(\mathrm{sSet}_J)$ is going to be the category whose objects are (small) ∞-categories and whose morphisms are J-homotopy classes of maps between ∞-categories. When we take the one-truncation of $L(\mathrm{sSet}_J)$, we are going to get $\mathrm{Ho}(\mathrm{sSet}_J)$ exactly!