8 Meeting October 31st, 2024

Title: Universal Characterization of Algebraic K-Theory
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8.1 Why Do We Care?

There are a series of conjectures in mathematics that are intricately related to the study of algebraic K-theory!

1. For people in algebraic number theory, there is a conjecture called **Kummer-Vandiver conjecture** that is very relevant to algebraic K-theory. Let $\mathbb{Q}(\zeta_p)$ be a number field, where ζ_p is a primitive p-th root of unity. In other words, $\mathbb{Q}(\zeta_p)$ is the p-th cyclotomic field.

   Conjecture 8.1. For all maximal real subfield F of $\mathbb{Q}(\zeta_p)$, let $h(F)$ be the class number of F, then p does not divide $h(F)$.

   There is an incredible result by a combination of Kurihara and Voevodsky that showed that

   Theorem 8.2. The Kummer-Vandier conjecture is true if and only if $K_{4n}(\mathbb{Z}) = 0$ for all n.

2. For people interested in geometry and topology, there is also a notion of s-cobordism theorem that relates to algebraic K-theory.

   Definition 8.3. Let W be a cobordism between M and N, we say this is an h-cobordism if the two inclusion maps $M \to W$ and $N \to W$ are homotopy equivalences.

   Note that an obvious h-cobordism is when $M = N$ and $W = M \times [0, 1]$, so W is a cylinder. One fundamental question is ask when an h-cobordism is a cylinder.

   Theorem 8.4. Let $X \hookrightarrow W$ be an h-cobordism, then the obstruction of W to cylinder lies in $K_1(\mathbb{Z}[\pi_1 X])$, this is sometimes also called the **Whitehead group**.

3. For people interested in algebraic geometry, there is also the Lichtenbaum-Quillen conjecture that relates algebraic K-theory and étale cohomology. We will not get into the details of this conjecture, but roughly speaking, the conjecture asserts that the algebraic K-theory does not satisfy étale descent. However, for large i, we have that
$$K_i(S, \mathbb{Z}/n) \cong H^{-i}_{\text{ét}}(S, F^{\text{ét}}/n).$$
   Here n is invertible in S, and $F^{\text{ét}}$ is the sheafification of the functor F, where F assigns each X to $K(X)$.

4. In fact for the algebraic geometers, there is a motivic spectral sequence (Thomason, 1985) of the form
$$H^*_{\text{ét}}(X, \underline{\pi}^{\text{ét}}_* K/p^v[\beta^{-1}]) \implies \pi_* K/p^v(X)[\beta^{-1}].$$
   Here β is called the **Bott element**. The specific details of what is on this item are omitted from this, but the key idea the audience should keep in mind is that there is a way to compute homotopy groups of K using a certain étale cohomology.

8.2 What is Algebraic K Theory?

Here we give a very concise introduction to algebraic K-theory. For a more thorough treatment, we refer to Wiebel's K book!

In this section, we fix R to be an associate unital ring with $1_R \neq 0_R$.

Definition 8.5. For $n > 0$, we define $K_n(R) = \pi_n(\mathrm{BGL}(R)^+)$. Here the plus sign "+" is Quillen's plus construction.. For $n = 0$, we define $K_0(R)$ as

$$\mathbb{Z}[\text{isomorphism classes of finite projective (left) } R\text{-modules}]/\sim$$

Here the equivalence relation is generated by $[P \oplus Q] \sim [P] + [Q]$.

We also deine $K(R)$ as $K_0 R \times \mathrm{BGL}(R)^+$.

Here are some examples of algebraic K-theory.

Example 8.6. The first major non-trivial calculation in algebraic K-theory is the K-theory of finite fields. In general, we have that

1. Let $\mathbb{F}$ be any field, then $K_0(\mathbb{F}) = \mathbb{Z}$.
2. Let $\mathbb{F}_q$ be a finite field of order q, then

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/(q^i - 1), n = 2i - 1 \\ 0, \text{ else} \end{cases}$$

   The rough idea of the proof was to use "certain operators" $\psi^q - 1 : BU \to BU$ and look at the fibers.

Remark 8.7. The previous construction of algebraic K-theory is done using Quillen's plus-construction. There are two alternative constructions via (1) Quillen's Q-construction and (2) Waldhausen's $S_\bullet$-construction. It turns out that the three constructions are equivalent!

For the sake of brevity, in this talk we will focus on Waldhausen's $S_\bullet$-construction, which is one in the setting of ∞-category. We also write Cat_∞^{st} as the ∞-categories of small stable ∞-categories, whose morphisms are exact functors (ie. preserve finite limits/colimits). In this section, we fix $\mathcal{C}$ as an object in Cat_∞^{st}.

Since $\mathcal{C}$ is stable, it is by definition pointed, and we use $*$ to denote the zero object in $\mathcal{C}$.

Definition 8.8. We define Waldhausen's $S_\bullet$-construction as follows - $S_\bullet\mathcal{C}$ is a simplicial category of the form:

- $S_0\mathcal{C} = *$.
- $S_1\mathcal{C}$ are diagrams of the form

$$\begin{array}{ccc} * & \longrightarrow & X \in \mathcal{C} \\ & & \downarrow \\ & & * \end{array}$$

  It turns out that $S_1\mathcal{C} \simeq \mathcal{C}$.
- $S_2\mathcal{C}$ are diagrams of the forward

$$\begin{array}{ccccc} * & \longrightarrow & X_{0,0} & \longrightarrow & X_{0,1} \\ & & \downarrow & & \downarrow \\ & & * & \longrightarrow & X_{1,1} \\ & & & & \downarrow \\ & & & & * \end{array}$$

where the square here is co-Cartesian (ie. pushout). It turns out that $S_1\mathcal{C} \simeq \text{Fun}(\Delta^1, \mathcal{C})$.

- In general, $S_n\mathcal{C}$ are diagrams are of the form

$$\begin{array}{ccccccccc}
* & \longrightarrow & X_{0,0} & \longrightarrow & X_{0,1} & \longrightarrow & \dots & \longrightarrow & X_{0,n-1} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & X_{1,1} & \longrightarrow & \dots & \longrightarrow & X_{1,n-1} \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & * & & \ddots & & \vdots \\
 & & & & & & & & \downarrow \\
 & & & & & & & & X_{n-1}, n \\
 & & & & & & & & \downarrow \\
 & & & & & & & & *
\end{array}$$

(Pictorially, they look like upper triangular matrices), such that each square is co-Cartesian/pushout.

Here are some important facts about this $S_\bullet$ construction:

1. All $S_n\mathcal{C}$ are stable, ie. $S_n\mathcal{C} \in \text{Cat}_\infty^{st}$.
2. We can construct the algebraic K-theory spectrum $K\mathcal{C}$ such that
$$K\mathcal{C}_n \coloneqq |(S_\bullet^{(n)}\mathcal{C})^{\simeq}|$$
Here $S_\bullet^{(n)}\mathcal{C} \coloneqq (S_\bullet \circ \dots \circ S_\bullet)(\mathcal{C})$ where we iterate the operator n times, and $(S_\bullet^{(n)}\mathcal{C})^{\simeq}$ is the sub-groupoid completion. Furthermore, the structure map is induced by
$$\Sigma(-)^{\simeq} \to |(S_\bullet\mathcal{C})^{\simeq}|$$
by restriction to 1-skeleton. Thus, we have $\Omega^\infty K\mathcal{C} \simeq \Omega|(S_\bullet\mathcal{C})^{\simeq}|$.
3. This construction K outlines a functor $K : \text{Cat}_\infty^{st} \to \text{Sp}$ that is Lax Symmetric Monoidal.
4. $K(\mathcal{C}) = K(\text{Sp}\,\mathcal{C})$. Here $\text{Sp}\,\mathcal{C}$ is the ∞-category of spectrum objects in $\mathcal{C}$, which is an ∞-functor $X : \mathbb{Z}\times\mathbb{Z} \to \mathcal{C}$ such that for all $i \neq j$, $X(i,j) = 0 \in \mathcal{C}$.

Remark 8.9. Like $K : \text{Cat}_\infty^{st} \to \text{Sp}$, there is a similar construction $\mathbb{K} : \text{Cat}_\infty^{st} \to \text{Sp}$ that produces a "non-connective" spectrum, meaning that the spectrum can have non-trivial negative homotopy groups. We omit the details of its construction in this talk here.

Remark 8.10. There is a **Dwyer-Kan (DK) simplicial localization** as follows - let $\mathcal{C}$ be a model category, then there is a way to map
$$\mathcal{C} \to N(\text{FibReplacement}(DK(\mathcal{C}, w\mathcal{C}))).$$
It turns out that the algebraic K theory spectrum produced in this Waldhausen construction may be decomposed in terms of its Dwyer-Kan simplicial localization and can lead to many interesting studies. This was the principal approach done by Blumberg and Mandell in [BM11].

8.3 Universal Property of Algebraic K-Theory

Let $\mathrm{Cat}_\infty^{perf} \subseteq \mathrm{Cat}_\infty^{st}$ be a full subcategory spanned by the **idempotent complete small stable ∞-categories**. In this case, since the ∞-categories are idempotent, we also have an adjunction

$$\mathrm{Idem} : \mathrm{Cat}_\infty^{st} \rightleftharpoons \mathrm{Cat}_\infty^{perf} : \mathrm{Forget}$$

To explain the terminology:

1. Recall that when $\mathcal{C}$ is a classical 1-category:

> **Definition 8.11.** Let $X, Y \in \mathcal{C}$, Y is called **a retract of** X if there is a diagram of the form
>
> $$\begin{array}{ccc} & & X \\ & \overset{\iota}{\nearrow} & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{id} & Y \end{array}$$
>
> Here ι is a monomorphism and r is an epimorphism. In this case, we say that $\iota \circ r$ is **idempotent**. This corresponds to our usual notion of idempotence because $(\iota \circ r)^2 = \iota \circ r$.

2. In the ∞-category sense - now let $\mathcal{C} \in \mathrm{Cat}_\infty^{st}$:

> **Definition 8.12.** Let $X, Y \in \mathcal{C}$, we say Y is a retract of X if Y is a retract of X in $h\mathcal{C}$ (the one-truncation). This is the same as saying there exists a 2-simplex $\Delta^2 \to \mathcal{C}$ corresponding to the diagram:
>
> $$\begin{array}{ccc} & & X \\ & \overset{\iota}{\nearrow} & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{id} & Y \end{array}$$
>
> We also define Idem^+ as the collection of simplicial sets such that for any finite $J \neq \emptyset$ that is totally ordered,
>
> $$\mathrm{Hom}_{Set}(\Delta^J, \mathrm{Idem}^+) = \{(J_0, \sim) : J_0 \subseteq J, \text{ and } \text{“} \sim \text{”satisfies for } i \leq j \leq k \in J, i, k \in J_0, i \sim k$$
>
> $$\text{implies } j \in J_0 \text{ and } i \sim j \sim k\}.$$
>
> From here we let $\mathrm{Idem} \subseteq \mathrm{Idem}^+$ denote the simplicial sets such that $J_0 = J$ in pairs $(J_0, \sim)$.

3. Finally, we can define what we mean by “idempotent complete”.

> **Definition 8.13.** Let $\mathcal{C} \in \mathrm{Cat}_\infty^{st}$, we say $\mathcal{C}$ is **idempotent complete** if for all $F \in \mathrm{Fun}(\mathrm{Idem}, \mathcal{C})$, F is effective. By effective, we mean that F can be extended to $\mathrm{Fun}(\mathrm{Idem}^+, \mathcal{C})$.

4. The Idem functor sends $\mathcal{C}$ to its idempotent completion.

After explaining the terminologies, we need three more definitions.

> **Definition 8.14.** Let $f : \mathcal{C} \to \mathcal{D} \in \mathrm{Cat}_\infty^{st}$ be a functor. We say that <u>f is a Morita equivalence</u> if $\mathrm{Idem}\, f : \mathrm{Idem}\, \mathcal{C} \to \mathrm{Idem}\, \mathcal{D}$ is an equivalence.

Definition 8.15. Consider the composition of functors in Cat_∞^{st}

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{E}$$

We say that this is an **exact sequence** if:

1. f is fully-faithful.
2. $\mathcal{D}/\mathcal{C} \simeq \mathcal{E}$. Here by the quotient $\mathcal{D}/\mathcal{C}$, we mean the pushout in the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{D}/\mathcal{C} \end{array}$$

3. $g \circ f = 0$.

This exact sequence splits if there is a section $s : \mathcal{E} \to \mathcal{D}, s' : \mathcal{D} \to \mathcal{C}$ (here a section should be thought of as a right adjoint going back such that composition is equivalent to identity via the adjunction morphism).

Definition 8.16. A functor $F : \mathrm{Cat}_\infty^{st} \to \mathrm{Sp}$ (into the stable ∞-category of spectra) is called an **additive invariant** if:

1. F inverts Morita equivalence.
2. F preserves filtered colimits.
3. F maps split exact sequence to split cofiber sequence.

F is called an **localizing invariant** if the first two conditions above holds, and (3') F takes exact sequence to cofiber sequence.

Theorem 8.17. The algebraic K-theory construction K taking a small stable ∞-category to an algebraic K-theory spectra is an additive invariant. If we take the non-connective construction of the spectrum, the functor $\mathbb{K}$ is a localizing invariant.

Remark 8.18. Topological Hochschild homology is an additive invariant.

Notation: Let us write $\mathrm{PSh}^+_{Sp}(\mathrm{Cat}_\infty^{st})$ be the category of presheaves

$$F : ((\mathrm{Cat}_\infty^{st})^W)^{op} \to \mathrm{Sp}$$

such that Condition (3) (not (3')) in Definition 8.16 is satisfied. Also let $\mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W)$ be the category of presheaves to be the same as the plus version, but without requiring condition (3). Here $(\mathrm{Cat}_\infty^{st})^W$ is the full subcategory given by compact objects.

There is a forgetful functor

$$\mathrm{Forget} : \mathrm{PSh}^+_{Sp}(\mathrm{Cat}_\infty^{st}) \to \mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W)$$

where we just forget about Condition (3). It turns out this admits an adjoint $L^+ : \mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W) \to \mathrm{PSh}^+_{Sp}(\mathrm{Cat}_\infty^{st})$

Definition 8.19. Let L^+ be the adjoint as above, we can define a map M_+ as follows.

$$M_+ : \quad \mathrm{Cat}_\infty^{st} \xhookrightarrow{\text{Yoneda}} \mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W) \xrightarrow{L^+} \mathrm{PSh}_{Sp}^+(\mathrm{Cat}_\infty^{st})$$

$$\mathcal{C} \mapsto M_+(\mathcal{C})$$

This map is called the **additive non-commutative motive**.

Remark 8.20. There is a similar map M_{loc} we can define for **localizing non-commutative motive**.

Finally, we are ready to state the main theorem.

Theorem 8.21 (Blumberg-Gepner-Tabuada, 2013, [BGT13])**.** For all $\mathcal{C} \in \mathrm{Cat}_\infty^{perf}$, there exists two natural equivalences:

1. $\mathrm{Map}(M_+(Sp^W), M_+(\mathcal{C})) \simeq K(\mathcal{C})$.
2. $\mathrm{Map}(M_{loc}(Sp^W), M_{loc}(\mathcal{C})) \simeq \mathbb{K}(\mathcal{C})$.

Here Sp^W is the full subcategory given by compact objects. This is actually called the ∞**-category of finite spectra**. Equivalently, this means that

$$\Psi : \mathrm{PSh}_{Sp}^+(\mathrm{Cat}_\infty^{st}) \to \mathrm{Sp}$$

is co-representable. There is a similar story that happens with localizing invariants.

From here, we obtain three corollaries.

Corollary 8.22. For all $n \in \mathbb{Z}$,

$$K_n\mathcal{C} \simeq \mathrm{Hom}(M_+(Sp^W), \Sigma^{-n} M_+\mathcal{C})$$

Corollary 8.23. For any additive invariant F,

$$\mathrm{Map}(K, F) = F(\mathrm{Sp}^W)$$

This means that K is the universal additive invariant!

Similarly, we also have that

Corollary 8.24. For any localizing invariant F,

$$\mathrm{Map}(\mathbb{K}, F) = F(\mathrm{Sp}^W)$$

This means that $\mathbb{K}$ is the universal localizing invariant!

Now we give an outline for the proof of Theorem 8.21 for the additive case.

Proof Sketch of Theorem 8.21. We only sketch the proof of the additive case. Indeed, for all $\mathcal{A} \in \mathrm{Cat}_\infty^{st}$, $\mathcal{B} \in \mathrm{Cat}_\infty^{perf}$ with B compact in $\mathrm{Cat}_\infty^{perf}$, then one can show that $M_+(\mathcal{A}) \simeq K_\mathcal{A}$. Here $K_\mathcal{A}$ is defined as follows:

- $K_\mathcal{A} \in \mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W)$
- $K_\mathcal{A}(\mathcal{C}) = K(\mathrm{Fun}^{ex}(\mathcal{C}, \mathrm{Idem}\,\mathcal{A}))$. Here, Fun^{ex} denote the exact functors.
- Note that $K_\mathcal{A}(\mathrm{Sp}^W) = K(\mathcal{A})$.
- One can also show that the functor $K_\mathcal{A}$ is local. In other words, for all split exact sequence $\mathcal{B} \to \mathcal{C} \to \mathcal{D}$ in $(\mathrm{Cat}_\infty^{perf})^W$, there is an equivalence

  $$\mathrm{Map}(\psi(\mathcal{D}), K_\mathcal{A}) \simeq \mathrm{Map}(\psi(\mathcal{C})/\psi(\mathcal{A}), K_\mathcal{A}) \quad (\dagger).$$

  where $\psi : \mathrm{Cat}_\infty^{perf} \to \mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{perf})^W)$ is the Yoneda embedding.

Thus, we have that

$$\begin{aligned} \mathrm{Map}(M_+(\mathcal{B}), M_+(\mathcal{A})) &\simeq \mathrm{Map}(M_+(\mathcal{B}), K_\mathcal{A}) && \text{Recall } M_+(\mathcal{A}) \simeq K_\mathcal{A} \\ &= \mathrm{Map}(L_+ \circ \psi(\mathcal{B}), K_\mathcal{A}) \\ &= \mathrm{Map}(\psi(\mathcal{B}), K_\mathcal{A}). \end{aligned}$$

Here, the maps in the last line should be thought of as happening in $\mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W)$. We obtained the last equality using the adjunction between L_+ and Forget. Thus, we have that

$$\mathrm{Map}(M_+(\mathcal{B}), M_+(\mathcal{A})) \simeq \mathrm{Map}(\psi(\mathcal{B}), K_\mathcal{A})).$$

From here, since $\mathcal{B}$ is in $(\mathrm{Cat}_\infty^{perf})$ and is compact, $\psi(\mathcal{B})$ is representable.

It turns out there is a theorem called the **spectral Yoneda lemma**. In this case, when we apply the spectral Yoneda lemma to $\psi(B)$, we have that $\mathrm{Map}(\psi(\mathcal{B}), K_\mathcal{A})) \simeq K_\mathcal{A}(\mathcal{B})$. Plugging $\mathcal{B} = Sp^W$ then obtains the proof of the additive case. ■

Remark 8.25. While we did not define THH (topological Hochschild Homology), we do in fact that that

$$\pi_0 \,\mathrm{Map}(K, THH) = \pi_0(THH(Sp^W)) \simeq \pi_0(THH(\mathbb{S})) = \mathbb{Z}.$$

The element 1 in $\mathbb{Z}$ corresponds to a unique map $K \to$ THH which is called the **Dennis trace**.