9 Meeting November 7th, 2024

Speaker: Mattie Ji
Title: Introduction to Algebraic L-Theory

9.1 Motivations for Algebraic L-Theory

Algebraic L-theory is often called an analog of K-theory for modules equipped with quadratic forms. Let us first offer some geometric motivations for how studying quadratic forms arose in a geometric setting.

Let M^n be a closed orientable connected manifold. The classic Poincaré duality in algebraic topology asserts the following.

Theorem 9.1. For all $k \in \mathbb{Z}$, $H^{n-k}(M; \mathbb{Z}) \cong H_k(M; \mathbb{Z})$.

Poincaré duality is quite useful in the study of 4-manifolds in low-dimensional topology. More generally, when $n = 4k$ is a multiple of 4, Poincaré duality provides the following invariant on M.

Theorem 9.2. There exists an element $[M] \in H_n(M; \mathbb{R})$ such that the following is a non-degenerate symmetric bilinear form:

$$\langle \bullet, \bullet \rangle_M : \qquad H^{2k}(M; \mathbb{R}) \times H^{2k}(M; \mathbb{R}) \xrightarrow{\cup} H^{4k}(M; \mathbb{R}) \xrightarrow{\bullet \cap [M]} H_0(M; \mathbb{R}) \cong \mathbb{R}$$

The non-degenerate quadratic form associated to the manifold M is an invariant, and we can assign an invariant to it.

Definition 9.3. We can choose a basis $x_1, ..., x_a, y_1, ..., y_b$ ($a + b = \dim_{\mathbb{R}} H^{2k}(M; \mathbb{R})$) of $H^{2k}(M; \mathbb{R})$ such that $\langle x_i, x_i \rangle = 1$ (ie. positive eigenvalues) and $\langle y_i, y_i \rangle = -1$ (ie. negative eigenvalues). The difference $a - b$ is called the **signature of** M. Note that the sign of the signature depends on the choice of the fundamental class, so we refer to the signature modulo sign.

It is a standard fact in linear algebra that a non-degenerate quadratic form over the reals is completely determined by its dimension and signature.

When $n = 4k + 2$, in this case the middle cup product $H^{2k+1}(M; \mathbb{R}) \times H^{2k+1}(M; \mathbb{R}) \to H^{4k+2}(M; \mathbb{R})$ is no longer commutative but anti-commutative. Over characteristic 2, there is no distinction. Poincaré stills gives the following theorem.

Theorem 9.4. There exists an element $[M] \in H_n(M; \mathbb{Z}/2\mathbb{Z})$ such that the following is a non-degenerate symmetric bilinear form:

$$\langle \bullet, \bullet \rangle_M : \qquad H^{2k+1}(M; \mathbb{Z}/2\mathbb{Z}) \times H^{2k+1}(M; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cup} H^{4k+2}(M; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\bullet \cap [M]} H_0(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

There is an analog of signature for $4k$-manifolds in the case of $4k + 2$-manifolds known as the Kervaire/Arf-Invariant.

Definition 9.5. A theorem by Arf shows that there is a basis $\{e_1, f_1, ..., e_r, f_r, g_1, ..., g_s\}$ such that the quadratic form associated to $\langle \bullet, \bullet \rangle_M$ may be rewritten as

$$(x_1, y_1, ..., x_r, y_r, z_1, ..., z_s) \mapsto \sum_{i=1}^{r}(a_i x_i^2 + x_i y_i + b_i y_i^2) + \sum_{j=1}^{s} g_j z_j^2$$

The **Kervaire/Arf invariant** is defined as $\sum_{i=1}^{r} a_i b_i$.

Theorem 9.6 (Arf)**.** A non-degenerate bilinear form over $\mathbb{Z}/2\mathbb{Z}$ is completely determined by its dimension and Arf invariant.

Remark 9.7. Before we move on, we briefly discuss one more observation about symmetric bilinear forms and quadratic forms that is reformulated as follows. If R is a commutative ring and let $\mathrm{Proj}(R)$ be the category of finitely generated projective R-modules. We observe that for $P \in \mathrm{Proj}(R)$

1. $\mathrm{Hom}_{R\otimes R}(P \otimes P, R)$ is the collection of bilinear R-valued forms on P.
2. There is an obvious action of C_2 on $\mathrm{Hom}_{R\otimes R}(P \otimes P, R)$, from which we have two canonical identifications

$$\mathrm{Hom}_{R\otimes R}(P \otimes P, M)^{C_2} \text{ are the symmetric bilinear } R\text{-valued forms on } P,$$

$$\mathrm{Hom}_{R\otimes R}(P \otimes P, M)_{C_2} \text{ are the quadratic } R\text{-valued forms on } P.$$

More generally, we could replace R with an R-module M in the items listed above. If we are considering an involution as well, we could also produce skew-symmetric and skew-quadratic forms from this identification.

9.2 Symmetric Bilinear and Quadratic Functors

Thus, the study of quadratic forms and symmetric bilinear forms arises quite naturally in algebraic topology and low-dimensional topology. It is then natural to ask - is there an ∞-categorification of these concepts? The hope is that, perhaps by abstracting the theory, we can study broader problems with similar phenomenon and make previous concrete problems easier.

This is where **Algebraic L-theory** comes in, but, to explain what algebraic L-theory is, we should first define our suitable generalizations of symmetric bilinear forms and quadratic forms in ∞-category theory.

Throughout this section, every ∞-category is a **stable ∞-category**.

- Recall $\mathcal{C}$ being stable means that it is pointed, fibers and cofibers exist, and a triangle is a fiber sequence if and only if it is a cofiber sequence.
- Equivalently, a pointed category $\mathcal{C}$ is stable if it admits finite limits and colimits, and a square is a pushout if and only if it is a pullback (Definition 5.11 of Gallauer).
- There are two canonical functors in $\mathcal{C}$ known as the loop functor Ω and the suspension functor Σ.
- The stable ∞-category of spectra Sp is a canonical example of stable ∞-categories.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories, we say F is **reduced** if it sends the zero object to the zero object (ie. $F(0) = 0$). We say a reduced functor is **exact** if it takes fiber sequences to fiber sequences.

Recall a **fiber sequence** is given by the homotopy pull-back

$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

Let us recall that for any stable ∞-category $\mathcal{C}$ with object X. For any other object X, there is a sequence of mapping spaces $\{\mathrm{Map}_{\mathcal{C}}(Y, \Sigma^n X)\}$ that constitutes a spectrum we will write as $\mathrm{Mor}_{\mathcal{C}}(Y, X)$. A mapping space $Map_{\mathcal{C}}(c, d)$ is given by the pullback:

$$\begin{tikzcd} Map_{\mathcal{C}}(c,d) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "(c,d)"'] & \mathrm{Fun}(\partial\Delta^1, \mathcal{C}) \end{tikzcd}$$

Definition 9.8 (Symmetric and Non-degenerate Bilinear Functors)**.** A **bilinear functor** is a functor $B : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ such that for all $c \in \mathcal{C}$, the following two functors are both exact,

$$d \mapsto B(c, d), d \mapsto B(d, c).$$

We use $\mathrm{Fun}^b(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp})$ to denote the full subcategory given by the bilinear functors.

Symmetric: There is a C_2 action on $\mathrm{Fun}^b(\mathcal{C})$ by flipping the two entries, we use

$$\mathrm{Fun}^s(\mathcal{C}) = [\mathrm{Fun}^b(\mathcal{C})]^{hC_2}$$

to denote the ∞-category of C_2-equivariant objects in $\mathrm{Fun}^b(\mathcal{C})$. A bilinear functor $B \in \mathrm{Fun}^s(\mathcal{C})$ is called **symmetric**. Note that this is also called the homotopy fixed point spectra with respect to C_2.

Non-degenerate: Let $B \in \mathrm{Fun}^b(\mathcal{C})$ be a bilinear functor.

1. We say that B is **right non-degenerate** if for each $Y \in \mathcal{C}$, the functor $B(-, Y)$ is representable by an object in $\mathcal{C}$. In other words, we can write

$$B(X, Y) \simeq \mathrm{Mor}_{\mathcal{C}}(X, D^{right} Y).$$

Here $D^{rigt} : \mathcal{C}^{op} \to \mathcal{C}$ is a functor keeping track of the representation.

2. We say that B is **left non-degenerate** if for each $X \in \mathcal{C}$, the functor $B(X, -)$ is representable by an object in $\mathcal{C}$. In other words, we can write

$$B(X, Y) \simeq \mathrm{Mor}_{\mathcal{C}}(Y, (D^{left})^{op} X) \simeq \mathrm{Mor}_{\mathcal{C}^{op}}(D^{left} X, Y).$$

Here $D^{left} : \mathcal{C} \to \mathcal{C}^{op}$ is a functor keeping track of the representation.

3. We say B is **non-degenerate** if it is both left and right non-degenerate. From definition, we can see that D^{left} and D^{right} are adjoint as

$$\mathrm{Mor}_{\mathcal{C}^{op}}(D^{left} X, Y) \simeq B(X, Y) \simeq \mathrm{Mor}_{\mathcal{C}}(X, D^{right} Y).$$

4. When B is symmetric and non-degenerate (notation: $B \in \mathrm{Fun}^{sn}(\mathcal{C})$), we write D as D^{right}. We note that D^{left} is actually D^{op} since

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}^{op}}(D^{left} X, Y) &\simeq B(X, Y) \\ &\simeq B(Y, X) \\ &\simeq \mathrm{Mor}_{\mathcal{C}}(Y, DX) \\ &\simeq \mathrm{Mor}_{\mathcal{C}^{op}}(D^{op} X, Y). \end{aligned}$$

Perfect: Let $B \in \mathrm{Fun}^{sn}(\mathcal{C})$, from the discussion above we know that D^{op} is adjoint to D. The unit of this adjunction gives an **evaluation map**:

$$\mathrm{ev} : \mathrm{id} \implies DD^{op}.$$

We say that B is **perfect** if ev is an equivalence.

Example 9.9 (Spainer-Whitehead Duality). Let $\mathcal{C} = \mathrm{Sp}$ and B be

$$B(X, Y) = \mathrm{Mor}_{Sp}(X \wedge Y, S).$$

B is a symmetric non-degenerate bilinear functor on $\mathcal{C}$. The corresponding duality functor D is called the **Spainer-Whitehead Duality**.

The restriction of B to Sp^{ω} (full subcategory spanned by the compact objects in Sp, in other words, the finite spectra) is **perfect**.

We also want to establish the analog of a quadratic form in ∞-categories. Motivated by the story in linear algebra, we consider the following construction.

Construction 9.10. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a reduced functor. For $X, Y \in \mathcal{C}$, we have consider maps

$$Q(X) \oplus Q(Y) \to_f Q(X \oplus Y) \to_g Q(X) \oplus Q(Y)$$

We note that up to equivalence, Remark 1.1.3.5 of Lurie tells us that $Q(X) \oplus Q(Y)$ is both the product and coproduct of $Q(X)$ and $Q(Y)$. From universal property, we have the following maps:

$$\begin{tikzcd}
 & & X & & \\
X \arrow[urr, "id_X"] \arrow[rr, "i_X"] \arrow[drr, "0"'] & & X \oplus Y \arrow[u, dashed, "f_X"'] \arrow[d, dashed, "f_Y"] & & Y \arrow[ull, "0"'] \arrow[ll, "i_Y"'] \arrow[dll, "id_Y"] \\
 & & Y & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & X \arrow[dll, "id_X"'] \arrow[d, dashed, "g_X"'] \arrow[drr, "0"] & & \\
X & & X \oplus Y \arrow[ll, "q_X"'] \arrow[rr, "q_Y"] & & Y \\
 & & Y \arrow[ull, "0"] \arrow[u, dashed, "g_Y"] \arrow[urr, "id_Y"'] & &
\end{tikzcd}$$

Since Q is contravariant, we obtain maps

1. $Q(f_X) : Q(X) \to Q(X \oplus Y)$ and $Q(f_X) : Q(X) \to Q(X \oplus Y)$, which induces the map $f : Q(X) \oplus Q(Y) \to Q(X \oplus Y)$ by universal property.
2. $Q(g_X) : Q(X \oplus Y) \to Q(X)$ and $Q(g_Y) : Q(XY) \to Q(Y)$, which induces the map $g : Q(X \oplus Y) \to Q(X) \oplus Q(Y)$ by universal property.

Schematically, we can think of $g \circ f$ as the matrix

$$\begin{pmatrix} Q(id_X) & Q(0) \\ Q(0) & Q(id_Y) \end{pmatrix}$$

Proposition 9.11. The composition $g \circ f$ is the identity, and this makes $Q(X) \oplus Q(Y)$ a direct summand of $Q(X \oplus Y)$. In particular, this gives a symmetric (in its arguments) functor $B : C^{op} \times C^{op} \to \mathrm{Sp}$ such that

$$Q(X \oplus Y) \simeq Q(X) \oplus Q(Y) \oplus B(X, Y).$$

B is called the **polarization of** Q.

> **Remark 9.12.** The proposition is really an analog of the following idea in linear algebra - if $q(x)$ is a quadratic form, then the term $q(x+y) - q(x) - q(y)$ is a symmetric bilinear function.

Proof Idea. We said earlier that $g \circ f$ should be schematically thought of as the matrix

$$\begin{pmatrix} Q(id_X) & Q(0) \\ Q(0) & Q(id_Y) \end{pmatrix}.$$

Since Q is a reduced functor, this matrix becomes

$$\begin{pmatrix} id_{Q(X)} & 0 \\ 0 & id_{Q(Y)} \end{pmatrix},$$

which is clearly the identity. More rigorously, the universal property tells us that the identity map is the unique map satisfying

$$\begin{tikzcd}
Q(X) \arrow[d, "j_X"'] \arrow[dr, "j_X"] & \\
Q(X) \oplus Q(Y) \arrow[r, dashed, "id"] & Q(X) \oplus Q(Y) \\
Q(Y) \arrow[u, "j_Y"] \arrow[ur, "j_Y"'] &
\end{tikzcd}$$

It suffices for us to show this diagram holds when we replace the identity map by $g \circ f$. Now we currently have a diagram of the form

$$\begin{tikzcd}
Q(X) \arrow[d, "j_X"] \arrow[dr, "Q(f_X)"] & & Q(X) \\
Q(X) \oplus Q(Y) \arrow[r, "f"] & Q(X \oplus Y) \arrow[r, "g"] \arrow[ur, "Q(g_X)"] \arrow[dr, "Q(g_Y)"'] & Q(X) \oplus Q(Y) \arrow[u, "r_X"'] \arrow[d, "r_Y"] \\
Q(Y) \arrow[u, "j_Y"] \arrow[ur, "Q(f_Y)"'] & & Q(Y)
\end{tikzcd}$$

Let us try to compute the term $g \circ f \circ j_X$. Now we see that

$$r_X \circ (g \circ f \circ j_X) = Q(g_X) \circ Q(f_X) = Q(g_X \circ f_X) = id_{Q(X)}$$

$$r_Y \circ (g \circ f \circ j_X) = Q(g_Y) \circ Q(f_X) = Q(0) = 0.$$

Thus, $g \circ f \circ j_X$ is the induced map in the diagram

$$\begin{tikzcd}
Q(X) & \\
Q(X) \oplus Q(Y) \arrow[u, "r_X"] \arrow[d, "r_Y"'] & Q(X) \arrow[l, dashed] \arrow[ul, "id_{Q(X)}"'] \arrow[dl, "0"] \\
Q(Y) &
\end{tikzcd}$$

But j_X is the other map that satisfies this ($Q(X) \oplus Q(Y)$ is both the product and the coproduct), so we have that $g \circ f \circ j_X = j_X$. Similarly, we also have that $g \circ f \circ j_Y = j_Y$, so we conclude that $g \circ f$ is the identity.

Showing that $Q(X) \oplus Q(Y)$ is a direct summand of $Q(X \oplus Y)$ follows more generally from the following fact - let $f : X \to Y, g : Y \to X$ between spectrum such that $g \circ f$ is the identity, then X is a direct summand of Y. To see why, let C_f be the cofiber of $f : X \to Y$ and Z be any spectrum, we have an exact sequence

$$0 \to [Z, X] \to [Z, Y] \to [Z, C_f] \to 0.$$

The existence of g can show that this is in fact split injective! Since we are really looking at cohomology here, this gives us

$$[Z, Y] \cong [Z, X] \oplus [Z, C_f].$$

On the other hand, recall that the coproduct is wedge sum, so we have that

$$[Z, X \vee C_f] \cong [Z, X] \oplus [Z, C_f].$$

Since this holds for all Z, the Yoneda lemma implies that $X \vee C_f \simeq Y$. ■

There is a canonical map we are interested in between Q and B. To reach there we first need to briefly discuss the notion of homotopy fixed points and homotopy orbit. We will not go too into details for the definition, so is life, but we will give two examples to help parse with the definition. We will also only talk about the specific case for C_2.

Definition 9.13. Let X be a spectrum equipped with a C_2-action, in a natural way compatible to X. Then, the **homotopy fixed point spectrum** of X with respect to C_2 is

$$X^{hC_2} = \mathrm{Fun}_G(\Sigma^\infty (EC_2)_+, X)$$

is the mapping space of C_2-equivariant maps between the two spectra.

The **homotopy orbit spectrum** of X with respect to C_2 is

$$X_{hC_2} = \Sigma^\infty (EC_2)_+ \wedge_{C_2} X.$$

Here the wedge product is taken with respect in C_2-spectra.

Example 9.14. Here are two examples whose proofs might not be that obvious

1. Let KU and KO be the complex and real K-theory spectra respectively. There is a C_2-action on KU by replacing a complex vector bundle with its complex conjugate bundle, and $KU^{hC_2} = KO$.
2. On the level of spaces, the homotopy orbit of a one-point space $*$ under C_2 is $\mathbb{R}P^\infty$.

Construction 9.15. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a reduced functor with polarization B. The diagonal map $\Delta : X \to X \oplus X$ and codiagonal map $\nabla : X \oplus X \to X$ induces maps

$$Q(X \oplus X) \xrightarrow{Q(\Delta)} Q(X) \xrightarrow{Q(\nabla)} Q(X \oplus X)$$

There is an inclusion map $i : B(X, X) \to Q(X \oplus X)$ and a projection map $\pi : Q(X \oplus X) \to B(X, X)$, so we can extend this sequence to

$$B(X, X) \xrightarrow{i} Q(X \oplus X) \xrightarrow{Q(\Delta)} Q(X) \xrightarrow{Q(\nabla)} Q(X \oplus X) \xrightarrow{\pi} B(X, X)$$

There is a canonical C_2 action on $B(X, X)$ roughly described as follows. B is symmetric in the higher categorical sense, meaning we are given an isomorphism between $B(X, Y)$ and $B(Y, X)$. When $X = Y$, this becomes an automorphism on $B(X, X)$ that defines a C_2 action. An alternative way to phrase this is that $\Delta(B)$ is a C_2 object of $\mathrm{Fun}(\mathcal{C}^{op}, Sp)$, where $\Delta : \mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{C}^{op}, Sp) \to \mathrm{Fun}(\mathcal{C}^{op}, Sp)$ is the restriction to the diagonal.

Furthermore, since $Q(\nabla)$ and $Q(\Delta)$ are both C_2-equivariant, the diagram above factors through as

$$B(X, X)_{hC_2} \longrightarrow Q(X) \longrightarrow B(X, X)^{hC_2}$$

Remark: The composition here is the norm map.

Thus, we have showed that every reduced functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ can produce an associated functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$. The definition of a quadratic functor is given as follows:

Definition 9.16. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a reduced functor with polarization B. We say Q is **quadratic** if any of the two equivalent conditions is true

1. B is bilinear and the functor $X \mapsto \mathrm{fib}(Q(X) \to B(X,X)^{hC_2})$ is exact.
2. B is bilinear and the functor $X \mapsto \mathrm{cofib}(B(X,X)_{hC_2} \to Q(X))$ is exact.

Furthermore, Q is perfect if its polarization B is perfect.

Remark 9.17. A **quadratic functor** in our talk is really what Thomas Goodwillie would call "a (reduced) and 2**-excisive functor**" in the framework of Goodwillie Calculus.

Example 9.18. Here are some examples of quadratic functors:

1. Any exact functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ is quadratic. In fact, they correspond to all the quadratic functors whose polarization vanishes.
2. Let $\mathcal{C}$ be a stable ∞-category and $B \in \mathrm{Fun}^{bs}(\mathcal{C})$, then

$$Q^q_B(X) = B(X,X)_{hC_2} \text{ and } Q^s_B(X) = B(X,X)^{hC_2}$$

   are quadratic functors. Q^q_B is the analog of **quadratic form** and Q^s_B is the analog of **symmetric bilinear form**.

These two constructions should be reminiscent of Remark 9.7.

9.3 L-Theory of Poincare Category

In this section, we will be working to define the L-theory of a Poincare ∞-category $(\mathcal{C}, Q)$, where Q is **perfect**.

Definition 9.19. A Poincare ∞-category is a pair $(\mathcal{C}, Q)$ where Q is perfect.

Definition 9.20. Let E be an Ω-spectrum, we define $\Omega^\infty E = E_0$ (the 0-th space). For a general spectrum E', there is a canonical way to produce an associated Ω-spectrum E of E' by specifying

$$E_n = \mathrm{colim}_k\, \Omega^k E'_{n+k}.$$

In this case, we define $\Omega^\infty E'$ as $\Omega^\infty E$.

Remark 9.21. We justify the notation Ω^∞ as follows. There is a classical correspondence between an Ω-spectrum and an infinite loop space. Given an infinite loop space X, we can think of X as a sequence of delooping $X_0 = X \to X_1 \to \ldots$ with weak equivalences $X_n \simeq \Omega X_{n+1}$. Thus, given an Ω-spectrum, its sequence of spaces naturally produces an infinite delooping of the 0-th space.

Definition 9.22. A **quadratic object** of $(\mathcal{C}, Q)$ is a pair $(X \in \mathcal{C}, q \in \Omega^\infty Q(X))$.

Recall there is a map $f : Q(X) \to B(X,X)^{hC_2}$, so q determines a point $f(q) \in B(X,X)_0^{hC_2}$ (the 0-th space). Since Q is non-degenerate, we recall that $B(X,X) \simeq \mathrm{Mor}_{\mathcal{C}}(X, DX)$, so $f(q)$ determines a map $X \to DX$. We say that (X,q) is a **Poincaré object** if $X \to DX$ is invertible. We use $\mathrm{Poin}(\mathcal{C}, Q)$ to denote the collection of Poincare objects.

According to Lurie - the intuition to have in mind is that Q is a functor that assigns each object $X \in \mathcal{C}$ a "spectrum of quadratic forms". A quadratic object (X,q) can be thought of as a specific choice of quadratic form for X. A Poincare object (X,q) is a specific choice of a nondegenerate quadratic form.

Example 9.23. The formation of this mapping spectra in $\mathcal{C}$ gives a quadratic functor

$$Q_{hyp} : \mathcal{C} \times \mathcal{C}^{op} \to \mathrm{Sp}, (X,Y) \mapsto \mathrm{Mor}_{\mathcal{C}}(X,Y).$$

In this case, $(\mathcal{C} \times \mathcal{C}^{op}, Q_{hyp})$ is Poincare with duality given by $(X,Y) \mapsto (Y,X)$. $(\mathcal{C} \times \mathcal{C}^{op}, Q_{hyp})$ is called the **hyperbolic ∞-category** associated to $\mathcal{C}$.

Here we give a concrete example of how Poincare objects relate to the geometric setting of manifolds.

Definition 9.24. Let A be an associative ring, **the perfect derived ∞-category** of A is an ∞-category $D^{perf}(A)$ is the full subcategory of $D(A)$ spanned by compact objects. Here $D(A)$ is the derived ∞-category of A (ie. $D(A) = N\,\mathrm{Ch}(A)[\text{quasi-iso}^{-1}]$). Concretely, $D^{perf}(A)$ is roughly constructed as follows:

1. The 0-simplicies of $D^{perf}(A)$ are bounded chain complexes of finitely generated projective left A-modules.
2. A 1-simplex of $D^{perf}(A)$ is the map of chain complexes $f : P_\bullet \to Q_\bullet$.
3. A 2-simplex of $D^{perf}(A)$ is a (not necessarily commutative) diagram of chain complexes

$$\begin{tikzcd} & Q_\bullet \arrow[dr, "g"] & \\ P_\bullet \arrow[ur, "f"] \arrow[rr, "h"'] & & R_\bullet \end{tikzcd}$$

   with a chain homotopy from h to $g \circ f$.
4. Higher dimensional simplicies are given analogously with higher-order chain homotopies.

Note that $D^{perf}(A)$ is clearly stable.

Example 9.25. Informally, recall there is a canonical mapping spectrum Mor attached to any stable ∞-category. Specifically we consider B on $D^{perf}(R)$ given by

$$B^i(X,Y) = \mathrm{Mor}_{R \otimes R}(X \otimes_R Y, R[-i]).$$

Here $R[-i]$ is the chain complex that is everywhere zero except for a single copy of R concentrated at the $-i$ degree.

There is an obvious duality given by the Hom-Tensor adjunction, and the associated $Q_R^{q,i}$ and $Q_R^{s,i}$ are both (perfect) quadratic functors. Here we append an index i to indicate that we are considering morphisms into $R[-i]$. We also write $Q_R^q = Q_R^{q,0}$ and $Q_R^s = Q_R^{s,0}$.

The following example is arguably the most important example of this talk. If the reader should get anything out of this talk, it should be this key example.

Example 9.26. Let $\mathcal{C} = D^{\mathrm{perf}}(\mathbb{Z})$ and define

$$Q(X) \coloneqq \mathrm{Mor}_{D^{\mathrm{perf}}(\mathbb{Z})}(X \otimes X, \mathbb{Z}[-n])^{hC_2}.$$

(Note that Q is $Q_{\mathbb{Z}}^{s,n}$ from our earlier example).
Let M^n be a closed oriented manifold. The singular cochain complexes $C^*(M,\mathbb{Z})$ is an object of $D^{perf}(\mathbb{Z})$. There is a quadratic functor Q on $D^{\mathrm{perf}}(\mathbb{Z})$ given by

$$Q(X) \coloneqq \mathrm{Mor}_{D^{\mathrm{perf}}(\mathbb{Z})}(X \otimes X, \mathbb{Z}[-n])^{hC_2}.$$

Here $\mathbb{Z}[-n]$ is a chain complex that is all zero except for a single copy of $\mathbb{Z}$ at degree $-n$.
In this case, we have a symmetric intersection pairing on M:

$$(C^*(M;\mathbb{Z}) \otimes C^*(M;\mathbb{Z}))_{hC_2} \to C^*(M;\mathbb{Z}) \to_{[M]} \mathbb{Z}[-n]$$

is a point $q_M \in \Omega^\infty Q(C^*(M;\mathbb{Z}))$. In this example, the statement of Poincare duality may be reformulated as follows:

Theorem 9.27 (Poincare Duality Reformulated)**.** $(C^*(M;\mathbb{Z}), q_M)$ is a Poincare object of $(D^{perf}(\mathbb{Z}), Q)$.

Note that we shifted the index to $-n$ because we defined everything at the 0-th space. In general, a Poincare object of dimension n is a Poincare object of dimension 0 with the index shifted down by n.

Our goal is to now construct a suitable algebraic structure on the collection of Poincaré objects to study them.

Definition 9.28. Let (X,q) and (X',q') be two quadratic (resp. Poincare) objects on $(\mathcal{C},Q)$. We define

$$(X,q) \oplus (X',q') \coloneqq (X \oplus X', q \oplus q').$$

Here $X \oplus X'$ is the standard (co)product of X and X', and $q \oplus q'$ is the image of (q,q') under the canonical map $Q(X) \oplus Q(X') \to Q(X \oplus X')$. It is a fact that $(X \oplus X', q \oplus q')$ is quadratic (resp. Poincare).

The operation $\oplus$ only gives a commutative monoid structure on the collection of Poincare objects. We want a suitable notion of equivalence so that this becomes a group structure.

Definition 9.29. Let $(\mathcal{C},Q)$ be as before, and $(X,q),(X',q')$ be two Poincare objects. An (algebraic) **cobordism** from (X,q) to (X',q') is the following data:

1. An object $L \in \mathcal{C}$ with maps $\alpha : L \to X$ and $\alpha' : L \to X'$.
2. Q induces maps $Q(X) \to Q(L)$ and $Q(X') \to Q(L)$. Let $\alpha^*(q), (\alpha')^*(q')$ be the images of q and q' be the images in the space $\Omega^\infty Q(L)$. We also want a path p joining $\alpha^*(q)$ and $(\alpha')^*(q')$.
3. (Non-degeneracy condition): The path gives a homotopy between the two maps $L \to D(L)$ given by:

$$\begin{tikzcd} X \arrow[d, "\text{induced by q}"'] & L \arrow[l, "\alpha"'] \arrow[r, "\alpha'"] & X' \arrow[d, "\text{induced by q'}"] \\ DX \arrow[r, "D(\alpha)"'] & DL & DX' \arrow[l, "D(\alpha')"] \end{tikzcd}$$

The diagram commutes up to a homotopy determined by the path p. Thus, the induced map $\mathrm{fib}(\alpha) \to L \to_{\alpha'} X' \to DX' \to DL$ is null-homotopic. Thus, there is an induced map of fibers $u : \mathrm{fib}(\alpha) \to \mathrm{fib}(D(\alpha'))$. We require u to be invertible.

We say (X, q) and (X', q') are cobordant if there is a cobordism from (X, q) to (X', q').

Theorem 9.30. Being cobordant is an equivalence relation $\sim$ on $\mathrm{Poin}(\mathcal{C}, Q)$, the Poincare objects of $(\mathcal{C}, Q)$. Furthermore, $\oplus$ is a well-defined abelian group operation on $\mathrm{Poin}(\mathcal{C}, Q)/\sim$.

Definition 9.31. We define $L_0(\mathcal{C}, Q) = \mathrm{Poin}(\mathcal{C}, Q)/\sim$. For $n > 0$, we define $L_n(\mathcal{C}, Q) := L_0(\mathcal{C}, \Omega^n Q)$.

Remark 9.32. The usual approach to defining higher degrees of L-theory is to construct a L-theory spectrum $\mathcal{L}(\mathcal{C}, Q)$ associated to a Poincaré category, and the n-th L-theory would be the n-th homotopy group of this spectrum. It turns out that this is canonically isomorphism to our definition. Due to the time constraint of this talk, we decided to stay with the current approach.

Remark 9.33. Although we have not focused on the classical theory much, we remark that L-theory indeed did not originate from higher algebra but had more concrete foundations. In the specific case where we have $(D^{perf}(R), Q^q_M)$ (with values in an R-module M, possibly with involution), we recover the classical **Wall-Ranicki** quadratic L-groups. Similarly with the symmetric case.

9.4 L-Theory of $\mathbb{Z}$ and Geometric Connections

Definition 9.34. The quadratic and symmetric L-theory of $\mathbb{Z}$ is given by $L_n(D^{perf}(\mathbb{Z}), Q^s_{\mathbb{Z}})$ and $L_n(D^{perf}(\mathbb{Z}), Q^q_{\mathbb{Z}})$ respectively. As a short hand, we denote them as $L^s(\mathbb{Z})$ and $L^q(\mathbb{Z})$ respectively.

Remark 9.35. This is not how this was defined in Lurie. We should have used finitely presented R-module spectra, but it turns out there is no difference with using perfect R-module spectra in this case.

The story of quadratic L-groups of $\mathbb{Z}$ is very important in the world of low-dimensional topology.

Theorem 9.36. $L^q_*(\mathbb{Z})$ may be computed as follows:

$$L^q_n(\mathbb{Z}) = \begin{cases} 8\mathbb{Z}, n = 4k \text{ (signature)} \\ 0, n = 4k+1 \\ \mathbb{Z}/2\mathbb{Z}, n = 4k+2 \text{ (Kervaire invariant)}, \\ 0, n = 4k+3 \end{cases}$$

Theorem 9.37. $L_*^s(\mathbb{Z})$ may be computed as follows:

$$L_n^s(\mathbb{Z}) = \begin{cases} \mathbb{Z}, n = 4k \text{ (signature)} \\ \mathbb{Z}/2, n = 4k+1 \text{ (de Rham invariant)} \\ 0, n = 4k+2 \\ 0, n = 4k+3 \end{cases}$$

Definition 9.38. The de Rham invariant of M^{4k+1} is the rank of 2-torsion in $H_{2k}(M)$ modulo 2, or equivalently the product of two Stiefel Whitney numbers $w_2 w_{4k-1}$.

The geometric connections between a compact oriented manifold of L-groups of $\mathbb{Z}$ are given as follows.

Theorem 9.39. Let M^n be a compact oriented manifold and $n = 4k$. Recall we explained earlier that $(C^*(M;\mathbb{Z}), q_M)$ is a Poincare object of $(D^{perf}(\mathbb{Z}), Q_{\mathbb{Z}}^{s,-n})$ (shifted by n-indices down). Thus, M gives an element element of $L_n^s(\mathbb{Z})$, which is exactly its signature.

Finally, we will end our talk with a brief discussion on the Kervaire invariant one question?

Question 9.40. What manifolds have Kervaire invariant 1?

Theorem 9.41.

1. For $n = 6, 14, 30, 62$, there exists a Keivarie invariant one manifold (this was known in the last century).
2. (Hill-Hopkins-Ravenal), If $n = 2^{J+1} - 2$ for $J \geq 7$, there are no Keivarie invariant one manifold.
3. This only leaves $2^7 - 2 = 126$, which is proved this year (2024) by Lin-Wang-Xu to be positive.