∞-Category Theory Reading Seminar Notes

Maintained by Mattie Ji

Fall 2024 Semester

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1 Meeting September 5th, 2024

Speaker: Kartik Tandon
The introductory talk is mainly on the motivation of "**why ∞-categories**"?

1. Why homotopy and category?
2. Why in particular ∞-categories?

To be more technical, while ∞-categories meant something broadly before, it now specifically refers to the "quasi-categories" of $(\infty, 1)$-categories developed by Boardman, Vought, Joyal, and Lurie. This is a bit far away for now, and we will focus on more concrete notions for now.

Recall that in category theory, there is the notion of natural transformations.

$$\begin{array}{c} \text{Natural Transformations} \\ \downarrow \\ \text{Functor} \\ \downarrow \\ \text{Objects and Morphisms} \end{array}$$

On the other hand, there are also the notions of universal properties, which relate to limits and colimits. There is in fact a notion of homotopy limits and colimits.

Example 1.1.

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & * \simeq CX \\ \downarrow & & \downarrow \\ * \simeq CX & \longrightarrow & \Sigma X \end{array}$$

colimits homotopy colimits

1.1 Homotopy and Homotopical Category

Here are some examples of homotopy categories:

1. hTop - This is the category obtained from `Top` by modding out the equivalence relation of homotopy equivalences.
2. $Ch[q{-}iso]^{-1}$ - This is the category obtained from $\mathrm{Ch}^*(Ab)$ by moddig out what are called "quasi-isomorphisms".

The notions of ordinary limits and colimits do not typically exist in these categories! Even if they exist, they typically do not agree with limits and colimits.

Example 1.2. Take the **Triangulated categories**:

- Cone is not functorial in this category. This can be partially remedied by the Octahedral Axiom.
- Colimits do not need to be functorial.

Attempts to fix these issues came into the idea of **homotopical categories**. This originally came from **Gabriel-Zismen** in 1967.

- They are of the form $(\mathcal{C}, W)$ where $W \subseteq \mathrm{Mor}(C)$.

They are some issues with this specific framework:

- (i) Formal constructions are technical.
- (ii) Too general of a framework, includes too many things.

1.2 Quillen's Development

This is where Quillen came in. In 1967, he introduced the idea of **model categories**, which are still prominently used today.

Definition 1.3. Model categories are complete and cocomplete categories of the data

$$(\mathcal{C}, W, \mathrm{Cof}, \mathrm{Fib}),$$

where $W, \mathrm{Cof}, \mathrm{Fib} \subseteq \mathrm{Mor}(\mathcal{C})$. W is typically called **weak equivalences**, Cof cofibrations, and Fib fibrations. They satisfy the following axioms such that

- (i) W has all isomorphisms and is closed under 2 out of 3 in the sense of

$$\begin{array}{ccccc} x & \longrightarrow & y & \longrightarrow & z \\ & & \xrightarrow{\qquad\qquad} & & \end{array}$$

- (ii) $(W \cap \mathrm{Cof}, \mathrm{Fib})$ and $(\mathrm{Cof}, W \cap \mathrm{Fib})$ are both weak factorization systems.

Remark 1.4. $W \cap \mathrm{Cof}$ is called acyclic cofibrations. $W \cap \mathrm{Fib}$ is called acylic fibrations.

Definition 1.5. A **weak factorization system** $(\mathcal{L}, R)$ is a pair such that the lifting property holds. In other words, for all f, g, the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i\in L}\downarrow & \exists \nearrow & \downarrow{\scriptstyle \alpha\in R} \\ B & \xrightarrow{g} & Y \end{array}$$

Example 1.6. In the category of Top, you can take (W, Fib) where W are the weak equivalences and Fib are the actual fibrations. They are two popular examples of fibrations:

$$\begin{array}{ccc} D^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ D^n \times I & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ X \times I & \longrightarrow & Z \end{array}$$

Serre fibrations Hurewicz fibrations

Exercise 1.7. What are the cofibrations in the example above.

Example 1.8. Now if we take the category of chain complexex over R-modules, we can take

1. W as quasi-isomorphisms.
2. Projective: cofibrations are level-wise monic with projective cokernels.
3. Injective: fibrations are level-wise epic with injective kernels.

In general, the **strategy** is to

- Perform a fibrant/cofibrant replacements with $\{$projective resolutions$\}$.
- Do normal category theory.

1.3 Infinity Categories

So far, we still have not gotten into ∞-categories yet. The analogy between model categories and ∞-categoeis are thought of as follows:

1. Model categories are to picking a basis.
2. ∞-categories are coordinate-free.
3. "The space of choices is contractible".

We can consider the ∞-Category of Spectra,the reason why is outlined as follows

1. There are lots of model categories on Spectra Sp.
2. Lewis's Theorem in 1971 asserted that there does not exist a **convenient** category of spectra (meaning it fails 5 nice properties we hope it to have).
3. We do have a twist map $\tau : A \wedge B \to B \wedge A$. But it might not be homotopic to the identity.
4. Even though the introduction is somewhat technical, the main punchline is that there are some things normal category theory is lacking. In fact, Lewis's Theorem does not hold on ∞-categories (they do really satisfy the 5 nice properties).

Remark 1.9. *Introduction to stable homotopy theory* by Denis Nardin is a great reference - ∞-categories from the start.

Question 1.10. Model categories are not good enough for some things, but what about enriched categories?

There is also a notion of **enriched** categories, perhaps more rooted in physics. They appear in the forms of

1. Top, sSet (simplicial sets).
2. A_∞-categories.
3. dg-categories, etc.

It turns out in fact that A_∞-categories and dg-categories are notions of what's called k-linear stable ∞-categories! Unlike settings outside of enriched category, where you could have statements like "$x \otimes -$ is flat", enriched categories want to start by showing the entire category C is flat, but you would still run into issues.

1.4 Crash Course on Simplicial Sets

Definition 1.11. We define Δ as the category where

1. Objects are $[n] = \{0, ..., n\}$.
2. Morphisms are **order-preserving**.

From here, the category of simplicial sets are the contravariant functors out of Δ into Set. In other words, $sSet = Set^{\Delta^{op}}$. There is a geometric realization of a simplicial set given as a functor

$$|-| : sSet \to Top, [n] \mapsto \Delta^n.$$

A more concrete interpretation of the definition above is as follows.

Definition 1.12. A simplicial set is a graded set over the natural number $\mathbb{N}$, with maps

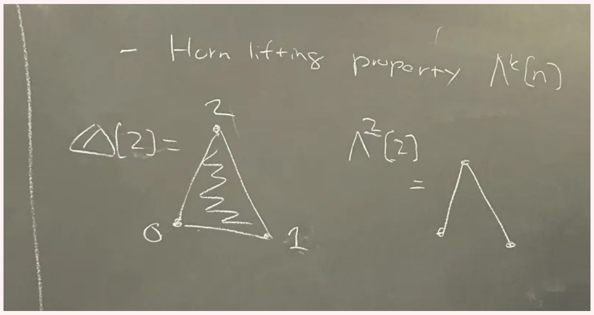
1. Face Maps: $d_m : X_m \to X_{m-1}$
2. Degeneracy Maps; $s_m : X_m \to X_{m+1}$

(If we want to be technical, they should have upper indices). satisfying the conditions

- $d_i d_j = d_{j-1} d_i$ for $i < j$.
- $d_i s_j = s_{j-1} d_i$ for $i < j$.
- If $i = j$ or $i = j + 1$, $d_i s_j = id$.
- $d_i s_j = s_j d_{i-1}$ if $i > j + 1$.
- $s_i s_j = s_{j+1} s_i$ if $i \leq j$.

Definition 1.13 (Kan Complexes). The **Kan complexes** are special cases of simplicial sets that satisfies:

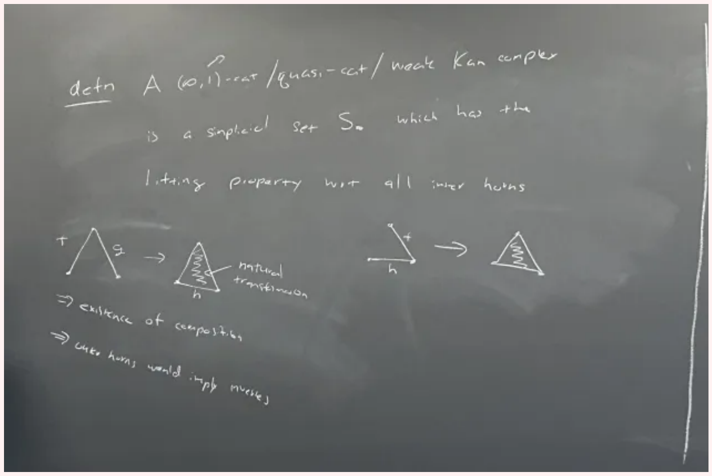
1. The **horn lifting property**. We give a pictorial definition $\Lambda^k[n]$ as, for example when $n = 2$,



   In this case, we note that

2. $\text{Hom}(-, [n]) \in Set^{\Delta^{op}}$. One should think of $\text{Hom}(-, [n])$ as "standard n-simplex". THis also turns out is not a Kan complex.
3. Let $\eta : CAT \to sSet$ be the **nerve functor** of the form $\text{Hom}([n], -)$. $n(C)$ are Kan complexes, fortunately!

Definition 1.14. A $(\infty, 1)$-category/quasicategory/weak Kan complex is a simplical set $S_\bullet$ which has the lifting property with respect to all inner horns. Here is a pictorial representation of what is going on:



1.5 Unique and Interesting Applications of ∞-Categories!!!

Here are some interesting applications:

1. Spectra Sp gives the notion of stable ∞-category enriched in Sp.
   - Cones are all functorial now.
2. Limits and colimits are nice enough. The "ordinary" notion of (co)limits now coincide with the "homotopical" notion of (co)limits.
3. There are some classical constructiosn that are now realizable as new colimit/limit interpretations. (ex. Thom space corresponds to Thom spectra. The classical construction is very technical, but it is very simplified in this new framework . More specifically,

$$M_+X := \mathrm{colim}(M^+ \to \mathrm{Pic}(R) \to \mathrm{Mod}_R).$$

4. Universal Properties in (∞)-category.
   - Thom Spectra.
   - K-theory - BGT (2013) showed that

     Theorem 1.15. There is a functor $\mathbb{K} : Cat_\infty^{Set} \to \mathrm{Sp}$ such that K-theory is the universal additive invariant. In other words,

     $$A \to B \to C \implies K(A) \oplus K(C) \cong K(B).$$

   - Here is another interesting related theorem

     Theorem 1.16 (Beilinson)**.** There is a semi-orthogonal decomposition $\mathcal{D}(\mathbb{P}^1_R) = \langle \mathcal{O}_X, \mathcal{O}_X(-1) \rangle$.

   As a corollary, it shows that

Corollary 1.17. $K(\mathbb{P}^1_R) \cong K(R) \oplus K(R)$.

- Descent - recall for the projective line in algebraic geometry, we can glue it as following pushout

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{Z}[x, x^{-1}] & \longrightarrow & \operatorname{Spec}\mathbb{Z}[u] \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{Z}[t] & \longrightarrow & \mathbb{P}^1 \end{array}$$

The question is, if we look at their derived categories, we have the following:

$$\begin{array}{ccc} \mathcal{D}(\mathbb{P}^2) & \longrightarrow & \mathcal{D}(\mathbb{Z}[u]) \\ \downarrow & & \downarrow \\ \mathcal{D}(\mathbb{Z}[t]) & \longrightarrow & \mathcal{D}(\mathbb{Z}[x, x^{-1}]) \end{array}$$

Is this a pullback?

The answer turns out to be NO for very complicated reasons. There is, however, a remedy in ∞-category, we have that

Theorem 1.18 (Barr-Beck-Lurie)**.** $\mathcal{D}(-)$ is a Cat_∞^{St}-valued sheaf in Sch_{Zar}.

2 Meeting September 12th, 2024

Speaker: Riley Shahar

2.1 Simplicial Sets

Definition 2.1. The **simplex category** Δ is a small category where

- The objects are the finite ordinals $[n] = \{0 < \ldots < n\}$ are the totally ordered notes on $n + 1$ elements.
- The morphisms are order-preserving set functions.

Δ is presented by "co-face maps" of the form

$$d^i : [n] \to [n + 1],$$

where d^i is the unique order preserving map from $[n] \to [n + 1]$ that misses i. Furthermore, s^i are the "co-degeneracy maps" of the form

$$s^i : [n] \to [n - 1]$$

as the unique order preseriving surjective map that repeeats the index i twice. In particular, these maps generate all the non-identity morphisms in Δ. They will also satisfy certain identities that we will write out for simplicial sets later.

Definition 2.2. A **simplicial set** is a presheaf of sets on the category Δ. In other words, it is a contravariant functor from Δ to Set. The category of simplicial sets, denoted sSet, is the functor category $\mathrm{Set}^{\Delta^{op}}$ where the objects are the functors and the morphisms are the natural transformations.

Let us unpack this definition. Let X be a simplicial set. This means that for each n, we have a set X_n, which we want to think of as the collection of "n-simplicies". Since a simplicial set is a functor, we also have maps

- $d_i : X_n \to X_{n-1}$ are the face maps that corresponds to d^i earlier. The face map d_i should be thought of as sending an n-simplex to its i-th face (the opposite face of the i-th vertex).
- $s_i : X_n \to X_{n+1}$ are the degeneracy maps that corresponds to s^i earlier. The degeneracy map s_i should be thought of as sending an n-simplex to a degenerate $n + 1$-simplex, with the i-th vertex repeated.

Here we give the following characterization of a simplicial set that one can use to detect whether a collection $\{X_n\}$ with morphisms form a simplicial set.

Proposition 2.3. Let X be a simplicial set, then the face and degeneracy maps satisfy the following identities.

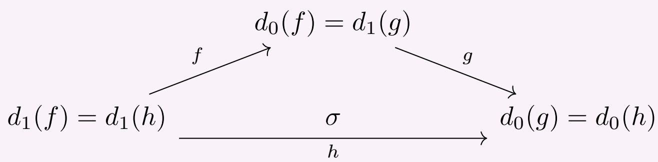
- $d_i d_j = d_{j-1} d_i$ for $i < j$
- $s_i s_j = s_{j+1} s_i$ for $i \leq j$
- If $i = j$ or $i = j + 1$, $d_i s_j = id$.
- $d_i s_j = s_{j-1} d_i$ if $i < j$.
- $d_i s_j = s_j d_{i-1}$ if $i > j + 1$.

Conversely, suppose $\{X_n\}_{n=0,1,2,\ldots}$ is a sequence of sets equipped with functions $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ satisfying the five identities above (for all i and all n), then this is a simplicial set.

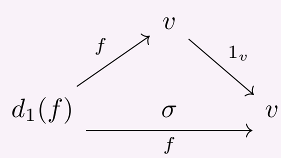
To make sure we are comfortable with the geometric picture of a simplicial set in our head. We investigate the following example in detail.

Example 2.4. Pictorially, what is a 2-simplex $\sigma \in X_2$? Well, for $n = 2$, there are three face maps $d_0, d_1, d_2 : X_2 \to X_1$. Write $g = d_0(\sigma), h = d_1(\sigma), f = d_2(\sigma)$, these can be regarded as the faces of σ.

g, h, f are 1-simplicies in X_1, which furthermore have face maps going to their two ends in X_0. Now, we observe for example that the first identity of simplicial set in the previous proposition implies that $d_0(h) = d_0(d_1(\sigma)) = d_0(d_0(\sigma)) = d_0(g)$, so g and h should have the same target (since d_0 corresponds to the opposite edge). Similarly, $d_0(f) = d_0(d_2(\sigma)) = d_1(d_0(\sigma)) = d_1(g)$ means that the target of f is the source of g. Finally, we can also get that f and h have the same source. Thus, the 2-simplex σ may be drawn as



There are also cases where the 2-simplex could be degenerate, ex. two of its vertices v, w are the same. In this case, tehe morphism between them is denoted 1_v and draw it as



Definition 2.5. Let X be a simplicial set. A simplex $x \in X_n$ is **degenerate** if it is in the image of a degeneracy map s^i.

Here we give a simple example of a simplicial set first.

Definition 2.6. Consider the functor $\Delta(-, [n]) : \Delta \to Set$ (HOM functor into $[n]$). This is a contravariant functor and hence a simplicial set. This is called the **standard n-simplex** and is denoted Δ^n.

Proposition 2.7. Let X be a simplicial set. By the Yoneda Lemma, there is a natural isomorphism

$$\mathrm{Hom}(\Delta^n, X) \cong X_n.$$

Thus, the standard n-simplicies can be thought of as representable presheaves. The **density theorem** in category theory implies the following result.

Proposition 2.8. Let X be a simplicial set, then X is the colimit of standard simplicies indexed by its category of elements, ie.

$$\mathrm{colim}_{x \in X_n} \Delta^n \cong X.$$

Let E be a cocomplete and locally small category and $F : \Delta \to E$ be any functor. We in fact have a left Kan extension of the form

$$\begin{array}{ccc} sSet & \xrightarrow{\operatorname{Lan}_y F} & E \\ \uparrow & \nearrow_{F} & \\ \Delta & & \end{array}$$

that commutes up to natural isomorphism. If we want this to genuinely commute, F needs to be co-continuous. Here the map $y : \Delta \to sSet$ sends S to the functor $\mathrm{Delta}(-, S)$. The left Kan extension is left-adjoint to the functor $R : E \to \mathrm{sSet}$ acting by $x \mapsto n \mapsto E(Fn, x)$.

Example 2.9. There is also a canonical covariant functor from $\Delta \to \mathrm{Top}$ by sending an ordinal $[n]$ to the standard simplex Δ^n. The construction above gives a left Kan extension of this functor to $\mathrm{sSet} \to \mathrm{Top}$. We denote this functor as

$$|\bullet| : sSet \to Top.$$

This is called **the geometric realization of a simplicial set**.

From the discussion above, we also know that $|\bullet|$ admits a left adjoint, which we will call Sing :

$$\mathrm{Sing} : Top \to sSet.$$

This is the singular complex on a topological space given by $\mathrm{Sing}(X)_n = \mathrm{Top}(|\Delta^n|, X)$.

A canonical example of a simplicial set arises in what is called the "nerve of a category".

Definition 2.10 (The Nerve of a Category)**.** Let $F : \Delta \to \mathrm{Cat}$ send $[n]$ in the abstract simplex to the ordinary category $[n]$, the nerve functor N is the right adjoint functor given by the left Kan extension of F to the category of simplicialy sets (in particular, as n-simplicies, $F\mathcal{C}$ has functors $F_n \to \mathcal{C}$

There is a concrete interpretation of the nerve as follows. Let $\mathcal{C}$ be any small category, the nerve of $\mathcal{C}$, denoted $N\mathcal{C}$ is a simplicial set

$$N\mathcal{C}_0 = \mathrm{Obj}\,\mathcal{C}, N\mathcal{C}_1 = \bigsqcup_{x_0, x_1} \mathcal{C}(x_0, x_1),$$

and in general $N\mathcal{C}_k$ composes of composable k-tuples $(\varphi_1, ..., \varphi_k)$. This means that, each φ_i is a morphism in $\mathcal{C}$, and the source of φ_1 is the target of φ_2, etc...

The face and degeneracy maps are defined as follows. The degeneracy map $s_i : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ takes a sequence of n composable maps

$$c_0 \to c_1 \to ... \to c_n$$

and it inserts an identity at c_i maps in the i-th spot, ie. s_i of the sequence above becomes

$$c_0 \to c_1 \to ... \to c_i \to_{id} c_i \to c_{i+1} \to ... \to c_n.$$

The face map $\partial_i : N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ takes a sequence of n composable maps

$$c_0 \to c_1 \to ... \to c_i \to_{f_i} c_{i+1} \to_{f_{i+1}} ... \to c_n$$

and just composes the i-th and $i + 1$-th arrow for $0 < i < n$ (leaving out the first and last arrow). In other words, it becomes

$$c_0 \to c_1 \to ... \to c_i \to_{f_{i+1} \circ f_i} c_{i+2} \to ... \to c_n.$$

Finally, before we move on the the definition of ∞-categories, we discuss a few more technical construction.

Definition 2.11. There is also a notion of **boundary map** on a standard simplex. Specifically, we let $\partial\Delta^n$ to denote the **simplicial n-sphere**, which is a simplicial set consisting of the boundary of the standard n-simplex, ie.

$$\partial\Delta^n = \bigcup_i d_i(\Delta^n).$$

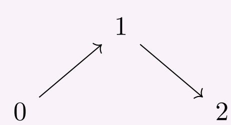
An **n-sphere** in a simplicial set X is a map $\partial\Delta^n \to X$.

Definition 2.12. The **simplicial horn** Λ^n_k is the **simplicial set** consisting of all faces of the standard n-simplex except for the i-th face, ie.

$$\Lambda^n_k = \bigcup_{i \neq k} d_i(\Delta^n).$$

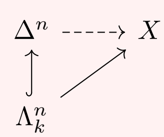
A **horn** in a simplicial set X is a map $\Lambda^n_k \to X$. A horn is inner if $0 < k < n$. A horn is outer if $k = 0$ or $k = n$.

Example 2.13. Here is an example for Λ^2_1.



Given a horn, there is a notion of filling the horn that should correspond to our geometric intuitions.

Definition 2.14 (Filling a Horn). Let X be a simplicial set and $\Lambda^n_k \to X$ be a horn. A **filler** of this horn is a lift to a standard n-simplex. In other words, it is a commutative diagram of the form



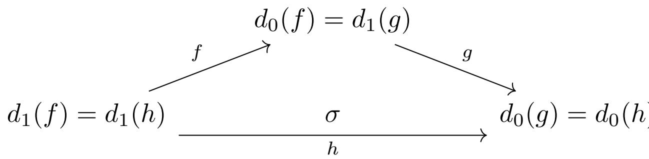
2.2 ∞-Category

Notation: For this section, here are some notations we will refer to in our construction:

- An **object** is a 0-simplex.
- A **morphism** is a 1-simplex.
- An n-cell (if $n = 2$, this is a natural transformation, etc.) is an n-simplex.
- The source of a morphism f is $d_1(f) = x$, the target is $d_0(f) = y$.
- The identity at an object x is the morphism $s_0(x)$.
- A **composable pair** is an inner 2-horm $\Lambda^2_1 \to X$. Why? Well drawing the Λ^2_1, we clearly see that it is just

$$x \to_f y \to_g z.$$

- Given a 2-simplex σ of the form



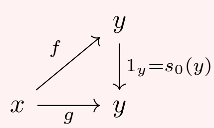
  We say that h is a **composite** of f and g. We write this as $h \simeq g \circ f$. Note that a composite is a filler of a composable pair. (ex. for Λ_1^2, we can fill it to a 2-simplex by a 2-cell and a 1-cell).

Now we are ready to finally define an ∞-category!!!

Definition 2.15 (Definition of an ∞-category)**.** An ∞-category (also known as a quasicategory or a weak Kan complex) is a simplicial set in which all inner horns have (not necessarily unique) fillers.

Remark 2.16. As a remark, for an ∞-category $\mathcal{C}$, every composable pair has a composite. The category of quasicategories is the full subcategory of sSet spanned by quasicategories.

Definition 2.17. Two maps $f : x \to y$ and $g : x \to y$ are **homotopic** if there is a 2-cell σ of the form



Here s_0 is the previous degeneracy map. In this case, we write $f \simeq g$.

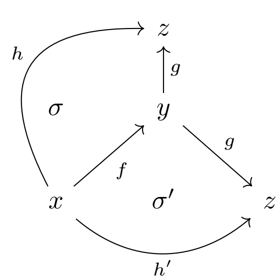
Note that $\simeq$ itself is an equivalence relation on morphisms in a quasicategory. The following proposition seems unique but needs to be checked.

Proposition 2.18 (Composites in a quasicategory are unique up to equivalence)**.** Let $f : x \to y, g : y \to z$ and $h, h' : x \to z$ be morphisms such that

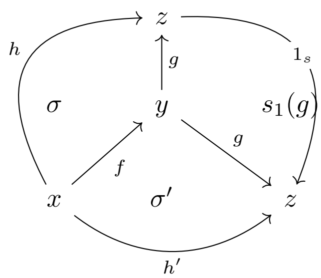
$$h \simeq_\sigma g \circ f \text{ and } h' \simeq_{\sigma'} g \circ f.$$

Here $\simeq_\sigma$ means h fills the composable pair into a 2-simplex σ. Then, h and h' are equivalent.

Proof. The proof is surprisngly geometric. We first note that it suffices for us to try to fill the horn given by h and h'. Now we consider the diagram



Showing that $h \simeq h'$ amounts down to showing that we can fill the horn on the upper right side. In this case, $s_1(g)$ is the desired 2-simplex, and we have that



■

2.3 Nerve of a Category and Homotopy Category

We saw previously that there is a nerve

$$N : \text{Cat} \to \text{sSet}$$

The nerve of a category is an example of an infinite category!

Proposition 2.19. 1. The nerve functor is a fully faithful functor.
2. The essential image of N is simplicial sets where every inner horn has a unique filler (ie. the ∞-categories with unique filler to every inner horn).

It turns out that the nerve functor has a left-adjoint, which is called the one-truncation or the homotopy category (denoted τ_1 or h) respectively. Concretely, it is given as follows.

Definition 2.20. The **homotopy category** $h(\mathcal{C})$ of an ∞-category $\mathcal{C}$ is a 1-category such that

1. objects: the 0-cells from $\mathcal{C}$.
2. morphisms: homotopy equivalence classes of morphisms in $\mathcal{C}$.

This is also called a 1-truncation of $\mathcal{C}$ and denoted $\tau_1(\mathcal{C})$.

Because the nerve functor is fully faithful, the counit of the adjunction between the nerve functor and 1-truncation functor yields the following isomorphism:

Theorem 2.21. There is a canonical isomorphism of 1-categories

$$h(N(C)) \cong \mathcal{C}.$$

Here, $\mathcal{C}$ is a 1-category.

Definition 2.22 (Isomorphisms in ∞-categories)**.** A map $f : x \to y$ in an ∞-category is an isomorphism if there is a map $g : y \to x$ such that we have equivalences $1_x \simeq g \circ f$ and $1_y \simeq f \circ g$.

Clearly from the construction of homotopy categories, we also have the following.

Proposition 2.23. f is an isomorphism if and only if $[f]$ is an isomorphism in $h(C)$.

2.4 ∞-groupoid

Definition 2.24. An ∞-groupoid is an ∞-category in which all morphisms are isomorphisms.

Remark 2.25. All cells above dimension 1 are automatically reversible, by equivalence of homotopy. That is why we only need the groupoid definition to specify inversion of 1-cells.

Definition 2.26. A Kan complex is a simplicial set in which every horn (not just inner) has a filler.

The following is a very deep theorem.

Theorem 2.27. The following are equivalent:

1. $\mathcal{C}$ is an ∞-groupoid.
2. $\mathcal{C}$ is a Kan complex.

Corollary 2.28. The singular simplicial complex $\mathrm{Sing}(X)$ of a topological space X is an ∞-groupoid.

Proof. It suffices for us to check that it is a Kan complex. So we want to solve to following extension problem for every horn.

$$\begin{array}{ccc} \Delta^n & \dashrightarrow & \mathrm{Sing}(X) \\ \uparrow & \nearrow & \\ \Lambda^n_k & & \end{array}$$

It suffices for us to take this to the geometric realization land (because $|\bullet|$ is adjoint to Sing), we are looking at

$$\begin{array}{ccc} |\Delta^n| & \dashrightarrow & X \\ \uparrow & \nearrow & \\ |\Delta^n_1| & & \end{array} .$$

This is solvable in the world of topology, because a horn is an obvious geometric retract of a geometric n-simplex. ■

2.5 Cardinalities in ∞-Category Theory

Let us first discuss this in the world of 1-categories.

Question 2.29. What is the category Set?

It is certainly not a set. The foundations for interpreting this is given by what is called the **NBG (Von Neumann - Bernays- Godel) set theory**.

Definition 2.30. The idea is that - a class is a formula with free variables where the quantifiers range only over sets, with 2 extra axioms.

It turns out this has two very nice properties:

1. Conservative extension of the ZF system.

2. Finitely axiomatizable.

Saunder MacLane thought this was enough to do 1-category theory and claimed everything in his book could be done using this set-up as long as you add the word "small" in front of your category. However, when you want to study larger categories, it becomes theoretically more challenging. An example of this question is looking at

$$y : \text{Top} \to \hat{\text{Top}}$$

where $\hat{\text{Top}}$ is the presheaf category on Top.

This is where the idea of MK (or MT) set theory, which roughly speaking.

Definition 2.31. MK (or MT) set theory is NBG set theory with quantifiers ranging over classes.

Remark 2.32. It is impossible to prove the consistency of this theory from ZF, because this theory implies the consistency of ZF.

This is fine with 1-categories. This does not, however, work well with ∞-categories. This is where the idea of **Grothendieck universe** comes in.

Definition 2.33. A **Grothendieck universe** is a set U closed under:

- Membership, meaning $x \in y \in U \implies x \in U$.
- Pairing $(\{x, y\} \in U)$
- Unions indexed by U.
- Power sets.

A U-set is a set that is in U. We can put an ∞-category of spaces in the realm of a Grothendieck universe which is an appropriate arena to work under.

However, it turns out that a Grothendieck universe is equivalent to what is called an **inaccessible cardinal**.

Definition 2.34. An inaccessible cardinal is a cardinal (set) that cannot be reached by unions and power-sets.

Example 2.35. The cardinality $|\mathbb{N}|$ is inaccessible relative to the cardinality of finite sets (You might ask - why can't I just take an infinite union of finite sets, but to do that, you see to use $|\mathbb{N}|$, whch is not allowed). The cardinality of $|\emptyset|$ is vacuously inaccessible, because, well, it has no sets.

Now, a even more generalization is what is so called the **Tarski-Grothendieck set theory**, which is built on the notion of "every x is contained in a universe". Now a logician will make a statement that everyone else will find confusing:

- It is enough to have a countable number of universes $U_0 \in U_1 \in U_2 \ldots$. This is enough to do all of category theory.
- BUT, to prove this, you need to construct a cardinal $|\mathbb{N}|$, which requires you to use more than a countable number of universes.

But, there is an argument to be made that, perhaps type theory is more suitable for category theory than this.

In the proposal of Tarski-Grothendieck set theory, Lurie thought everything he claimed in the book could be done using MK. TG set theory was just a more convenient proof. In this sense, the process of finding out what are the minimal set of axioms required to prove something is called **Reverse Mathematics**.

Example 2.36. McLacty in 2011 proved that FOA (fintie order arithmetic) is enough to prove all of EGA and SGA by Grothendieck.

Theorem 2.37 (Levy Reflection Theorem)**.** If ZFC $\vDash \phi(x_1, ..., x_n)$, then ZFC $\vDash (\exists V, V| = \phi(x_1, ..., x_n))$.

This theorem, essentially, implies that if we only prove a theorem with a finite number of axioms, we can find a smaller universe that can also prove this theorem.

Example 2.38. Angus Macintyre in the 2000s sketched a proof that the Peano Axioms imply Fermat's Last Theorems. No formal proof was every written down.

3 Meeting September 19th, 2024

The ∞-category of spaces

Speaker: Mats Hansen

Today we will be looking at the ∞-category of spaces. We also want to talk about functors between ∞-categories, mapping spaces, how to identify constructions as examples of ∞-categories. Ultimately, it will hopefully give us a tower of abstractions to climb.

Motivation: The ∞-category of spaces is an analogus construction to the category of sets in the world of 1-categories.

Let us recall some properties of the category Set:

1. They have free cocompletion of a singleton.
2. The morphism of any two objects in a locally small 1-category take value in Set.
3. There is a standard Yoneda embedding for functors from a locally small category $\mathcal{C}$ into Set.

3.1 Functors of ∞-Category

We first need to make sense a notion of functor between ∞-categories.

Definition 3.1. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories, a functor of ∞-categories is a morphism of simplicial sets (ie. it is a natural transformation between the two functors compatible with the face maps d_i and the degeneracy maps s_i).

Definition 3.2. Let K be a simplicial set and $\mathcal{C}$ an ∞-category. We define a new simplicial set $\mathrm{Fun}(K, \mathcal{C})$ concretely as follows:

- $\mathrm{Fun}(K, C)_n = \mathrm{Hom}_{sSet}(K \times \Delta^n, \mathcal{C})$.
- The face and degeneracy maps are induced by

$$d^i : \Delta^{n-1} \to \Delta^n \text{ and } s^i : \Delta^n \to \Delta^{n+1}.$$

Note that this is an internal hom adjunction in sSet.

$\mathrm{Fun}(K, \mathcal{C})$ is called the <u>∞-category of functors from K to $\mathcal{C}$</u>.

Theorem 3.3. Let K be a simplicial set and $\mathcal{C}$ an ∞-category, the simplicial set $\mathrm{Fun}(K, \mathcal{C})$ is an ∞-category.

Proof. We will use the lifting property of maps of simplicial sets. One wants to show that there is a solution to the following lifting problem.

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathrm{Fun}(K, C) \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

The usual way one approach these kind of lifting problems is to apply adjunctions in a smart way. Using the internal hom adjunction, this is equivalent indeed to solving

$$\begin{array}{ccc} \Lambda^n_i \times K & \longrightarrow & C \\ \big\downarrow & \nearrow & \\ \Delta^n \times K & & \end{array}$$

We can augment this to a diagram of the form

$$\begin{array}{ccc} \Lambda^n_i \times K & \longrightarrow & C \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n \times K & \longrightarrow & \Delta^0 \end{array}$$

where we note the morphism $C \to \Delta^0$ is necessarily unique.

We pause the proof to introduce a definition in the middle.

Definition 3.4. A map $f : X \to Y$ is called an **inner fibration** if it satisfies the right lifting property with respect to all hom inclusions. In other words, we have a lift of the form

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

Hence, we see that the proof amounts to showing that the map $C \to \Delta^0$ is an inner fibration. The proof of this is in fact not categorical at all but is rather an extremely combinatorial proof. The proof follows from the fact that $\mathrm{Fun}(-, K)$ preserves inner fibrations if and only if the claim of maps having the left lifting property (LLP) wr.t. the inner fibrations are closed under $\times K$. ■

Remark 3.5. It turns out that C is an ∞-category if and only if $C \to \Delta^0$ is an inner fibration.

3.2 Mapping Spaces

Motivation: When $\mathcal{C}$ is a 1-category and $d, c \in \mathrm{Obj}(\mathcal{C})$. We can consider the pull back of the form

morphisms in $\mathcal{C}$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(c,d) & \longrightarrow & \mathrm{Fun}([1],\mathcal{C}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (0,1),\text{return source and target}} \\ \Lambda^0 & \xrightarrow{(c,d)} & \mathcal{C}\times\mathcal{C} \end{array}$$

terminal 1-category

This guides the definition of simplicial set of morphisms for ∞-categories.

Let us now recall the following proposition.

Proposition 3.6. For any simplicial set $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ is a Kan complex.
2. $\mathcal{C}$ is an ∞-groupoid.

We will obtain that the simplicial set of morphisms will be an $(\infty, 0)$-category (ie. a ∞-groupoid).

Definition 3.7. Let $\mathcal{C}$ be an ∞-catgeory, and $c, d \in \mathrm{Obj}(\mathcal{C})$. We define the **mapping space** as the pullback

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(c,d) & \longrightarrow & \mathrm{Fun}(\Delta^1,c) \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{(c,d)} & \mathrm{Fun}(\partial\Delta^1,c) \end{array}$$

Note that $\partial\Delta^n$ has two properties:

1. $\mathrm{Fun}(\partial\Delta^1, c)$ are pair of objects.
2. $\mathrm{Fun}(\partial\Delta^1, c) = c \times c$.

Here Δ^0 is a single point and $\Delta^0 \to \mathrm{Fun}(\partial\Delta^n, c)$ goes to the pair (c, d).

$\mathrm{Map}_{\mathcal{C}}(c, d)$ ocnsists of morphisms from c to d that restrict to c and d at the boundary of Δ^1.

Theorem 3.8. For $\mathcal{C}$ an ∞-category and $c, d \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Map}_{\mathcal{C}}(c, d)$ is a Kan complex, and hence an ∞-groupoid by the proposition.

Proof. One can show that

$$\mathrm{Fun}(\Delta^1, c) \longrightarrow \mathrm{Fun}(\partial\Delta^1, c)$$

is an inner fibration that is stable under pullback. From here, we can deduce that the map $\mathrm{Map}_{\mathcal{C}}(c,d) \to \Delta^0$ is a "conservative" inner fibration. In particular, this implies that it satisfies the filling condition, and it implies that it is a Kan complex. ■

3.3 The ∞-category of Spaces

Construction: Take some integer $0 \leq i \leq j$ where $i, j \in \mathbb{N}_0$. From here we define

$$P_{i,j} = \{I \subseteq \{i, ..., j\} \mid \min(I) = i \text{ and } \max(I) = j\}.$$

$P_{i,j}$ itself has a partial order given by inclusion.

Definition 3.9. From here we define the simplicial category $C[\Delta^n]$ given by

1. Objects are the numbers $0, 1, ..., n$
2. The morphisms are simplicial Hom Sets of the form

$$\mathrm{Hom}_{C[\Delta^k]}(i,j) = N(P_{i,j})$$

   Here, $N(P_{i,j})$ is the nerve of the poset $P_{i,j}$.

3. Compositions are given by the union.

Example 3.10. When $n = 0$, $C[\Delta^0]$ is the terminimal simplicial category that has one simple object 0 and singleton simplicial mapping space.

When $n = 1$, $C[\Delta^1]$ has objects $0, 1$ and all the simplicial mapping spaces are again trivial (empty or singletons).

The interesting case occurs when $n = 2$. In this case, $C[\Delta^2]$ has three objects $0, 1, 2$. The non-trivial simplicial hom-set is $N(P_{0,2})$. There are two elements in $P_{0,2}$ here, $\{0, 2\} \subseteq \{0, 1, 2\}$.

It turns out this construction is in fact functorial in n. Hence, we can define a **simplicial set of spaces** Spc as the following.

Definition 3.11. A simplicial set of spaces Spc is given by the simplicial set

- $\mathrm{Spc}_n = \mathrm{hom}_{sCat}(C[\Delta^n], \mathrm{Kan})$ where Kan stand for the category of Kan complexes with simplicial hom sets.
- Note that this can be equivalently written as $\mathrm{Hom}_{sSet}(\Delta^n, \mathrm{Spc})$.

Remark 3.12. The construction of Spc is an example of what is called a **homotopy coherent nerve construction**, which is a construction given to any simplicial category $\mathcal{C}$.

Example 3.13. We note that the 0-simplicies of Spc are exactly the objects of Kan. The 1-simplicies are the maps of Kan complexes. The 2-simplicies are in bijection with maps of Kan complexes $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$ together with a homotopy $g \circ f \simeq h$.

4 Meeting September 26th, 2024

Speaker: Elle Pishevar

Today we will be talking about limits, colimits, and adjunctions in the setting of ∞-categories. Let us first establish some terminologies for this lecture:

1. $\mathcal{C}$ will denote an ∞-catgeory.
2. For $a, b \in obj(\mathcal{C})$, we denote the mapping space $\mathrm{Map}_{\mathcal{C}}(a, b)$
3. We use $I \in \mathrm{sSet}$ to denote an indexing diagram - this will be used later to index limits (For the purposes of this talk, we say sSet is a small category).
4. $\mathcal{C}^I = \mathrm{Fun}(I \to \mathcal{C})$ is the functor category from I to $\mathcal{C}$.

4.1 Limits

Definition 4.1. A cone of a functor F consists of a pair (y, η), where $\eta : c_y \to F$ is a natural transformation from c_y to F. Here c_y descends as the map

$$c_y : I \to \Delta^0 \to_y \mathcal{C}$$

and is the constant functor. From here we can define a map $Map_{\mathcal{C}}(x, y) \to Map_{\mathcal{C}^I}(c_x, F)$ as the composition of

$$\mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{\quad c \quad} \mathrm{Map}_{\mathcal{C}^I}(c_x, c_y) \xrightarrow{\quad \eta_* \quad} \mathrm{Map}_{\mathcal{C}^I}(c_x, F)$$

Definition 4.2. The cone (y, η) is a **limit cone** if the map $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{C}^I}(c_x, F)$ specified previously is a homotopy equivalence. In this case, we call y the **limit** of F. We write this notationally as

$$y = \lim_I F = \lim_{i \in I} F(i).$$

Proposition 4.3. If we plug in $N(I)$ and $N(\mathcal{C})$ (nerves of 1-categories), then the previous definition are recovers the ordinary 1-limits.

Example 4.4. Here are some common examples for limits:

1. In the specific case where $I = \emptyset$, then $\mathrm{Fun}(I, \mathcal{C}) = \mathrm{pt}$. In this case, y is a limit if for all x

$$\mathrm{Map}_{\mathcal{C}}(x, y) \simeq \star$$

   In this case, we call y the **terminal object**.

2. Suppose I is a discrete set, ie. I is the disjoint union of some collection of points. Write the elements of $F(I)$ as $F(i)$ for each $i \in I$. Then, we observe that

$$\mathrm{Fun}(I, \mathcal{C}) = \prod_I \mathcal{C},$$

   where the right hand side are taken as products in sSet. $y \in \mathcal{C}$ is a limit to this diagram if there is a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(x, y) \to \prod_{i \in I} \mathrm{Map}_{\mathcal{C}}(x, F(i)).$$

Proposition 4.5. One can check that

$$\mathrm{Map}_{\mathcal{C}}(x, \prod_{i \in I} F(i)) \simeq \prod_{i \in I} \mathrm{Map}_{\mathcal{C}}(x, F(i)).$$

Hence $y \in \mathcal{C}$ is a limit to the discrete diagram is we have the equivalence

$$\mathrm{Map}_{\mathcal{C}}(x, y) \simeq \mathrm{Map}_{\mathcal{C}}(x, \prod_{i \in I} F(i))$$

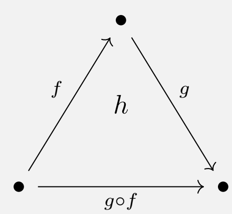
From here we can conclude that $y \simeq \prod_{i \in I} F(i)$.

The definition of limit in ∞-categories, as expected, are also unique.

Proposition 4.6. If y, y' are limits to the same F, then they are equivalent.

Remark 4.7 (Quick Divergence by Nir Gadish)**.** We have that

- $\mathrm{Maps}(\Delta^2, \mathcal{C})$ is the space of all composites in $\mathcal{C}$. This is intuitively the idea of a homotopy:



- $\mathrm{Maps}(\Delta^1, \mathcal{C}) \times \mathrm{Maps}(\Delta^1, \mathcal{C})$ is the space of composable morphisms.
- There is a canonica homomotopy equivalence

$$\mathrm{Maps}(\Delta^2, \mathcal{C}) \to \mathrm{Maps}(\Delta^1, \mathcal{C}) \times \mathrm{Maps}(\Delta^1, \mathcal{C})$$

- This is in fact a homotopy equivalence of Kan complexes, so we can get a section s back

$$\mathrm{Maps}(\Delta^2, \mathcal{C}) \leftarrow \mathrm{Maps}(\Delta^1, \mathcal{C}) \times \mathrm{Maps}(\Delta^1, \mathcal{C})$$

Theorem 4.8. The full subcategory spanned by limits in the functor category (this is sometiems called the sSet of limits) is either empty or contractible (ie. trivial).

The last example of a limit we want to talk about is the **pullback**. This actually acts differently than how they are typically in 1-categories.

Example 4.9. Suppose we have the diagram I being

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ 0' & \longrightarrow & 1 \end{array}$$

In this case $\mathrm{Fun}(I, C)$ are diagrams of the form

$$\begin{array}{ccc} & & b \\ & & \downarrow{\scriptstyle h} \\ c & \xrightarrow[k]{} & d \end{array}$$

Let $b \times_d c$ denote the limit of this diagram. This is called the pullback.

Lemma 4.10. A map $a \in \mathrm{obj}(\mathcal{C}) \mapsto b \times_d c$ is equivalent to the data of

$$i : a \to b, j : a \to k$$

so that we have the homotopy equivalence $h \circ i \simeq k \circ j$ (diagram commutes up to homotopy).

Theorem 4.11. Taking limits commute with mapping spaces, ie. there is a homotopy equivalence of the form

$$\mathrm{Map}_{\mathcal{C}}(x, \lim_i F(i)) \simeq \lim_i \mathrm{Map}_{\mathcal{C}}(x, F(i)).$$

Proof. This is a corollary of a deeper theorem (that we will not prove) that there is an equivalence between

$$(y, \eta) \iff (y^c(y), y^c(\eta)).$$

■

4.2 Colimits

The discussions for the colimits are a lot shorter.

Definition 4.12. For any ∞-category $\mathcal{C}$, there is a canonical notion of an opposite ∞-category $\mathcal{C}^{op}$. The colimit of $\mathcal{C}$ is the limit in the $\mathcal{C}^{op}$. There is a correspondence

$$\{\text{colimits}\, F : I \to \mathcal{C}\} = \{\lim F^{op} : I^{op} \to \mathcal{C}^{op}\}.$$

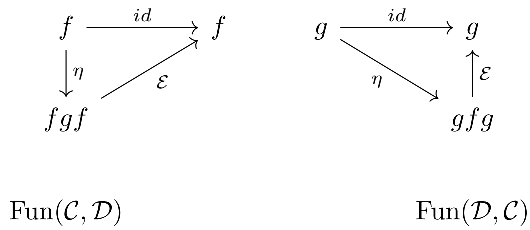
Dually to the notion of a terminal object, we can define an initial object as the colimit over the empty set. Pushouts are dual to the pullbacks.

4.3 Adjunction

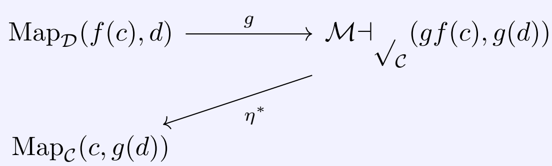
Let $f : \mathcal{C} \to \mathcal{D}, g : \mathcal{D} \to \mathcal{C}$ be functors respectively. They have an adjunction if there is a pair of natural transformations (sometimes aptly called unit maps)

$$\eta : id_{\mathcal{C}} \to gf, \mathcal{E} : fg \to \mathrm{id}_{\mathcal{D}}$$

such that the following diagrams hold up to homotopy

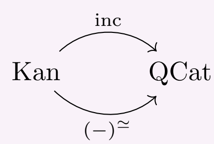


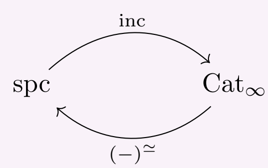
Theorem 4.13. An adjunction gives rise to a homotopy equivalence of spaces



for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$.

Example 4.14. We have the following two examples of adjunctions.





Example 4.15. Consider the category $\mathcal{C}$ to $\mathrm{Fun}(BG, C)$. The left adjoints are the homotopy orbits, and the right adjoints are the homotopy fixed points.

Theorem 4.16 (Fundamental Theorem of Adjoint Functors). Left adjoints preserve colimits, and right adjoints preserve limits.

5 Meeting October 10th, 2024

Topic: Stable ∞-categories
Speaker: Colton Griffin

5.1 Definition of Stable ∞-categories

Today we will be talking about the stable ∞-categories. Most of what we are talking about is a mix of Maximillen's notes, Gallagher's, and Lurie's higher algebra (with an emphasis on the last source). We will concretely investigate two specific examples of them:

1. The ∞-category of Spectra.
2. Derived categories.

Note that while we could form a derived category for any abelian category, the general construction of a "stable" derived category is very general.

Definition 5.1. An ∞-category $\mathcal{C}$ is pointed if there exists an object 0 that is both initial and final. This just means that

$$\mathrm{Hom}_{\mathcal{C}}(0, X) \simeq * \simeq \mathrm{Hom}_{\mathcal{C}}(X, 0)$$

for all objects $X \in \mathrm{obj}(\mathcal{C})$.

Remark 5.2. We remark that $\mathcal{C}$ is pointed if and only if there exists an initial object $\emptyset$, a final point $*$, and a one-morphism $* \to \emptyset$. These conditions imply the $\emptyset$ agrees with $*$ already.

Definition 5.3. Let $\mathcal{C}$ be a pointed ∞-category, a **triangle** is a square $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z \end{array}$$

Here 0 is the initial and final object.

1. A triangle is a fiber (resp. cofiber) sequence if it is a pullback (resp. pushout). Note that Maximilien Péroux calls this exact and coexact instead.
2. Let $g : X \to Y$ be a morphism, a kernel/fiber of g is a fiber sequence of the form

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Y \end{array}$$

3. Let $g : X \to Y$ be a morphism, a cokernel/cofiber of g is a cofiber sequence of the form

$$\begin{array}{ccc} X & \xrightarrow[g]{} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & W \end{array}$$

Now we are ready to definition a stable ∞-category.

Definition 5.4. A pointed ∞-category $\mathcal{C}$ is **stable** if it satisfies the following 2 conditions

1. For every morphism $g : X \to Y$, its fibers and cofibers exist.
2. Every triangle has the property that - it is a fiber sequence if and only if it is a cofiber sequence.

We can regard this definition as a sort of generalization of triangulated categories. The motivation behind why we want to look at stable ∞-categories because triangulated categories requires sort of a choice rather than an intrinsic property that stable ∞-categories offer.

5.2 Spectra

Definition 5.5. A spectrum E is a collection of pointed spaces $(E_n)_{n\geq 0}$ with structure maps

$$\Sigma E_n \to E_{n+1}$$

There is also a morphism of spectra from $E \to E'$ given by $E_n \to E'_n$ for all n that respects structure maps.

Definition 5.6. There is also a notion of Ω-spectrum where we require that the adjoints of the structure maps are weak equivalences.

Example 5.7. Let X be a pointed space, the suspension spectrum $\Sigma^\infty X$ given by $\Sigma^\infty X_n = \Sigma^n X$, and the morphisms of the structure maps are the identity. A specific example of the suspension spectrum is the sphere spectrum $\mathbb{S}$ when we take $X = S^0$.

There is a suitable notion of homotopy groups of a spectrum.

Definition 5.8. Let E be a spectrum, we define

$$\pi_n(E) \coloneqq \operatorname{colim}_k \pi_{n+k}(E_k).$$

In the specific case where E is the sphere spectrum $\mathbb{S}$, $\pi_n(\mathbb{S})$ is exactly the n-th stable homotopy group of spheres.

Example 5.9. Here is another example of spectrum. Let G be an abelian group, we can form the Eilenberg-Maclane spectrum HG where $HG_n = K(G, n)$. There is a canonical weak equivalence given by

$$K(G, n) \simeq \Omega K(G, n + 1),$$

which gives the structure map in suspension. Taking the homotopy groups of HG gives the singular homology is coefficient G.

There is a remarkable theorem that relates spectra to cohomology theories.

Theorem 5.10 (Brown Representability)**.** There is a correspondence between Ω-spectra and cohomology theories.

Definition 5.11. A weak equivalence of spectra E and E' is a morphsim $f : E \to E'$ that induces isomorphism on all of their homotopy groups. SH is the localization of (Spectra) by weak equivalence.

5.3 Loop Space and Suspension

We can define a suitable notion of suspension and loop functor in pointed ∞-categories.

Definition 5.12. Let $\mathcal{C}$ be a pointed ∞-category. Let M^Σ (resp. M^Ω) to be the full subcategory of squares that look like the following

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & X' \end{array}$$

such that the square is a pushout (resp. pullback). Here $0, 0'$ are zero objects.

We have the following theorem that is not at all easy.

Theorem 5.13. Assume that fibers and cofibers all exist. Then, there exists a trivial Kan fibration $M^\Sigma \to \mathcal{C}$ with section $s : \mathcal{C} \to M^\Sigma$. Let $e : M^\Sigma \to \mathcal{C}$ return the object X' - the bottomr right corner of the square. From here we define the suspension functor as

$$\Sigma = e \circ s.$$

We can similarly define ΩX. From here, we get the squares:

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

Lemma 5.14 (Loop-Suspension Adjunction). Σ is left adjoint to Ω. Furthermore, when $\mathcal{C}$ is stable, the functors Σ, Ω gives an equivalence.

We have talked about spectra and stable ∞-categories. Now we will try to relate the two.

Definition 5.15. If $c \in \mathrm{obj}(\mathcal{C})$ is some final object, we can define $\mathcal{C}_*$ the ∞-category of pointed objects to be the full subcategory with morphisms of the form $c \to d$.

Definition 5.16 (Stabilization). We define $\mathrm{Sp}(\mathcal{C})$ as the limit of the sequence

$$\mathcal{C}_* \xleftarrow{\Omega} \mathcal{C}_* \xleftarrow{\Omega} \mathcal{C}_* \longleftarrow \dots$$

In the specific case when $\mathcal{C} = \mathrm{Spc}$, we call $\mathrm{Sp}(\mathrm{Spc})$ the stable ∞-category of spectra.

Proposition 5.17. If $\mathcal{C}$ has finite limits, then $\mathrm{Sp}(\mathcal{C})$ is stable.

5.4 Derived Category

The construction $\mathrm{Sp}(\bullet)$ gives a lot of ways to construct stable ∞-categories. We will look at another major example in the world of derived categories. The general results that motivate this construction is as follows:

Theorem 5.18. Let $\mathcal{C}$ be a stable ∞-category, then its homotopy category $h\mathcal{C}$ has the structure of a triangulated category.

Let us clarify some terminologies first.

Definition 5.19 (Additive Category)**.** An **additive** category $\mathcal{C}$ is a category equipped with the following additional data...

- For $A, B \in \mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(A, B)$ is given the structure of an abelian group.

satisfying...

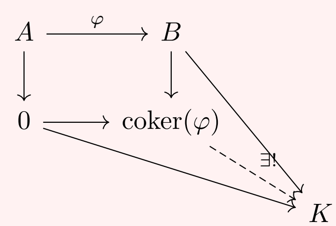
1. Composition distributes over addition, ie.

$$(f + g) \circ h = (f \circ h) + (g \circ h) \text{ and } f \circ (g + h) = (f \circ g) + (f \circ h)$$

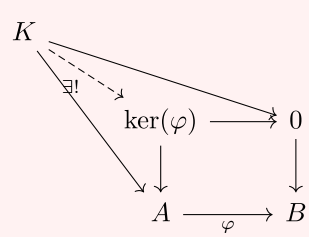
2. $\mathcal{C}$ has a zero object, meaning that it is both the initial and final object.
3. $\mathcal{C}$ has finite products.

An additive category is called **abelian** if...

4. kernels and cokernels exist. In the sense that if we have a morphism $\varphi : A \to B$, the cokernel of this morphism $\mathrm{coker}(\varphi)$,



Similarly for kernel, ie. they are pushouts or pullbacks.



5. Every monomorphism is the kernel of its cokernel. In the sense that for a monomorphism $\varphi : A \to B$, consider the map $A \to B \to \mathrm{coker}(\varphi)$, then the kernel of this morphism $B \to \mathrm{coker}(\varphi)$ is (A, φ).
6. Every epimorphism is the cokernel of its kernel.

Definition 5.20. An additive category $\mathcal{C}$ is **triangulated** if we have

1. A morphism $T : X \in \mathrm{obj}(\mathcal{C}) \to X$ given by $X \mapsto X[1]$.

2. A collection of distinguished triangles of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

such that they satisfy some axioms which we omit for this talk.

A sad fact about triangulated categories is that they are generally very hard to work with.

Remark 5.21. For the stable ∞-category $\mathcal{C}$, it has the structure of a triangulated category if we take T to be the suspension functor.

There is a general procedure to produce a derived category of abelian category, which will be examples of triangulated categories.

Definition 5.22. Let A be an abelian category. We say that A "has enough projectives" (or injectives) if every object admits a projective (or injective) resolution.

Remark 5.23. Let A be an abelian category with enough projectives (or injectives). We can produce a category $D^{\pm}(A)$ as a stable ∞-category such that its homotopy category $hD^{\pm}(A)$ is the usual derived category.

Definition 5.24. Let K be a commutative ring with unity, a dg-category $\mathcal{C}$ (roughly speaking) consists of

1. $\mathrm{Obj}(\mathcal{C})$ - object class
2. For all objects X and Y, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are the chain complexes of K-modules with a notion of tensor product and composition.

We can do this over any ring, but in the specific case where $K = \mathbb{Z}$ is the ring of integers, we have the following construction.

Definition 5.25 (dg Nerve)**.** Let $\mathcal{C}$ be a dg category and $n \geq 0$, we can (roughly speaking) define $N_{dg}(\mathcal{C})_n$ to be set of pairs of the form

$$(\{X_i\}_{i=0}^n, \{f_I\})$$

such that

1. X_i is an object in $\mathcal{C}$ for all i.
2. For all $I = \{i_- < i_n < ... < i_1 < i_+\} \subseteq [n]$, $f_I \in \mathrm{Hom}(X_{i_-}, X_{i_+})_m$ satisfying

$$df_I = \sum_{i=0}^{m} (-1)^j (...)$$

   where ... is some combinatorial arrangement.

Example 5.26. In the dg Nevre construction, the 0-simplex are the objects, and 1-simplex are degree 0 morphisms $f : X \to Y$ with $df = 0$, and so on.

Lemma 5.27. We have the following:

1. $N_{dg}(\mathcal{C})$ is an ∞-category.
2. Let A be an additive category, the category $\mathrm{Ch}(A)$ of chain complexes on A is a dg-category.

Definition 5.28. Let A be an additive category. We define $Ch^{-}(A)$ as the category of chain complexes where $M_n = 0$ for $n << 0$. We similarly define $Ch^{+}(A)$ as the category of chain complexes where $M_n = 0$ for $n >> 0$.

Definition 5.29 (∞-Derived Categories)**.** Let A be an abelian category with enough injectives (resp. projectives), we can define $D^{+}(A)$ (resp. $D^{-}(A)$) as the dg-nerve of $\mathrm{Ch}^{+}(A_{inj})$ (resp. $\mathrm{Ch}^{-}(A_{proj})$).

It turns out that both categories are stable ∞-categories, which follows from the following general fact in Lurie.

Proposition 5.30. Let A be an additive category, then $N_{dg}(Ch(A))$ is stable.

6 Meeting October 17th, 2024

Title: Presentability of ∞-categories
Speaker: Fangji Liu

A natural question to ask for the title is.

> **Question 6.1.** What is a **presentable** ∞-categories? Why do we need a presentable ∞-category?

Most of the talk today will be devoted to defining this category. The intuition is that a presentable category should satisfy the notion of:

1. The simplest kind of categories are small categories, but most categories are not small.
2. The idea of a presentable category is - although it is not small, it should be "generated" by some small subcategories.

There are some interests in why we need presentable ∞-categories too! For instance,

- Presentable ∞-categories are more tractable and hence easier to study.
- Another motivation came from the universal characterization of K-theory (by BGT). The construction utilized some additive/localizing invariants in $\mathrm{Cat}_\infty^{ex} \to D$ where we required D to go into some presentable ∞-category

$$\mathrm{Cat}_\infty^{ex} \to D \hookrightarrow \text{presentable } \infty\text{-category}$$

- There is a recent development called **continuous K-theory** which is a functor

$$K : \{\text{dualizable presentable } \infty\text{-categories}\} \to \mathrm{Sp}$$

  which extends the standard functor we have

$$K : \mathrm{Cat}_{small} \to \mathrm{Sp}\,.$$

- Adjoint functor theorem.
- There is a correspondence between presentable ∞-categories and combinatorial model categories.

6.1 Cocompletion and Ind-completion

To discuss the construction, we will first talk about cocompletion and ind-completion. For an ordinary category $\mathcal{C}$, it need not be cocomplete (meaning that it admits all small colimits). There is, however, a very natural way to produce a cocompletion of $\mathcal{C}$ (it can be thought of as an analog of free group).

> **Theorem 6.2.** The **free cocompletion of** $\mathcal{C}$ is the presheaf category of $\mathcal{C}$, ie.
>
> $$\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, Set).$$
>
> The fully-faithful embedding of $\mathcal{C}$ in $\mathcal{P}(\mathcal{C})$ is given by the Yoneda embedding, ie
>
> $$i : \mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C}), c \mapsto [-, c]$$
>
> We call the essential image of $\mathcal{C}$ as the **representables** in $\mathcal{P}(\mathcal{C})$.
>
> In other words, let $\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), D)$ be all the functors that preserve colimits, then there is an equivalence of category given by restriction
>
> $$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Proof. Let $H \in \mathcal{P}(\mathcal{C})$, we essentially want to show that

$$H = \text{colimit of some representables}\,.$$

There is a very explicit construction of this colimit. We take the category C/H where

- The objects of C/H are objects $x \in H(c)$ for all c.
- The morphisms from $x \in H(c) \to x' \in H(c')$ is a morphism

$$f : c \to c' \text{ such that } H(f) \cdot x' = x.$$

- In other words, C/H is the full-subcategory of $\mathcal{C}$ spanned by the representables of $\mathcal{P}(C)/H$ (slice category).

One can check that

$$H = \operatorname{colim}_{C/H} F$$

Here each functor $F : C/H \to \mathcal{P}(C)$ sends $x \in H(c) \mapsto i(c)$ (recall i is the Yoneda embedding). ■

This is the discussion for 1-category, but the construction generalizes to ∞-categories!

Theorem 6.3. Let $\mathcal{C}$ be an ∞-category, then the free cocompletion of $\mathcal{C}$ is exactly

$$\mathcal{P}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{op}, \text{Spc}).$$

Proof Sketch. The idea is to find an ∞-category analog of a slice category and apply similar arguments. The slice category is given by the homotopy pullback

$$\begin{array}{ccc} C/H & \longrightarrow & * \\ \downarrow & & \downarrow \\ \text{Fun}(\Delta^1, \mathcal{C}) & \xrightarrow{\Delta} & \text{Fun}(\Delta^0, \mathcal{C}) \end{array}$$

In this case, we will have again that $H = \operatorname{colim}_{C/H} i(c)$. ■

On the other hand, Ind completion is given by the concept of filtered colimits.

Definition 6.4 (Filtered Categories)**.** A 1-category $\mathcal{C}$ is a **filtered category** if

- For any finite list of objects $\{c_i\}_{i=1}^n$, there exists $d \in \text{obj}(c)$ with morphisms $c_i \to d$ for all $i = 1$ to n.
- For any finite collection of morphisms $h_i : c \to c'$ for $i = 1$ to n, there exists a morphism $f : c' \to d$ such that

$$f \circ h_i = f \circ h_j \text{ for all } i, j.$$

Definition 6.5. A **filtered colimit** is a colimit whose index diagram is a filtered category.

The presheaf category is the free cocompletion, we want a suitable analog for cocompletion that only contains all filtered colimits.

Definition 6.6. For an ordinary category $\mathcal{C}$, we define $\text{Ind}(\mathcal{C})$ to be the full subcategory of $\mathcal{P}(\mathcal{C})$ consisting of H such that C/H is a filtered category (or equivalently that H is a filtered colimit of C).

Of course, from here, we have the following.

Proposition 6.7. $\mathrm{Ind}(\mathcal{C})$ is the free filtered cocompletion (also called an Ind completion) of $\mathcal{C}$. In other words, we have an equivalence

$$\mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where the LHS is the filtered-colimit preserving functors.

This is the construction for 1-categories, but the catch is that the same construction does not quite work for ∞-categories. Let us however analyze some properties of filtered categories to see if they can motivate a definition.

Proposition 6.8. A 1-category is filtered if and only if for all finite simplicial sets I, for a map $I \to N(\mathcal{C})$, there exists an extension $I^{\triangle} \to N(\mathcal{C})$. Here $I^{\triangle}$ refers to the cocone (this is just saying every map has a cocone).

Definition 6.9. We say that an ∞-category $\mathcal{C}$ is filtered if for all finite simplicial set I, a map $I \to \mathcal{C}$ extends to $I^{\triangle} \to \mathcal{C}$.

6.2 Compactness

Once we have the notion of filtered colimit, there is a notion of a compact object.

Definition 6.10. An object $d \in \mathcal{C}$, where $\mathcal{C}$ is an ordinary category, is called **compact** if the functor

$$[d, -] : \mathcal{C} \to \mathrm{Sets}$$

preserves filtered colimits. Let C^{ω} be the full subcategory spanned by compact objects.

Here are some examples of compact objects.

Category	Compact Objects
Set	Finite Sets
Vect_k	Finite dimensional vector space
Mod_R	Finitely presented modules
Grps	Finitely presented groups
Top	Fintie Sets with discrete topology
$\mathrm{Open}(X)$	compact open sets in X
sSet	Finite simplicial sets

Table 1: Some examples of categories and their compact objects.

Note that the compact objects are Top are not exactly all the compact spaces...

Proposition 6.11. We make two observations for every category $\mathcal{C}$ (with the exception of Top) in Table 1:

1. $\mathcal{C}$ is generated by compact objects (being colimits of compact objects).
2. The subcategory of compact objects in $\mathcal{C}$ is small.

Definition 6.12. A cardinal κ is called regular if for a collection $\{A_i\}_{i \in I}$ where I has cardinal $< \kappa$ and each A_i has cardinal $< \kappa$, the union $\bigcup_{i \in I} A_i$ has cardinal $< \kappa$.

Example 6.13. $0, \omega$, and the continuum are examples of a regular cardinal. Here ω refers to the cardinality of the natural numbers.

Definition 6.14. For any regular cardinal κ, we can define a κ-filtered category whose collection of objects and morphisms in the definition are no longer finite, but of cardinality $< \kappa$ (they are called κ-small). We can also define κ-compact sets similarly, and $\mathrm{Ind}_\kappa(\mathcal{C})$ similarly. These notions extend similarly to ∞-categories.

6.3 Presentable ∞-category

We are finally able to define a presentable ∞-category.

Definition 6.15. An ∞-category $\mathcal{C}$ is called **presentable** if there exists a regular cardinal κ, a small ∞-category C', such that C' admits κ-small colimits, and

$$\mathcal{C} = \mathrm{Ind}_\kappa(C')$$

Definition 6.16. A functor $f : \mathcal{C} \to \mathcal{D}$ is a **localization** if it has a fully faithful right adjoint. A functor $f : \mathcal{C} \to \mathcal{D}$ is **accessible** if there exists regular cardinal κ, $\mathcal{C}, \mathcal{D}$ admits κ-filtered colimits, and f preserves them.

Theorem 6.17. The following are equivalent:

1. $\mathcal{C}$ is presentable.
2. $\mathcal{C}$ is equivalent to $\mathrm{Ind}_\kappa(C^\kappa)$, where C^κ is the full subcategory of κ-compact objects, and C^κ is essentially small (note no κ!), and admits κ-small colimits.
3. $\mathcal{C}$ is equivalent to $\mathrm{Ind}_\kappa(C')$ such that C' is small and $\mathcal{C}$ (no $'$) admits colimits.
4. There exists a small ∞-category C' and an "accessible localization" in the sense there is a localization $\mathcal{P}(C') \to \mathcal{C}$ whose fully faithful right adjoint is accessible.
5. $\mathcal{C}$ is locally small, cocomplete, and there exists a regular cardinal κ, a set S consisting of κ-compact objects, such that S generates C under small colimits.

Remark 6.18. The condition that $\mathcal{C}$ is equivalent to $\mathrm{Ind}_\kappa(C')$ in (3) such that C' is small is called being "**accessible**".

Example 6.19. For a small category $\mathcal{C}$ that is cocomplete. $\mathcal{C}$ is presentable if and only if $\mathcal{C}$ is idempotent complete.

We will end the meeting with a discussion on the adjoint functor theorem.

Theorem 6.20. Presentable ∞-categories are complete and cocomplete.

Theorem 6.21 (Adjoint Functor Theorem)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories.

1. F is a left adjoint if and only if F preserves colimits.
2. F is a right adjoint if and only if F preserves limits and is accessible.

Remark 6.22 (Remark by Nir Gadish)**.** If every object is the colimit of compact objects, then we can compute the hom-set $[x, y]$ as

$$\begin{aligned} [x, y] &= [\operatorname{colim}_I c, \operatorname{colim}_J d] \\ &= \lim_I [c, \operatorname{colim}_J d] \\ &= \lim_I \operatorname{colim}_J [c, d] \end{aligned}$$

Thus, every morphism can also be hit by morphisms in the compact subcategory.

7 Meeting October 24th, 2024

Title: Homotopy theory of ∞-categories
Speaker: Saul Hilsenrath

Today we will be talking about the homotopy theory of ∞-categories.

7.1 Setup

Let us recall a few constructions from earlier talks.

- We have the nerve functor $N : \text{Cat} \to \text{sSet}$ that is full and faithful.
- We also have the 1-truncation functor $\tau : \text{sSet} \to \text{Cat}$ that extends the inclusion of the simplex category $\Delta \to \text{Cat}$.
- From the previous talk, we now know that $\Delta \to \text{Cat}$ can be extended by taking colimits.
- The one-truncation τ is left-adjoint to the nerve functor N.

We also recall a categorical lemma.

Lemma 7.1. Suppose F is left adjoint to a functor U, and U is full and faithful. Then there is a natural isomorphism given by the co-unit

$$F \circ U \simeq id$$

As a corollary of this categorical lemma, we have that

Corollary 7.2. We have a natural isomorphism of the form $\tau \circ N \simeq 1_{Cat}$.

As the first instance of using this construction, we have the following lemma.

Lemma 7.3. τ preserves binary products.

Proof Sketch. Recall that the nerve $N([n]) = \Delta^n$, so

$$\begin{aligned}
\tau(\Delta^m \times \Delta^n) &\cong \tau(N([m]) \times N([n])) \\
&\cong \tau(N([m] \times [n])) && \text{Nerve is right-adjoint and hence preserves products} \\
&\cong [m] \times [n] && \text{By the preceding Corollary} \\
&\cong \tau(\Delta^m) \times \tau(\Delta^n).
\end{aligned}$$

The proof then concludes by extending this using colimits. ∎

7.2 Concrete Homotopy Theory (on Simplicial Sets)

Everything we discuss in this section applies to all of sSet. Recall in topology, a homotopy is of the form $H : I \times X \to Y$. We want an analog of the interval. In this section, we first establish some items of terminology (note these are not canonical):

1. For a presheaf X over A, let X_a be the image of $a \in \text{Obj}(A)$ by X. We call X_a the **fiber over** a.
2. For a well-ordered, non-empty, finite poset category E, we use Δ^E to denote the nerve of E, $N(E)$.

Definition 7.4. We use J to denote the nerve of the category $0 \iff 1$ (this is a category with two objects 0 and 1, one isomorphism between 0 and 1, and no additional morphisms besides the identities). In this case, we define $\partial J := \partial\Delta^1$. Note that we can write

$$\partial J = \Delta^{\{0\}} \cup \Delta^{\{1\}}.$$

Here the union is taken fiber-wise, and $\Delta^{\{0\}}, \Delta^{\{1\}}$ are both isomorphic to Δ^0.

Remark 7.5. Δ^0 is the terminal object in sSet. As a result, the product $\Delta^0 \times X$ is isomorphic to X.

Lemma 7.6. For $j = 0, 1$, let $i_j : X \to \partial J \times X$ be the embedding of X as $\Delta^{\{j\}} \times X$. Then

$$(\partial J \times X, i_0, i_1) \text{ is the coproduct } X + X.$$

Proof. The idea is just that

$$\partial J \times X = (\Delta^{\{0\}} \times X) \sqcup (\Delta^{\{1\}} \times X).$$

∎

Definition 7.7. In the category of simplicial sets,

1. A J-homotopy is a morphism $J \times X \to Y$.
2. We say that $f, g : X \to Y$ are J-homotopic, written as $f \sim_J g$, if there exists a lift to the problem

$$\begin{array}{ccc} \partial J \times X & \xrightarrow{[f,g]} & Y \\ \big\downarrow & \nearrow_{\exists h} & \\ J \times X & & \end{array}$$

Note that equivalently, this is saying that

$$\begin{array}{ccc} X + X & \xrightarrow{[f,g]} & Y \\ \big\downarrow & \nearrow_{\exists h} & \\ J \times X & & \end{array}$$

3. We say that $f : X \to Y$ is a J-homotopy equivalence if there exists $g : Y \to X$ such that

$$g \circ f \sim_J 1_X, f \circ g \sim_J 1_Y.$$

We state the following lemma whose verification is left to the reader.

Lemma 7.8. $\sim_J$ is reflexive and symmetric. It is generally not transitive, though.

Proposition 7.9. Let $f, g : X \to Y$. If $f \sim_J g$, then $\tau(f)$ is naturally isomorphic to $\tau(g)$ (here $\tau(f), \tau(g)$ are functors in Cat).

Proof. Recall that τ preserves coproducts and we showed that it preserves binary products. Thus, we have the following diagram

$$\begin{array}{ccc} \tau(X)+\tau(X) & \xrightarrow{[\tau(f),\tau(g)]} & \tau(Y) \\ \downarrow & \nearrow_{\tau(h)} & \\ \tau(J)\times\tau(X) & & \end{array}$$

Here h is given by the $\sim_J$ definition. In particular, by how J is defined, $\tau(J) \cong (0 \iff 1)$. Thus, $\tau(J)$ is going to look like a category of the form $(a_0 \iff a_1)$.

Let $f : a_0 \to a_1$ be the only morphism from a_0 to a_1 in that category. We can consider

$$\eta = (\tau(h)(f \times 1_x))_{x \in \mathrm{Obj}(\tau(X))}$$

which will give the desired natural isomorphism. ■

Corollary 7.10. As a corollary, if $f : X \to Y$ is a J-homotopy equivalence, then $\tau(f)$ is an equivalence of categories.

Proof. From the previous proposition, if $g : Y \to X$ is a J-homotopy inverse, then

$$\tau(g)\tau(f) \sim 1_{\tau(X)} \text{ and } \tau(f)\tau(g) \sim 1_{\tau(Y)}.$$

■

Here we introduce another notation: We have that i (resp. p) has the left (resp. right) lifting property with respect to p (resp. i) if for every commutative square of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

there is a lift in the diagonal of the square, as seen above. We denote this situations as i LLP p or p RLP i.

Definition 7.11. Given a collection I of morphisms, we write $r(I), \ell(I)$ for

$$r(I) = \{p \mid p \text{ RLP } i,\ \forall i \in I\}$$

$$\ell(I) = \{i \mid i \text{ LLP } p,\ \forall p \in I\}.$$

Definition 7.12. We use Cof (cofibrations) to denote the collection of monomorphisms in sSet. We use tFib (called trivial fibrations) to denote $r(\mathrm{Cof})$.

Proposition 7.13. Let $f : X \to Y$ be a trivial fibration. Then f is a J-homotopy equivalence.

Proof. We use $\pi : J \times X \to X$ to denote the projection. Let $\emptyset$ denote the empty presheaf (i.e., the presheaf whose fibers are all empty). Then we have a lift of the form

$$\begin{array}{ccc} \emptyset & \hookrightarrow & X \\ \downarrow & \nearrow^{\exists s} & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[=]{} & Y \end{array}$$

and a lift of the form

$$\begin{array}{ccc} \partial J \times X & \xrightarrow{[s\circ f, 1_X]} & X \\ \big\downarrow & \overset{\exists h}{\nearrow} & \big\downarrow f \\ J \times X & \xrightarrow[f\circ\pi]{} & Y \end{array}$$

(In both cases, we have lifts because the left sides are monomorphisms.) In particular, the first diagram tells us that $f \circ s = 1_Y$, and the second diagram tells us that $s \circ f \sim_J 1_X$. ■

We establish yet another notation: For simplicial sets A, X, we write $X^A := \mathrm{Fun}(A, X) = \mathrm{Hom}_{\mathrm{sSet}}(h(-) \times A, X)$. Here $h(-)$ is given by the Yoneda embedding $\Delta \hookrightarrow \mathrm{sSet}$.

Thus, a map $f : A \to B$ induces a morphism $X^f : X^B \to X^A$. In particular, the n-th component of X^f is the natural transformation

$$(X^f)_n = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times f, X),$$

given by pre-composition by $1_{\Delta^n} \times f$.

Here we state a proposition that we won't prove for the sake of time.

Proposition 7.14. Let $f : A \to B$ be a J-homotopy equivalence. Then X^f is also a J-homotopy equivalence. In particular, this implies that $\tau(X^f)$ is an equivalence of categories.

7.3 Abstract Homotopy Theory (on ∞-Categories)

Definition 7.15. Let A, B be simplicial sets. We say that $f : A \to B$ is a **categorical weak equivalence** if for all ∞-categories X, $\tau(X^f)$ is an equivalence of categories. We use W to denote the class of categorical weak equivalences.

Example 7.16.

1. J-homotopy equivalences are categorical weak equivalences.
2. As a special case, trivial fibrations are categorical weak equivalences.

Lemma 7.17. W has the 2-out-of-3 property. In other words, if $h = g \circ f$, and two out of f, g, h are in W, then the third morphism is in W.

Proof Sketch. The class of equivalences of categories in Cat has the 2-out-of-3 property. Now use the functoriality of $\tau \circ X^{(-)}$. ■

Definition 7.18. The class of trivial cofibrations tCof is given by $\mathrm{Cof} \cap W$. Further, we write the class of categorical fibrations as

$$\mathrm{Fib} = r(\mathrm{tCof}).$$

Lemma 7.19. Let $I_1 = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$ (note that this is a set). Then

$$\mathrm{Cof} = \ell(r(I_1)).$$

Definition 7.20. Let κ be any cardinal. A simplicial set has size $< \kappa$ if $|\operatorname{Mor}(\Delta/A)| < \kappa$.

Lemma 7.21. There exists a cardinal κ such that if I_2 is the set of trivial cofibrations between simplicial sets of size $< \kappa$, then

$$\mathrm{tCof} = \ell(r(I_2)).$$

Remark 7.22. By definition, the class of trivial fibrations is $r(\mathrm{Cof})$, so we have that

$$\mathrm{tFib} = r(\mathrm{Cof}) = r(\ell(r(I_1))) = r(I_1).$$

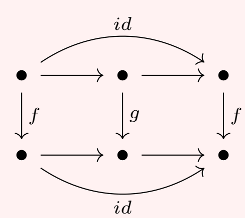
By the same argument

$$\mathrm{Fib} = r(I_2).$$

Finally, $\mathrm{tFib} = r(\mathrm{Cof}) \subset r(\mathrm{tCof}) = \mathrm{Fib}$. Thus, the trivial fibrations form a sub-class of the categorical fibrations.

Definition 7.23. Let $\mathcal{C}$ be a 1-category. A **weak factorization system** (WFS) is a pair (A, B) such that

1. $A, B \subset \operatorname{Mor}(\mathcal{C})$.
2. A, B are closed under retracts. Here, if we have a commutative diagram



   we say that "f is a retract of g".

3. $A \subset \ell(B)$.
4. For any $f \in \operatorname{Mor}(\mathcal{C})$, there exist $i \in A, p \in B$ such that $f = p \circ i$.

Remark 7.24. In the very concrete world of compactly generated weakly Hausdorff spaces, the mapping cylinder would give a weak factorization system.

Lemma 7.25 (Small Object Argument). Let $\mathcal{C}$ be a 1-category and $I \subset \operatorname{Mor}(\mathcal{C})$ be a **set** (note the importance of this being a set) such that for all $i \in I$, $c = \operatorname{dom}(i)$ is small, in the sense that there exists a cardinal κ such that

$$\operatorname{Hom}_{\mathcal{C}}(c, -) \text{ preserves colimits of } \kappa\text{-filtered well-ordered sets.}$$

Then, $(\ell(r(I)), r(I))$ is a weak factorization system.

Lemma 7.26. If A is a small category, then all objects of the presheaf category $\mathrm{Set}^{A^{\mathrm{op}}}$ are small. Here by small, we mean the definition of small in the small object argument.

In particular, this lemma implies that we don't have to worry about the smallness criterion of the small object argument. In particular, we obtain as a corollary

Proposition 7.27. $(\mathrm{Cof}, \mathrm{tFib})$ and $(\mathrm{tCof}, \mathrm{Fib})$ are weak factorization systems.

We know that $\mathrm{tCof} = \mathrm{Cof} \cap W$. We would like a similar result for fibrations.

Lemma 7.28. $\mathrm{tFib} = \mathrm{Fib} \cap W$.

Proof. We know from an earlier remark that $\mathrm{tFib} \subset \mathrm{Fib}$, and we know that every trivial fibration is a categorical weak equivalence. Thus, $\mathrm{tFib} \subset \mathrm{Fib} \cap W$. For the other direction, let $f \in \mathrm{Fib} \cap W$. Since $(\mathrm{Cof}, \mathrm{tFib})$ is a WFS, there exist $i \in \mathrm{Cof}, p \in \mathrm{tFib}$ such that

$$f = p \circ i.$$

Now, $f, p \in W$, so the 2-out-of-3 property gives us that $i \in W$. Now, this implies that $i \in \mathrm{tCof}$.

Thus, f has the RLP with respect to i. A standard category theory argument shows that f is a retract of p, but since WFS is closed under retracts, we have that $f \in \mathrm{tFib}$. ■

With all the setup we have built earlier, we may give a model category structure.

Definition 7.29. A **model category** is a 1-category $\mathcal{C}$ with $W, \mathrm{Fib}, \mathrm{Cof} \subset \mathrm{Mor}(\mathcal{C})$ such that

- $\mathcal{C}$ has all finite limits and colimits.
- W has the 2-out-of-3 property.
- $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ and $(\mathrm{Cof} \cap W, \mathrm{Fib})$ are weak factorization systems.

Theorem 7.30. sSets with $W, \mathrm{Fib}, \mathrm{Cof}$ defined previously is a model category. This model category is called the **Joyal model category** and denoted sSet_J.

7.4 Fibrant Objects

Notation: We use iFib to denote the class of inner fibrations (defined in Mats' talk). We also recall from the same talk that X is an ∞-category if and only if $(!_X : X \to \Delta^0) \in \mathrm{iFib}$.

Lemma 7.31. $\mathrm{Fib} \subset \mathrm{iFib}$.

Proof Idea. One can show that the morphisms in $\ell(\mathrm{iFib})$ (a.k.a., the inner anodyne morphisms) are all trivial cofibrations. ■

For $f \in W$, we know that for all ∞-categories X, $\tau(X^f)$ is an equivalence of categories, so it is essentially surjective. For $f \in tCof$, we have a stronger result.

Lemma 7.32. If $f \in \mathrm{tCof}$, then $\tau(X^f)$ is surjective on objects for all ∞-categories X.

Definition 7.33. In a model category $\mathcal{C}$, an object c is **fibrant** if the unique morphism $!_c : c \to 1$ to the terminal object is a fibration.

Theorem 7.34. The fibrant objects in sSet_J are exactly the ∞-categories.

Proof. Suppose X is a fibrant object in sSet_J. Then $!_x \in \mathrm{Fib} \subset \mathrm{iFib}$, which we know from Mats' talk implies that X is an ∞-category.

Conversely, suppose X is an ∞-category, and let $f : A \to B$ be a trivial cofibration. Then we know that $\tau(X^f)$ is surjective on objects. In particular, recall that on ∞-categories, τ is the homotopy category construction, so the surjectivity condition is the exact same thing as saying that $\mathrm{Hom}_{\mathrm{sSet}}(f, X)$ (pre-composition by f) is surjective. In particular, all triangles of the following form have a lift $B \to X$.

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\big\downarrow & \overset{\exists}{\nearrow} & \\ B & & \end{array}$$

This implies that when we add the terminal object Δ^0 (which does not affect the rest of the diagram), we have a lift.

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\big\downarrow & \nearrow & \big\downarrow \\ B & \longrightarrow & \Delta^0 \end{array}$$

In particular, this implies that $!_X$ RLP f. Hence, we have that $!_X \in r(\mathrm{tCof}) = \mathrm{Fib}$. Thus, X is a fibrant object! ∎

7.5 Combinatorial Model Categories and Localization

Given a model 1-category, we can define an associated homotopy category by "localizing the weak equivalences", but how can we do that for ∞-categories? This is the goal of this section. We first establish some setup.

Definition 7.35.

1. A 1-category $\mathcal{C}$ is **presentable** if its nerve is presentable.
2. A model category $\mathcal{C}$ is **cofibrantly generated** if it is constructed using the small object argument, i.e., there are sets $I, J \subset \mathrm{Mor}(\mathcal{C})$ (I contains generating cofibrations, J contains generating trivial cofibrations) such that $\mathrm{Cof} = \ell(r(I))$ and $\mathrm{Fib} = r(J)$.
3. A **combinatorial model category** is a cofibrantly generated model category whose underlying category is also presentable.

Example 7.36.

1. The category Set is presentable.
2. For any small category A, the presheaf category $\mathrm{Set}^{A^{\mathrm{op}}}$ is presentable. In particular, this means that sSet is presentable.

3. The Joyal model category sSet_J is a combinatorial model category.

Definition 7.37. Let C be an ∞-category and $i : \tau(C)' \to \tau(C)$ be a subcategory of its one-truncation. We can define a simplicial subset C' as the pullback:

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ N(\tau(C)') & \longrightarrow & N(\tau(C)) \end{array}$$

Here, C' is an ∞-category, and it is called the **subcategory of C spanned by** $\tau(C)'$. If $\tau(C)'$ is full, then C' is the **full subcategory** of C spanned by $\tau(C)'$.

Now we are prepared to define the notion of localization.

Definition 7.38. Let $W \subset A$ be simplicial sets. For each ∞-category X, $\mathrm{Fun}_W(A, X) \subset \mathrm{Fun}(A, X)$ is the full subcategory of functors $A \to X$ taking W to invertible morphisms.

A **localization** of A by W is the data $(L(A), \gamma : A \to L(A))$ such that

1. $L(A)$ is an ∞-category.
2. $\gamma : A \to L(A)$ takes W to invertible isomorphisms.
3. For every ∞-category X, X^γ induces an equivalence of ∞-categories $\mathrm{Fun}(L(A), X) \to \mathrm{Fun}_W(A, X)$.

When $\mathcal{C}$ is a model category, we let $L(C)$ denote the localization of $N(\mathcal{C})$ by the subcategory generated by the weak equivalences W.

Theorem 7.39. Let $\mathcal{C}$ be a combinatorial model category. Then $L(\mathcal{C})$ is presentable.

Corollary 7.40. The ∞-category of small ∞-categories $\infty\text{-Cat} = L(\mathrm{sSet}_J)$ is presentable.

Remark 7.41. Why is ∞-Cat "the same as" $L(\mathrm{sSet}_J)$? This is actually the definition of ∞-Cat in Cisinki's book, but we justify the intuition below.

In ordinary categories, it is a standard theorem in model category theory that given a model category $\mathcal{C}$, its associated homotopy category $\mathrm{Ho}(C)$ is categorically equivalent to $C_{cf}/\sim$, the category whose objects are the objects in C that are both fibrant and cofibrant and whose morphisms are suitable "homotopy classes of maps" in C. (These turn out to just be J-homotopy classes in the case of sSet_J.)

We proved in the previous section that the ∞-categories are all fibrant objects. What about cofibrant objects? Well, we observe that the simplicial set with empty fibers is the initial object and the unique map from the initial object to any other simplicial set is thus a monomorphism. This means that every object in sSet_J is cofibrant! Thus, in light of this standard theorem in model category theory, $\mathrm{Ho}(\mathrm{sSet}_J)$ is going to be the category whose objects are (small) ∞-categories and whose morphisms are J-homotopy classes of maps between ∞-categories. When we take the one-truncation of $L(\mathrm{sSet}_J)$, we are going to get $\mathrm{Ho}(\mathrm{sSet}_J)$ exactly!

8 Meeting October 31st, 2024

Title: Universal Characterization of Algebraic K-Theory
Speaker: Albert Yang

8.1 Why Do We Care?

There are a series of conjectures in mathematics that are intricately related to the study of algebraic K-theory!

1. For people in algebraic number theory, there is a conjecture called **Kummer-Vandiver conjecture** that is very relevant to algebraic K-theory. Let $\mathbb{Q}(\zeta_p)$ be a number field, where ζ_p is a primitive p-th root of unity. In other words, $\mathbb{Q}(\zeta_p)$ is the p-th cyclotomic field.

   Conjecture 8.1. For all maximal real subfield F of $\mathbb{Q}(\zeta_p)$, let $h(F)$ be the class number of F, then p does not divide $h(F)$.

   There is an incredible result by a combination of Kurihara and Voevodsky that showed that

   Theorem 8.2. The Kummer-Vandier conjecture is true if and only if $K_{4n}(\mathbb{Z}) = 0$ for all n.

2. For people interested in geometry and topology, there is also a notion of s-cobordism theorem that relates to algebraic K-theory.

   Definition 8.3. Let W be a cobordism between M and N, we say this is an h-cobordism if the two inclusion maps $M \to W$ and $N \to W$ are homotopy equivalences.

   Note that an obvious h-cobordism is when $M = N$ and $W = M \times [0, 1]$, so W is a cylinder. One fundamental question is ask when an h-cobordism is a cylinder.

   Theorem 8.4. Let $X \hookrightarrow W$ be an h-cobordism, then the obstruction of W to cylinder lies in $K_1(\mathbb{Z}[\pi_1 X])$, this is sometimes also called the **Whitehead group**.

3. For people interested in algebraic geometry, there is also the Lichtenbaum-Quillen conjecture that relates algebraic K-theory and étale cohomology. We will not get into the details of this conjecture, but roughly speaking, the conjecture asserts that the algebraic K-theory does not satisfy étale descent. However, for large i, we have that
$$K_i(S, \mathbb{Z}/n) \cong H^{-i}_{\text{ét}}(S, F^{\text{ét}}/n).$$
   Here n is invertible in S, and $F^{\text{ét}}$ is the sheafification of the functor F, where F assigns each X to $K(X)$.

4. In fact for the algebraic geometers, there is a motivic spectral sequence (Thomason, 1985) of the form
$$H^*_{\text{ét}}(X, \underline{\pi}^{\text{ét}}_* K/p^v[\beta^{-1}]) \implies \pi_* K/p^v(X)[\beta^{-1}].$$
   Here β is called the **Bott element**. The specific details of what is on this item are omitted from this, but the key idea the audience should keep in mind is that there is a way to compute homotopy groups of K using a certain étale cohomology.

8.2 What is Algebraic K Theory?

Here we give a very concise introduction to algebraic K-theory. For a more thorough treatment, we refer to Wiebel's K book!

In this section, we fix R to be an associate unital ring with $1_R \neq 0_R$.

Definition 8.5. For $n > 0$, we define $K_n(R) = \pi_n(\mathrm{BGL}(R)^+)$. Here the plus sign "+" is Quillen's plus construction.. For $n = 0$, we define $K_0(R)$ as

$$\mathbb{Z}[\text{isomorphism classes of finite projective (left) } R\text{-modules}]/\sim$$

Here the equivalence relation is generated by $[P \oplus Q] \sim [P] + [Q]$.

We also deine $K(R)$ as $K_0R \times \mathrm{BGL}(R)^+$.

Here are some examples of algebraic K-theory.

Example 8.6. The first major non-trivial calculation in algebraic K-theory is the K-theory of finite fields. In general, we have that

1. Let $\mathbb{F}$ be any field, then $K_0(\mathbb{F}) = \mathbb{Z}$.
2. Let $\mathbb{F}_q$ be a finite field of order q, then

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/(q^i - 1), n = 2i - 1 \\ 0, \text{ else} \end{cases}$$

   The rough idea of the proof was to use "certain operators" $\psi^q - 1 : BU \to BU$ and look at the fibers.

Remark 8.7. The previous construction of algebraic K-theory is done using Quillen's plus-construction. There are two alternative constructions via (1) Quillen's Q-construction and (2) Waldhausen's $S_\bullet$-construction. It turns out that the three constructions are equivalent!

For the sake of brevity, in this talk we will focus on Waldhausen's $S_\bullet$-construction, which is one in the setting of ∞-category. We also write Cat_∞^{st} as the ∞-categories of small stable ∞-categories, whose morphisms are exact functors (ie. preserve finite limits/colimits). In this section, we fix $\mathcal{C}$ as an object in Cat_∞^{st}.

Since $\mathcal{C}$ is stable, it is by definition pointed, and we use $*$ to denote the zero object in $\mathcal{C}$.

Definition 8.8. We define Waldhausen's $S_\bullet$-construction as follows - $S_\bullet\mathcal{C}$ is a simplicial category of the form:

- $S_0\mathcal{C} = *$.
- $S_1\mathcal{C}$ are diagrams of the form

$$\begin{array}{ccc} * & \longrightarrow & X \in \mathcal{C} \\ & & \downarrow \\ & & * \end{array}$$

  It turns out that $S_1\mathcal{C} \simeq \mathcal{C}$.

- $S_2\mathcal{C}$ are diagrams of the forward

$$\begin{array}{ccccc} * & \longrightarrow & X_{0,0} & \longrightarrow & X_{0,1} \\ & & \downarrow & & \downarrow \\ & & * & \longrightarrow & X_{1,1} \\ & & & & \downarrow \\ & & & & * \end{array}$$

where the square here is co-Cartesian (ie. pushout). It turns out that $S_1\mathcal{C} \simeq \mathrm{Fun}(\Delta^1, \mathcal{C})$.

- In general, $S_n\mathcal{C}$ are diagrams are of the form

$$\begin{array}{ccccccccc}
* & \longrightarrow & X_{0,0} & \longrightarrow & X_{0,1} & \longrightarrow & \dots & \longrightarrow & X_{0,n-1} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & X_{1,1} & \longrightarrow & \dots & \longrightarrow & X_{1,n-1} \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & * & & \ddots & & \vdots \\
 & & & & & & & & \downarrow \\
 & & & & & & & & X_{n-1}, n \\
 & & & & & & & & \downarrow \\
 & & & & & & & & *
\end{array}$$

(Pictorially, they look like upper triangular matrices), such that each square is co-Cartesian/pushout.

Here are some important facts about this $S_\bullet$ construction:

1. All $S_n\mathcal{C}$ are stable, ie. $S_n\mathcal{C} \in \mathrm{Cat}_\infty^{st}$.
2. We can construct the algebraic K-theory spectrum $K\mathcal{C}$ such that

$$K\mathcal{C}_n \coloneqq |(S_\bullet^{(n)}\mathcal{C})^{\simeq}|$$

Here $S_\bullet^{(n)}\mathcal{C} \coloneqq (S_\bullet \circ \dots \circ S_\bullet)(\mathcal{C})$ where we iterate the operator n times, and $(S_\bullet^{(n)}\mathcal{C})^{\simeq}$ is the sub-groupoid completion. Furthermore, the structure map is induced by

$$\Sigma(-)^{\simeq} \to |(S_\bullet\mathcal{C})^{\simeq}|$$

by restriction to 1-skeleton. Thus, we have $\Omega^\infty K\mathcal{C} \simeq \Omega|(S_\bullet\mathcal{C})^{\simeq}|$.

3. This construction K outlines a functor $K : \mathrm{Cat}_\infty^{st} \to \mathrm{Sp}$ that is Lax Symmetric Monoidal.
4. $K(\mathcal{C}) = K(\mathrm{Sp}\,\mathcal{C})$. Here $\mathrm{Sp}\,\mathcal{C}$ is the ∞-category of spectrum objects in $\mathcal{C}$, which is an ∞-functor $X : \mathbb{Z}\times\mathbb{Z} \to \mathcal{C}$ such that for all $i \neq j$, $X(i,j) = 0 \in \mathcal{C}$.

Remark 8.9. Like $K : \mathrm{Cat}_\infty^{st} \to \mathrm{Sp}$, there is a similar construction $\mathbb{K} : \mathrm{Cat}_\infty^{st} \to \mathrm{Sp}$ that produces a "non-connective" spectrum, meaning that the spectrum can have non-trivial negative homotopy groups. We omit the details of its construction in this talk here.

Remark 8.10. There is a **Dwyer-Kan (DK) simplicial localization** as follows - let $\mathcal{C}$ be a model category, then there is a way to map

$$\mathcal{C} \to N(\mathrm{FibReplacement}(DK(\mathcal{C}, w\mathcal{C}))).$$

It turns out that the algebraic K theory spectrum produced in this Waldhausen construction may be decomposed in terms of its Dwyer-Kan simplicial localization and can lead to many interesting studies. This was the principal approach done by Blumberg and Mandell in [BM11].

8.3 Universal Property of Algebraic K-Theory

Let $\mathrm{Cat}_\infty^{perf} \subseteq \mathrm{Cat}_\infty^{st}$ be a full subcategory spanned by the **idempotent complete small stable ∞-categories**. In this case, since the ∞-categories are idempotent, we also have an adjunction

$$\mathrm{Idem} : \mathrm{Cat}_\infty^{st} \rightleftharpoons \mathrm{Cat}_\infty^{perf} : \mathrm{Forget}$$

To explain the terminology:

1. Recall that when $\mathcal{C}$ is a classical 1-category:

Definition 8.11. Let $X, Y \in \mathcal{C}$, Y is called **a retract of** X if there is a diagram of the form

$$\begin{array}{ccc} & & X \\ & {\scriptstyle\iota}\nearrow & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{id} & Y \end{array}$$

Here ι is a monomorphism and r is an epimorphism. In this case, we say that $\iota \circ r$ is **idempotent**. This corresponds to our usual notion of idempotence because $(\iota \circ r)^2 = \iota \circ r$.

2. In the ∞-category sense - now let $\mathcal{C} \in \mathrm{Cat}_\infty^{st}$:

Definition 8.12. Let $X, Y \in \mathcal{C}$, we say Y is a retract of X if Y is a retract of X in $h\mathcal{C}$ (the one-truncation). This is the same as saying there exists a 2-simplex $\Delta^2 \to \mathcal{C}$ corresponding to the diagram:

$$\begin{array}{ccc} & & X \\ & {\scriptstyle\iota}\nearrow & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{id} & Y \end{array}$$

We also define Idem^+ as the collection of simplicial sets such that for any finite $J \neq \emptyset$ that is totally ordered,

$$\mathrm{Hom}_{Set}(\Delta^J, \mathrm{Idem}^+) = \{(J_0, \sim) : J_0 \subseteq J, \text{ and } \text{“} \sim \text{”satisfies for } i \leq j \leq k \in J, i, k \in J_0, i \sim k$$

$$\text{implies } j \in J_0 \text{ and } i \sim j \sim k\}.$$

From here we let $\mathrm{Idem} \subseteq \mathrm{Idem}^+$ denote the simplicial sets such that $J_0 = J$ in pairs $(J_0, \sim)$.

3. Finally, we can define what we mean by “idempotent complete”.

Definition 8.13. Let $\mathcal{C} \in \mathrm{Cat}_\infty^{st}$, we say $\mathcal{C}$ is **idempotent complete** if for all $F \in \mathrm{Fun}(\mathrm{Idem}, \mathcal{C})$, F is effective. By effective, we mean that F can be extended to $\mathrm{Fun}(\mathrm{Idem}^+, \mathcal{C})$.

4. The Idem functor sends $\mathcal{C}$ to its idempotent completion.

After explaining the terminologies, we need three more definitions.

Definition 8.14. Let $f : \mathcal{C} \to \mathcal{D} \in \mathrm{Cat}_\infty^{st}$ be a functor. We say that f is a Morita equivalence if $\mathrm{Idem}\, f : \mathrm{Idem}\, \mathcal{C} \to \mathrm{Idem}\, \mathcal{D}$ is an equivalence.

Definition 8.15. Consider the composition of functors in Cat_∞^{st}

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{E}$$

We say that this is an **exact sequence** if:

1. f is fully-faithful.
2. $\mathcal{D}/\mathcal{C} \simeq \mathcal{E}$. Here by the quotient $\mathcal{D}/\mathcal{C}$, we mean the pushout in the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{D}/\mathcal{C} \end{array}$$

3. $g \circ f = 0$.

This exact sequence splits if there is a section $s : \mathcal{E} \to \mathcal{D}, s' : \mathcal{D} \to \mathcal{C}$ (here a section should be thought of as a right adjoint going back such that composition is equivalent to identity via the adjunction morphism).

Definition 8.16. A functor $F : \mathrm{Cat}_\infty^{st} \to \mathrm{Sp}$ (into the stable ∞-category of spectra) is called an **additive invariant** if:

1. F inverts Morita equivalence.
2. F preserves filtered colimits.
3. F maps split exact sequence to split cofiber sequence.

F is called an **localizing invariant** if the first two conditions above holds, and (3') F takes exact sequence to cofiber sequence.

Theorem 8.17. The algebraic K-theory construction K taking a small stable ∞-category to an algebraic K-theory spectra is an additive invariant. If we take the non-connective construction of the spectrum, the functor $\mathbb{K}$ is a localizing invariant.

Remark 8.18. Topological Hochschild homology is an additive invariant.

Notation: Let us write $\mathrm{PSh}^+_{Sp}(\mathrm{Cat}_\infty^{st})$ be the category of presheaves

$$F : ((\mathrm{Cat}_\infty^{st})^W)^{op} \to \mathrm{Sp}$$

such that Condition (3) (not (3')) in Definition 8.16 is satisfied. Also let $\mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W)$ be the category of presheaves to be the same as the plus version, but without requiring condition (3). Here $(\mathrm{Cat}_\infty^{st})^W$ is the full subcategory given by compact objects.

There is a forgetful functor

$$\mathrm{Forget} : \mathrm{PSh}^+_{Sp}(\mathrm{Cat}_\infty^{st}) \to \mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W)$$

where we just forget about Condition (3). It turns out this admits an adjoint $L^+ : \mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W) \to \mathrm{PSh}^+_{Sp}(\mathrm{Cat}_\infty^{st})$

Definition 8.19. Let L^+ be the adjoint as above, we can define a map M_+ as follows.

$$M_+ : \quad \mathrm{Cat}_\infty^{st} \xrightarrow{\text{Yoneda}} \mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W) \xrightarrow{L^+} \mathrm{PSh}_{Sp}^+(\mathrm{Cat}_\infty^{st})$$

$$\mathcal{C} \mapsto M_+(\mathcal{C})$$

This map is called the **additive non-commutative motive**.

Remark 8.20. There is a similar map M_{loc} we can define for **localizing non-commutative motive**.

Finally, we are ready to state the main theorem.

Theorem 8.21 (Blumberg-Gepner-Tabuada, 2013, [BGT13])**.** For all $\mathcal{C} \in \mathrm{Cat}_\infty^{perf}$, there exists two natural equivalences:

1. $\mathrm{Map}(M_+(Sp^W), M_+(\mathcal{C})) \simeq K(\mathcal{C})$.
2. $\mathrm{Map}(M_{loc}(Sp^W), M_{loc}(\mathcal{C})) \simeq \mathbb{K}(\mathcal{C})$.

Here Sp^W is the full subcategory given by compact objects. This is actually called the ∞**-category of finite spectra**. Equivalently, this means that

$$\Psi : \mathrm{PSh}_{Sp}^+(\mathrm{Cat}_\infty^{st}) \to \mathrm{Sp}$$

is co-representable. There is a similar story that happens with localizing invariants.

From here, we obtain three corollaries.

Corollary 8.22. For all $n \in \mathbb{Z}$,

$$K_n\mathcal{C} \simeq \mathrm{Hom}(M_+(Sp^W), \Sigma^{-n} M_+\mathcal{C})$$

Corollary 8.23. For any additive invariant F,

$$\mathrm{Map}(K, F) = F(\mathrm{Sp}^W)$$

This means that K is the universal additive invariant!

Similarly, we also have that

Corollary 8.24. For any localizing invariant F,

$$\mathrm{Map}(\mathbb{K}, F) = F(\mathrm{Sp}^W)$$

This means that $\mathbb{K}$ is the universal localizing invariant!

Now we give an outline for the proof of Theorem 8.21 for the additive case.

Proof Sketch of Theorem 8.21. We only sketch the proof of the additive case. Indeed, for all $\mathcal{A} \in \mathrm{Cat}_\infty^{st}$, $\mathcal{B} \in \mathrm{Cat}_\infty^{perf}$ with B compact in $\mathrm{Cat}_\infty^{perf}$, then one can show that $M_+(\mathcal{A}) \simeq K_\mathcal{A}$. Here $K_\mathcal{A}$ is defined as follows:

- $K_\mathcal{A} \in \mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W)$
- $K_\mathcal{A}(\mathcal{C}) = K(\mathrm{Fun}^{ex}(\mathcal{C}, \mathrm{Idem}\,\mathcal{A}))$. Here, Fun^{ex} denote the exact functors.
- Note that $K_\mathcal{A}(\mathrm{Sp}^W) = K(\mathcal{A})$.
- One can also show that the functor $K_\mathcal{A}$ is local. In other words, for all split exact sequence $\mathcal{B} \to \mathcal{C} \to \mathcal{D}$ in $(\mathrm{Cat}_\infty^{perf})^W$, there is an equivalence

$$\mathrm{Map}(\psi(\mathcal{D}), K_\mathcal{A}) \simeq \mathrm{Map}(\psi(\mathcal{C})/\psi(\mathcal{A}), K_\mathcal{A}) \quad (\dagger).$$

  where $\psi : \mathrm{Cat}_\infty^{perf} \to \mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{perf})^W)$ is the Yoneda embedding.

Thus, we have that

$$\begin{aligned}
\mathrm{Map}(M_+(\mathcal{B}), M_+(\mathcal{A})) &\simeq \mathrm{Map}(M_+(\mathcal{B}), K_\mathcal{A}) && \text{Recall } M_+(\mathcal{A}) \simeq K_\mathcal{A} \\
&= \mathrm{Map}(L_+ \circ \psi(\mathcal{B}), K_\mathcal{A}) \\
&= \mathrm{Map}(\psi(\mathcal{B}), K_\mathcal{A}).
\end{aligned}$$

Here, the maps in the last line should be thought of as happening in $\mathrm{PSh}_{Sp}((\mathrm{Cat}_\infty^{st})^W)$. We obtained the last equality using the adjunction between L_+ and Forget. Thus, we have that

$$\mathrm{Map}(M_+(\mathcal{B}), M_+(\mathcal{A})) \simeq \mathrm{Map}(\psi(\mathcal{B}), K_\mathcal{A})).$$

From here, since $\mathcal{B}$ is in $(\mathrm{Cat}_\infty^{perf})$ and is compact, $\psi(\mathcal{B})$ is representable.

It turns out there is a theorem called the **spectral Yoneda lemma**. In this case, when we apply the spectral Yoneda lemma to $\psi(B)$, we have that $\mathrm{Map}(\psi(\mathcal{B}), K_\mathcal{A})) \simeq K_\mathcal{A}(\mathcal{B})$. Plugging $\mathcal{B} = Sp^W$ then obtains the proof of the additive case. ■

Remark 8.25. While we did not define THH (topological Hochschild Homology), we do in fact that that

$$\pi_0\,\mathrm{Map}(K, THH) = \pi_0(THH(Sp^W)) \simeq \pi_0(THH(\mathbb{S})) = \mathbb{Z}.$$

The element 1 in $\mathbb{Z}$ corresponds to a unique map $K \to \mathrm{THH}$ which is called the **Dennis trace**.

9 Meeting November 7th, 2024

Speaker: Mattie Ji
Title: Introduction to Algebraic L-Theory

9.1 Motivations for Algebraic L-Theory

Algebraic L-theory is often called an analog of K-theory for modules equipped with quadratic forms. Let us first offer some geometric motivations for how studying quadratic forms arose in a geometric setting.

Let M^n be a closed orientable connected manifold. The classic Poincaré duality in algebraic topology asserts the following.

Theorem 9.1. For all $k \in \mathbb{Z}$, $H^{n-k}(M;\mathbb{Z}) \cong H_k(M;\mathbb{Z})$.

Poincaré duality is quite useful in the study of 4-manifolds in low-dimensional topology. More generally, when $n = 4k$ is a multiple of 4, Poincaré duality provides the following invariant on M.

Theorem 9.2. There exists an element $[M] \in H_n(M;\mathbb{R})$ such that the following is a non-degenerate symmetric bilinear form:

$$\langle \bullet, \bullet \rangle_M : \qquad H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \xrightarrow{\cup} H^{4k}(M;\mathbb{R}) \xrightarrow{\bullet \cap [M]} H_0(M;\mathbb{R}) \cong \mathbb{R}$$

The non-degenerate quadratic form associated to the manifold M is an invariant, and we can assign an invariant to it.

Definition 9.3. We can choose a basis $x_1, ..., x_a, y_1, ..., y_b$ ($a + b = \dim_{\mathbb{R}} H^{2k}(M;\mathbb{R})$) of $H^{2k}(M;\mathbb{R})$ such that $\langle x_i, x_i \rangle = 1$ (ie. positive eigenvalues) and $\langle y_i, y_i \rangle = -1$ (ie. negative eigenvalues). The difference $a - b$ is called the **signature of** M. Note that the sign of the signature depends on the choice of the fundamental class, so we refer to the signature modulo sign.

It is a standard fact in linear algebra that a non-degenerate quadratic form over the reals is completely determined by its dimension and signature.

When $n = 4k + 2$, in this case the middle cup product $H^{2k+1}(M;\mathbb{R}) \times H^{2k+1}(M;\mathbb{R}) \to H^{4k+2}(M;\mathbb{R})$ is no longer commutative but anti-commutative. Over characteristic 2, there is no distinction. Poincaré stills gives the following theorem.

Theorem 9.4. There exists an element $[M] \in H_n(M;\mathbb{Z}/2\mathbb{Z})$ such that the following is a non-degenerate symmetric bilinear form:

$$\langle \bullet, \bullet \rangle_M : \qquad H^{2k+1}(M;\mathbb{Z}/2\mathbb{Z}) \times H^{2k+1}(M;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cup} H^{4k+2}(M;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\bullet \cap [M]} H_0(M;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

There is an analog of signature for $4k$-manifolds in the case of $4k+2$-manifolds known as the Kervaire/Arf-Invariant.

Definition 9.5. A theorem by Arf shows that there is a basis $\{e_1, f_1, ..., e_r, f_r, g_1, ..., g_s\}$ such that the quadratic form associated to $\langle \bullet, \bullet \rangle_M$ may be rewritten as

$$(x_1, y_1, ..., x_r, y_r, z_1, ..., z_s) \mapsto \sum_{i=1}^{r}(a_i x_i^2 + x_i y_i + b_i y_i^2) + \sum_{j=1}^{s} g_j z_j^2$$

The **Kervaire/Arf invariant** is defined as $\sum_{i=1}^{r} a_i b_i$.

Theorem 9.6 (Arf)**.** A non-degenerate bilinear form over $\mathbb{Z}/2\mathbb{Z}$ is completely determined by its dimension and Arf invariant.

Remark 9.7. Before we move on, we briefly discuss one more observation about symmetric bilinear forms and quadratic forms that is reformulated as follows. If R is a commutative ring and let $\mathrm{Proj}(R)$ be the category of finitely generated projective R-modules. We observe that for $P \in \mathrm{Proj}(R)$

1. $\mathrm{Hom}_{R\otimes R}(P \otimes P, R)$ is the collection of bilinear R-valued forms on P.
2. There is an obvious action of C_2 on $\mathrm{Hom}_{R\otimes R}(P \otimes P, R)$, from which we have two canonical identifications

$$\mathrm{Hom}_{R\otimes R}(P \otimes P, M)^{C_2} \text{ are the symmetric bilinear } R\text{-valued forms on } P,$$

$$\mathrm{Hom}_{R\otimes R}(P \otimes P, M)_{C_2} \text{ are the quadratic } R\text{-valued forms on } P.$$

More generally, we could replace R with an R-module M in the items listed above. If we are considering an involution as well, we could also produce skew-symmetric and skew-quadratic forms from this identification.

9.2 Symmetric Bilinear and Quadratic Functors

Thus, the study of quadratic forms and symmetric bilinear forms arises quite naturally in algebraic topology and low-dimensional topology. It is then natural to ask - is there an ∞-categorification of these concepts? The hope is that, perhaps by abstracting the theory, we can study broader problems with similar phenomenon and make previous concrete problems easier.

This is where **Algebraic L-theory** comes in, but, to explain what algebraic L-theory is, we should first define our suitable generalizations of symmetric bilinear forms and quadratic forms in ∞-category theory.

Throughout this section, every ∞-category is a **stable ∞-category**.

- Recall $\mathcal{C}$ being stable means that it is pointed, fibers and cofibers exist, and a triangle is a fiber sequence if and only if it is a cofiber sequence.
- Equivalently, a pointed category $\mathcal{C}$ is stable if it admits finite limits and colimits, and a square is a pushout if and only if it is a pullback (Definition 5.11 of Gallauer).
- There are two canonical functors in $\mathcal{C}$ known as the loop functor Ω and the suspension functor Σ.
- The stable ∞-category of spectra Sp is a canonical example of stable ∞-categories.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories, we say F is **reduced** if it sends the zero object to the zero object (ie. $F(0) = 0$). We say a reduced functor is **exact** if it takes fiber sequences to fiber sequences.

Recall a **fiber sequence** is given by the homotopy pull-back

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & Y \end{array}$$

Let us recall that for any stable ∞-category $\mathcal{C}$ with object X. For any other object X, there is a sequence of mapping spaces $\{\mathrm{Map}_{\mathcal{C}}(Y, \Sigma^n X)\}$ that constitutes a spectrum we will write as $\mathrm{Mor}_{\mathcal{C}}(Y, X)$. A mapping space $Map_{\mathcal{C}}(c, d)$ is given by the pullback:

$$\begin{array}{ccc} Map_{\mathcal{C}}(c,d) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow[(c,d)]{} & \mathrm{Fun}(\partial\Delta^1, \mathcal{C}) \end{array}$$

Definition 9.8 (Symmetric and Non-degenerate Bilinear Functors)**.** A **bilinear functor** is a functor $B : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ such that for all $c \in \mathcal{C}$, the following two functors are both exact,

$$d \mapsto B(c, d), d \mapsto B(d, c).$$

We use $\mathrm{Fun}^b(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{C}^{op}, \mathrm{Sp})$ to denote the full subcategory given by the bilinear functors.

Symmetric: There is a C_2 action on $\mathrm{Fun}^b(\mathcal{C})$ by flipping the two entries, we use

$$\mathrm{Fun}^s(\mathcal{C}) = [\mathrm{Fun}^b(\mathcal{C})]^{hC_2}$$

to denote the ∞-category of C_2-equivariant objects in $\mathrm{Fun}^b(\mathcal{C})$. A bilinear functor $B \in \mathrm{Fun}^s(\mathcal{C})$ is called **symmetric**. Note that this is also called the homotopy fixed point spectra with respect to C_2.

Non-degenerate: Let $B \in \mathrm{Fun}^b(\mathcal{C})$ be a bilinear functor.

1. We say that B is **right non-degenerate** if for each $Y \in \mathcal{C}$, the functor $B(-, Y)$ is representable by an object in $\mathcal{C}$. In other words, we can write

$$B(X, Y) \simeq \mathrm{Mor}_{\mathcal{C}}(X, D^{right} Y).$$

   Here $D^{rigt} : \mathcal{C}^{op} \to \mathcal{C}$ is a functor keeping track of the representation.

2. We say that B is **left non-degenerate** if for each $X \in \mathcal{C}$, the functor $B(X, -)$ is representable by an object in $\mathcal{C}$. In other words, we can write

$$B(X, Y) \simeq \mathrm{Mor}_{\mathcal{C}}(Y, (D^{left})^{op} X) \simeq \mathrm{Mor}_{\mathcal{C}^{op}}(D^{left} X, Y).$$

   Here $D^{left} : \mathcal{C} \to \mathcal{C}^{op}$ is a functor keeping track of the representation.

3. We say B is **non-degenerate** if it is both left and right non-degenerate. From definition, we can see that D^{left} and D^{right} are adjoint as

$$\mathrm{Mor}_{\mathcal{C}^{op}}(D^{left} X, Y) \simeq B(X, Y) \simeq \mathrm{Mor}_{\mathcal{C}}(X, D^{right} Y).$$

4. When B is symmetric and non-degenerate (notation: $B \in \mathrm{Fun}^{sn}(\mathcal{C})$), we write D as D^{right}. We note that D^{left} is actually D^{op} since

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}^{op}}(D^{left} X, Y) &\simeq B(X, Y) \\ &\simeq B(Y, X) \\ &\simeq \mathrm{Mor}_{\mathcal{C}}(Y, DX) \\ &\simeq \mathrm{Mor}_{\mathcal{C}^{op}}(D^{op} X, Y). \end{aligned}$$

Perfect: Let $B \in \mathrm{Fun}^{sn}(\mathcal{C})$, from the discussion above we know that D^{op} is adjoint to D. The unit of this adjunction gives an **evaluation map**:

$$\mathrm{ev} : \mathrm{id} \implies DD^{op}.$$

We say that B is **perfect** if ev is an equivalence.

Example 9.9 (Spainer-Whitehead Duality)**.** Let $\mathcal{C} = \mathrm{Sp}$ and B be

$$B(X,Y) = \mathrm{Mor}_{Sp}(X \wedge Y, S).$$

B is a symmetric non-degenerate bilinear functor on $\mathcal{C}$. The corresponding duality functor D is called the **Spainer-Whitehead Duality**.

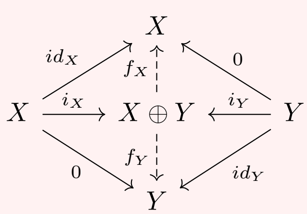
The restriction of B to Sp^{ω} (full subcategory spanned by the compact objects in Sp, in other words, the finite spectra) is **perfect**.

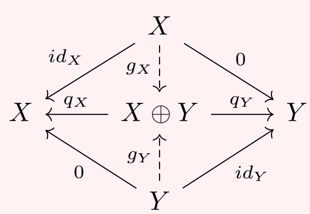
We also want to establish the analog of a quadratic form in ∞-categories. Motivated by the story in linear algebra, we consider the following construction.

Construction 9.10. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a reduced functor. For $X, Y \in \mathcal{C}$, we have consider maps

$$Q(X) \oplus Q(Y) \to_f Q(X \oplus Y) \to_g Q(X) \oplus Q(Y)$$

We note that up to equivalence, Remark 1.1.3.5 of Lurie tells us that $Q(X) \oplus Q(Y)$ is both the product and coproduct of $Q(X)$ and $Q(Y)$. From universal property, we have the following maps:





Since Q is contravariant, we obtain maps

1. $Q(f_X) : Q(X) \to Q(X \oplus Y)$ and $Q(f_X) : Q(X) \to Q(X \oplus Y)$, which induces the map $f : Q(X) \oplus Q(Y) \to Q(X \oplus Y)$ by universal property.
2. $Q(g_X) : Q(X \oplus Y) \to Q(X)$ and $Q(g_Y) : Q(XY) \to Q(Y)$, which induces the map $g : Q(X \oplus Y) \to Q(X) \oplus Q(Y)$ by universal property.

Schematically, we can think of $g \circ f$ as the matrix

$$\begin{pmatrix} Q(id_X) & Q(0) \\ Q(0) & Q(id_Y) \end{pmatrix}$$

Proposition 9.11. The composition $g \circ f$ is the identity, and this makes $Q(X) \oplus Q(Y)$ a direct summand of $Q(X \oplus Y)$. In particular, this gives a symmetric (in its arguments) functor $B : C^{op} \times C^{op} \to \mathrm{Sp}$ such that

$$Q(X \oplus Y) \simeq Q(X) \oplus Q(Y) \oplus B(X,Y).$$

B is called the **polarization of** Q.

> **Remark 9.12.** The proposition is really an analog of the following idea in linear algebra - if $q(x)$ is a quadratic form, then the term $q(x+y) - q(x) - q(y)$ is a symmetric bilinear function.

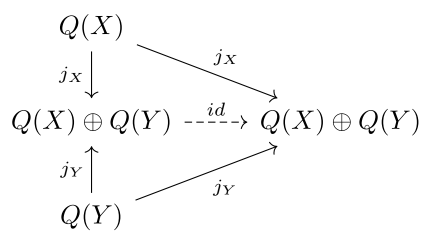
Proof Idea. We said earlier that $g \circ f$ should be schematically thought of as the matrix

$$\begin{pmatrix} Q(id_X) & Q(0) \\ Q(0) & Q(id_Y) \end{pmatrix}.$$

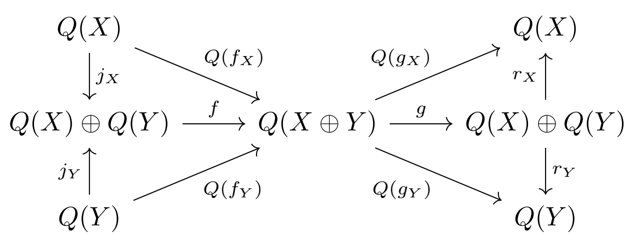
Since Q is a reduced functor, this matrix becomes

$$\begin{pmatrix} id_{Q(X)} & 0 \\ 0 & id_{Q(Y)} \end{pmatrix},$$

which is clearly the identity. More rigorously, the universal property tells us that the identity map is the unique map satisfying



It suffices for us to show this diagram holds when we replace the identity map by $g \circ f$. Now we currently have a diagram of the form

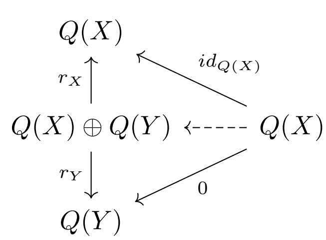


Let us try to compute the term $g \circ f \circ j_X$. Now we see that

$$r_X \circ (g \circ f \circ j_X) = Q(g_X) \circ Q(f_X) = Q(g_X \circ f_X) = id_{Q(X)}$$

$$r_Y \circ (g \circ f \circ j_X) = Q(g_Y) \circ Q(f_X) = Q(0) = 0.$$

Thus, $g \circ f \circ j_X$ is the induced map in the diagram



But j_X is the other map that satisfies this ($Q(X) \oplus Q(Y)$ is both the product and the coproduct), so we have that $g \circ f \circ j_X = j_X$. Similarly, we also have that $g \circ f \circ j_Y = j_Y$, so we conclude that $g \circ f$ is the identity.

Showing that $Q(X) \oplus Q(Y)$ is a direct summand of $Q(X \oplus Y)$ follows more generally from the following fact - let $f : X \to Y, g : Y \to X$ between spectrum such that $g \circ f$ is the identity, then X is a direct summand of Y. To see why, let C_f be the cofiber of $f : X \to Y$ and Z be any spectrum, we have an exact sequence

$$0 \to [Z, X] \to [Z, Y] \to [Z, C_f] \to 0.$$

The existence of g can show that this is in fact split injective! Since we are really looking at cohomology here, this gives us

$$[Z,Y] \cong [Z,X] \oplus [Z,C_f].$$

On the other hand, recall that the coproduct is wedge sum, so we have that

$$[Z, X \vee C_f] \cong [Z,X] \oplus [Z,C_f].$$

Since this holds for all Z, the Yoneda lemma implies that $X \vee C_f \simeq Y$. ■

There is a canonical map we are interested in between Q and B. To reach there we first need to briefly discuss the notion of homotopy fixed points and homotopy orbit. We will not go too into details for the definition, so is life, but we will give two examples to help parse with the definition. We will also only talk about the specific case for C_2.

Definition 9.13. Let X be a spectrum equipped with a C_2-action, in a natural way compatible to X. Then, the **homotopy fixed point spectrum** of X with respect to C_2 is

$$X^{hC_2} = \mathrm{Fun}_G(\Sigma^\infty (EC_2)_+, X)$$

is the mapping space of C_2-equivariant maps between the two spectra.

The **homotopy orbit spectrum** of X with respect to C_2 is

$$X_{hC_2} = \Sigma^\infty (EC_2)_+ \wedge_{C_2} X.$$

Here the wedge product is taken with respect in C_2-spectra.

Example 9.14. Here are two examples whose proofs might not be that obvious

1. Let KU and KO be the complex and real K-theory spectra respectively. There is a C_2-action on KU by replacing a complex vector bundle with its complex conjugate bundle, and $KU^{hC_2} = KO$.
2. On the level of spaces, the homotopy orbit of a one-point space $*$ under C_2 is $\mathbb{R}P^\infty$.

Construction 9.15. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a reduced functor with polarization B. The diagonal map $\Delta : X \to X \oplus X$ and codiagonal map $\nabla : X \oplus X \to X$ induces maps

$$Q(X \oplus X) \xrightarrow{Q(\Delta)} Q(X) \xrightarrow{Q(\nabla)} Q(X \oplus X)$$

There is an inclusion map $i : B(X,X) \to Q(X \oplus X)$ and a projection map $\pi : Q(X \oplus X) \to B(X,X)$, so we can extend this sequence to

$$B(X,X) \xrightarrow{i} Q(X \oplus X) \xrightarrow{Q(\Delta)} Q(X) \xrightarrow{Q(\nabla)} Q(X \oplus X) \xrightarrow{\pi} B(X,X)$$

There is a canonical C_2 action on $B(X,X)$ roughly described as follows. B is symmetric in the higher categorical sense, meaning we are given an isomorphism between $B(X,Y)$ and $B(Y,X)$. When $X = Y$, this becomes an automorphism on $B(X,X)$ that defines a C_2 action. An alternative way to phrase this is that $\Delta(B)$ is a C_2 object of $\mathrm{Fun}(\mathcal{C}^{op}, Sp)$, where $\Delta : \mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{C}^{op}, Sp) \to \mathrm{Fun}(\mathcal{C}^{op}, Sp)$ is the restriction to the diagonal.

Furthermore, since $Q(\nabla)$ and $Q(\Delta)$ are both C_2-equivariant, the diagram above factors through as

$$B(X,X)_{hC_2} \longrightarrow Q(X) \longrightarrow B(X,X)^{hC_2}$$

Remark: The composition here is the norm map.

Thus, we have showed that every reduced functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ can produce an associated functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$. The definition of a quadratic functor is given as follows:

Definition 9.16. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a reduced functor with polarization B. We say Q is **quadratic** if any of the two equivalent conditions is true

1. B is bilinear and the functor $X \mapsto \mathrm{fib}(Q(X) \to B(X,X)^{hC_2})$ is exact.
2. B is bilinear and the functor $X \mapsto \mathrm{cofib}(B(X,X)_{hC_2} \to Q(X))$ is exact.

Furthermore, Q is perfect if its polarization B is perfect.

Remark 9.17. A **quadratic functor** in our talk is really what Thomas Goodwillie would call "a (reduced) and 2**-excisive functor**" in the framework of Goodwillie Calculus.

Example 9.18. Here are some examples of quadratic functors:

1. Any exact functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ is quadratic. In fact, they correspond to all the quadratic functors whose polarization vanishes.
2. Let $\mathcal{C}$ be a stable ∞-category and $B \in \mathrm{Fun}^{bs}(\mathcal{C})$, then

$$Q_B^q(X) = B(X,X)_{hC_2} \text{ and } Q_B^s(X) = B(X,X)^{hC_2}$$

   are quadratic functors. Q_B^q is the analog of **quadratic form** and Q_B^s is the analog of **symmetric bilinear form**.

These two constructions should be reminiscent of Remark 9.7.

9.3 L-Theory of Poincare Category

In this section, we will be working to define the L-theory of a Poincare ∞-category $(\mathcal{C}, Q)$, where Q is **perfect**.

Definition 9.19. A Poincare ∞-category is a pair $(\mathcal{C}, Q)$ where Q is perfect.

Definition 9.20. Let E be an Ω-spectrum, we define $\Omega^\infty E = E_0$ (the 0-th space). For a general spectrum E', there is a canonical way to produce an associated Ω-spectrum E of E' by specifying

$$E_n = \mathrm{colim}_k\, \Omega^k E'_{n+k}.$$

In this case, we define $\Omega^\infty E'$ as $\Omega^\infty E$.

Remark 9.21. We justify the notation Ω^∞ as follows. There is a classical correspondence between an Ω-spectrum and an infinite loop space. Given an infinite loop space X, we can think of X as a sequence of delooping $X_0 = X \to X_1 \to \ldots$ with weak equivalences $X_n \simeq \Omega X_{n+1}$. Thus, given an Ω-spectrum, its sequence of spaces naturally produces an infinite delooping of the 0-th space.

Definition 9.22. A **quadratic object** of $(\mathcal{C}, Q)$ is a pair $(X \in \mathcal{C}, q \in \Omega^\infty Q(X))$.

Recall there is a map $f : Q(X) \to B(X,X)^{hC_2}$, so q determines a point $f(q) \in B(X,X)_0^{hC_2}$ (the 0-th space). Since Q is non-degenerate, we recall that $B(X,X) \simeq \mathrm{Mor}_{\mathcal{C}}(X, DX)$, so $f(q)$ determines a map $X \to DX$. We say that (X,q) is a **Poincaré object** if $X \to DX$ is invertible. We use $\mathrm{Poin}(\mathcal{C}, Q)$ to denote the collection of Poincare objects.

According to Lurie - the intuition to have in mind is that Q is a functor that assigns each object $X \in \mathcal{C}$ a "spectrum of quadratic forms". A quadratic object (X,q) can be thought of as a specific choice of quadratic form for X. A Poincare object (X,q) is a specific choice of a nondegenerate quadratic form.

Example 9.23. The formation of this mapping spectra in $\mathcal{C}$ gives a quadratic functor

$$Q_{hyp} : \mathcal{C} \times \mathcal{C}^{op} \to \mathrm{Sp}, (X,Y) \mapsto \mathrm{Mor}_{\mathcal{C}}(X,Y).$$

In this case, $(\mathcal{C} \times \mathcal{C}^{op}, Q_{hyp})$ is Poincare with duality given by $(X,Y) \mapsto (Y,X)$. $(\mathcal{C} \times \mathcal{C}^{op}, Q_{hyp})$ is called the **hyperbolic ∞-category** associated to $\mathcal{C}$.

Here we give a concrete example of how Poincare objects relate to the geometric setting of manifolds.

Definition 9.24. Let A be an associative ring, **the perfect derived ∞-category** of A is an ∞-category $D^{perf}(A)$ is the full subcategory of $D(A)$ spanned by compact objects. Here $D(A)$ is the derived ∞-category of A (ie. $D(A) = N\,\mathrm{Ch}(A)[\text{quasi-iso}^{-1}]$). Concretely, $D^{perf}(A)$ is roughly constructed as follows:

1. The 0-simplicies of $D^{perf}(A)$ are bounded chain complexes of finitely generated projective left A-modules.
2. A 1-simplex of $D^{perf}(A)$ is the map of chain complexes $f : P_\bullet \to Q_\bullet$.
3. A 2-simplex of $D^{perf}(A)$ is a (not necessarily commutative) diagram of chain complexes

$$\begin{array}{ccccc} & & Q_\bullet & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ P_\bullet & & \xrightarrow[h]{} & & R_\bullet \end{array}$$

   with a chain homotopy from h to $g \circ f$.
4. Higher dimensional simplicies are given analogously with higher-order chain homotopies.

Note that $D^{perf}(A)$ is clearly stable.

Example 9.25. Informally, recall there is a canonical mapping spectrum Mor attached to any stable ∞-category. Specifically we consider B on $D^{perf}(R)$ given by

$$B^i(X,Y) = \mathrm{Mor}_{R \otimes R}(X \otimes_R Y, R[-i]).$$

Here $R[-i]$ is the chain complex that is everywhere zero except for a single copy of R concentrated at the $-i$ degree.

There is an obvious duality given by the Hom-Tensor adjunction, and the associated $Q_R^{q,i}$ and $Q_R^{s,i}$ are both (perfect) quadratic functors. Here we append an index i to indicate that we are considering morphisms into $R[-i]$. We also write $Q_R^q = Q_R^{q,0}$ and $Q_R^s = Q_R^{s,0}$.

The following example is arguably the most important example of this talk. If the reader should get anything out of this talk, it should be this key example.

Example 9.26. Let $\mathcal{C} = D^{\text{perf}}(\mathbb{Z})$ and define

$$Q(X) := \text{Mor}_{D^{\text{perf}}(\mathbb{Z})}(X \otimes X, \mathbb{Z}[-n])^{hC_2}.$$

(Note that Q is $Q_{\mathbb{Z}}^{s,n}$ from our earlier example).
Let M^n be a closed oriented manifold. The singular cochain complexes $C^*(M,\mathbb{Z})$ is an object of $D^{perf}(\mathbb{Z})$. There is a quadratic functor Q on $D^{\text{perf}}(\mathbb{Z})$ given by

$$Q(X) := \text{Mor}_{D^{\text{perf}}(\mathbb{Z})}(X \otimes X, \mathbb{Z}[-n])^{hC_2}.$$

Here $\mathbb{Z}[-n]$ is a chain complex that is all zero except for a single copy of $\mathbb{Z}$ at degree $-n$.
In this case, we have a symmetric intersection pairing on M:

$$(C^*(M;\mathbb{Z}) \otimes C^*(M;\mathbb{Z}))_{hC_2} \to C^*(M;\mathbb{Z}) \to_{[M]} \mathbb{Z}[-n]$$

is a point $q_M \in \Omega^\infty Q(C^*(M;\mathbb{Z}))$. In this example, the statement of Poincare duality may be reformulated as follows:

Theorem 9.27 (Poincare Duality Reformulated). $(C^*(M;\mathbb{Z}), q_M)$ is a Poincare object of $(D^{perf}(\mathbb{Z}), Q)$.

Note that we shifted the index to $-n$ because we defined everything at the 0-th space. In general, a Poincare object of dimension n is a Poincare object of dimension 0 with the index shifted down by n.

Our goal is to now construct a suitable algebraic structure on the collection of Poincaré objects to study them.

Definition 9.28. Let (X,q) and (X',q') be two quadratic (resp. Poincare) objects on $(\mathcal{C},Q)$. We define

$$(X,q) \oplus (X',q') := (X \oplus X', q \oplus q').$$

Here $X \oplus X'$ is the standard (co)product of X and X', and $q \oplus q'$ is the image of (q,q') under the canonical map $Q(X) \oplus Q(X') \to Q(X \oplus X')$. It is a fact that $(X \oplus X', q \oplus q')$ is quadratic (resp. Poincare).

The operation $\oplus$ only gives a commutative monoid structure on the collection of Poincare objects. We want a suitable notion of equivalence so that this becomes a group structure.

Definition 9.29. Let $(\mathcal{C},Q)$ be as before, and $(X,q),(X',q')$ be two Poincare objects. An (algebraic) **cobordism** from (X,q) to (X',q') is the following data:

1. An object $L \in \mathcal{C}$ with maps $\alpha : L \to X$ and $\alpha' : L \to X'$.
2. Q induces maps $Q(X) \to Q(L)$ and $Q(X') \to Q(L)$. Let $\alpha^*(q), (\alpha')^*(q')$ be the images of q and q' be the images in the space $\Omega^\infty Q(L)$. We also want a path p joining $\alpha^*(q)$ and $(\alpha')^*(q')$.
3. (Non-degeneracy condition): The path gives a homotopy between the two maps $L \to D(L)$ given by:

$$\begin{array}{ccccc} X & \xleftarrow{\alpha} & L & \xrightarrow{\alpha'} & X' \\ {\scriptstyle\text{induced by q}}\downarrow & & & & \downarrow{\scriptstyle\text{induced by q'}} \\ DX & \xrightarrow[D(\alpha)]{} & DL & \xleftarrow[D(\alpha')]{} & DX' \end{array}$$

The diagram commutes up to a homotopy determined by the path p. Thus, the induced map $\mathrm{fib}(\alpha) \to L \to_{\alpha'} X' \to DX' \to DL$ is null-homotopic. Thus, there is an induced map of fibers $u : \mathrm{fib}(\alpha) \to \mathrm{fib}(D(\alpha'))$. We require u to be invertible.

We say (X, q) and (X', q') are cobordant if there is a cobordism from (X, q) to (X', q').

Theorem 9.30. Being cobordant is an equivalence relation $\sim$ on $\mathrm{Poin}(\mathcal{C}, Q)$, the Poincare objects of $(\mathcal{C}, Q)$. Furthermore, $\oplus$ is a well-defined abelian group operation on $\mathrm{Poin}(\mathcal{C}, Q)/\sim$.

Definition 9.31. We define $L_0(\mathcal{C}, Q) = \mathrm{Poin}(\mathcal{C}, Q)/\sim$. For $n > 0$, we define $L_n(\mathcal{C}, Q) := L_0(\mathcal{C}, \Omega^n Q)$.

Remark 9.32. The usual approach to defining higher degrees of L-theory is to construct a L-theory spectrum $\mathcal{L}(\mathcal{C}, Q)$ associated to a Poincaré category, and the n-th L-theory would be the n-th homotopy group of this spectrum. It turns out that this is canonically isomorphism to our definition. Due to the time constraint of this talk, we decided to stay with the current approach.

Remark 9.33. Although we have not focused on the classical theory much, we remark that L-theory indeed did not originate from higher algebra but had more concrete foundations. In the specific case where we have $(D^{perf}(R), Q^q_M)$ (with values in an R-module M, possibly with involution), we recover the classical **Wall-Ranicki** quadratic L-groups. Similarly with the symmetric case.

9.4 L-Theory of $\mathbb{Z}$ and Geometric Connections

Definition 9.34. The quadratic and symmetric L-theory of $\mathbb{Z}$ is given by $L_n(D^{perf}(\mathbb{Z}), Q^s_{\mathbb{Z}})$ and $L_n(D^{perf}(\mathbb{Z}), Q^q_{\mathbb{Z}})$ respectively. As a short hand, we denote them as $L^s(\mathbb{Z})$ and $L^q(\mathbb{Z})$ respectively.

Remark 9.35. This is not how this was defined in Lurie. We should have used finitely presented R-module spectra, but it turns out there is no difference with using perfect R-module spectra in this case.

The story of quadratic L-groups of $\mathbb{Z}$ is very important in the world of low-dimensional topology.

Theorem 9.36. $L^q_*(\mathbb{Z})$ may be computed as follows:

$$L^q_n(\mathbb{Z}) = \begin{cases} 8\mathbb{Z}, n = 4k \text{ (signature)} \\ 0, n = 4k+1 \\ \mathbb{Z}/2\mathbb{Z}, n = 4k+2 \text{ (Kervaire invariant)}, \\ 0, n = 4k+3 \end{cases}$$

Theorem 9.37. $L^s_*(\mathbb{Z})$ may be computed as follows:

$$L^s_n(\mathbb{Z}) = \begin{cases} \mathbb{Z}, n = 4k \text{ (signature)} \\ \mathbb{Z}/2, n = 4k+1 \text{ (de Rham invariant)} \\ 0, n = 4k+2 \\ 0, n = 4k+3 \end{cases}$$

Definition 9.38. The de Rham invariant of M^{4k+1} is the rank of 2-torsion in $H_{2k}(M)$ modulo 2, or equivalently the product of two Stiefel Whitney numbers $w_2 w_{4k-1}$.

The geometric connections between a compact oriented manifold of L-groups of $\mathbb{Z}$ are given as follows.

Theorem 9.39. Let M^n be a compact oriented manifold and $n = 4k$. Recall we explained earlier that $(C^*(M;\mathbb{Z}), q_M)$ is a Poincare object of $(D^{perf}(\mathbb{Z}), Q^{s,-n}_{\mathbb{Z}})$ (shifted by n-indices down). Thus, M gives an element element of $L^s_n(\mathbb{Z})$, which is exactly its signature.

Finally, we will end our talk with a brief discussion on the Kervaire invariant one question?

Question 9.40. What manifolds have Kervaire invariant 1?

Theorem 9.41.

1. For $n = 6, 14, 30, 62$, there exists a Keivarie invariant one manifold (this was known in the last century).
2. (Hill-Hopkins-Ravenal), If $n = 2^{J+1} - 2$ for $J \geq 7$, there are no Keivarie invariant one manifold.
3. This only leaves $2^7 - 2 = 126$, which is proved this year (2024) by Lin-Wang-Xu to be positive.

10 Meeting November 14th, 2024

Speaker: Kartik Tandon
Title: Monadicity in ∞-Categories.

For reference, what we are talking about in the lecture (at least when it gets to higher algebra) is adapted from Section 4.7 of Lurie's Higher Algebra.

10.1 The Classical Setting of Monads

There are a few fundamental questions we can ask that are all connected by the idea of Mondas.

Question 10.1.

1. When are two rings R and S Morita equivalent? In other words, there is an (additive) equivalence of category between Mod_R and Mod_S.
2. Let R be a ring, when is there an equivalence between $D(R)$ and $\mathrm{Mod}(HR)$ (in terms of infinity category theory).
3. Descent Theorems
4. Koszul duality, Serre's Affineness criterion

Here is a classical proposition in Morita equivalences:

Proposition 10.2. Let $R = k$ be a field, then $M_n(k)$ (the $n \times n$ matrix ring over k) is Morita equivalent to k.

More generally, we have that

Theorem 10.3. Let R be a ring, then R is Morita equivalent to $\mathrm{End}_{Mod_R}(Q^{op})$ if Q satisfies:

- Q is finitely presented (ie. $\mathrm{Map}(Q, -)$ commutes with filtered colimits. This should be thought of as a compactness condition - recall finitely presented R-modules are the compact objects in Mod_R).
- Q is projective (ie. $\mathrm{Map}(Q, -)$ preserves split coequalizers and additivity).
- Q is a generator (ie. $\mathrm{Map}(Q, -)$ is faithful. As a remark, this actually implies that $\mathrm{Ext}^n(Q, -) = 0$, for all $n > 0$).

Let $S = \mathrm{End}_{Mod_R}(Q^{op})$. After rewriting the three conditions in languages closer to our seminar, we see that the proof of Morita equivalence is obtained - if we have a lift:

$$\begin{array}{ccc} & & \mathrm{Mod}_S \\ & \nearrow & \downarrow{\scriptstyle \text{forget}} \\ \mathrm{Mod}_R & \xrightarrow{M \mapsto \mathrm{Hom}_R(Q,M)} & \mathrm{Mod}_{\mathbb{Z}} \end{array}$$

and if the three conditions given by the theorem guarantees this lift is an equivalence.

This is where we introduce the proof of monads. To give the definition of monad, we have the following:

Definition 10.4. A **monad** $T \in \mathcal{C}$ is a monoid in the category of endofunctors $\mathrm{End}(\mathcal{C})$. Specifically, T is the data $(T : \mathcal{C} \to \mathcal{C}, \mu : T^2 \implies T, \eta : 1_{\mathcal{C}} \implies T)$ and we want the following two diagrams to commute

$$\begin{array}{ccc} T\circ T\circ T & \xrightarrow{\mu T} & T\circ T \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T\circ T & \xrightarrow[\mu]{} & T \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{\eta T} & T\circ T \\ {\scriptstyle T\eta}\downarrow & \searrow^{id} & \downarrow{\scriptstyle \mu} \\ T\circ T & \xrightarrow[\mu]{} & T \end{array}$$

Remark 10.5. "A monad is a monoid in the category of endofunctors" is a long running joke in the world of Functional Programming.

Definition 10.6. An algebra A over a monad is the pair $(A \in \mathcal{C}, \alpha : TA \to A)$ such that it is a T-module in the $\mathrm{End}(\mathcal{C})$-tensored category $\mathcal{C}$. That is, we want the following two diagrams to commute:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle id_A}\searrow & \downarrow{\scriptstyle \alpha} \\ & & A \end{array} \qquad \begin{array}{ccc} T\circ T(A) & \xrightarrow{T\alpha} & TA \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \alpha} \\ TA & \xrightarrow[\alpha]{} & A \end{array}$$

A morphism $f : (A, \alpha) \to (B, \beta)$ between two T-algebras is a map $f : A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[f]{} & B \end{array}$$

Composition and identities are the same as in $\mathcal{C}$.

Together, this forms a category of T-algebras denoted $\mathrm{Alg}_T(\mathcal{C})$. This is also called the **Eilenberg-Moore Category**.

Remark 10.7. Monads arise from adjunctions. Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint and $G : \mathcal{D} \to \mathcal{C}$ be right adjoint, then $G \circ F$ is a monad! In this talk, F will always be left and G will always be right.

Proposition 10.8. Let (T, μ, η) be a monad in $\mathcal{C}$, then there is an adjunction between $\mathcal{C}$ and $\mathrm{Alg}_T(\mathcal{C})$ given by

$$F^T : \mathcal{C} \rightleftarrows \mathrm{Alg}_T(\mathcal{C}) : U^T$$

Here U^T is the forgetful functor and F^T (called the **free T-algebra functor**) is given by

$$F^T(A) = (TA, \mu_A : T^2A \to TA) \text{ and } F^T f = Tf.$$

Fruthermore, the adjunction $U^T \circ F^T$ recovers the monad (T, μ, η).

Motivated by the proposition, we also give the following definition

Definition 10.9. A free T-algebra $X \in \mathcal{C}$ is an object in the image of

$$X \mapsto (T(X), \mu_X : T \circ T(X) \to T(X))$$

Definition 10.10. An adjunction $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ is **monadic** if the monad $T = G \circ F$ induces an equivalence $\tilde{G} : \mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$ in the following sense:

$$\begin{array}{ccc} & & \mathrm{Alg}_T(\mathcal{C}) \\ & \overset{\tilde{G}}{\nearrow} & \big\downarrow \text{Forget} \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

Example 10.11. Here are some examples:

1. There exists a monad T on Set such that $\mathrm{Alg}_T(Set) \cong \mathrm{Grp}$ (so the algebra of T over sets are groups). T comes from the Free and Forgetful functor in the adjunction. The free T-algebras are the free groups.
2. There exists a monad T_R on abelian groups such that $\mathrm{Alg}_{T_R}(Ab) \cong \mathrm{RMod}$ (so the algebra of T_R over Ab are R-modules). T_R comes from the Tensor and Hom functors in adjunction. The free T-algebras are the free R-modules.

In this case, both adjunctions are monadic.

Given a pair of adjoint functors, we would like a sufficient criterion to determine if they would be monadic.

Theorem 10.12 (Barr-Beck Monadicity Theorem)**.** Let F, G be adjoints as before. Suppose $\mathcal{D}$ admits split coequalizers, then if

- i) G is conservative (if. $G \circ f$ is an equivalence, then f is an equivalence).
- ii) G preserves split coequalizers.

Then $F \dashv G$ is monadic.

Remark 10.13. Here even though we say the word "equivalence", they are isomorphisms as morphisms in the 1-category. It is in higher algebra, we end up calling isomorphisms as "equivalences".

Proof Idea. We have a lift of the following for F and G - Here is a picture for G:

$$\begin{array}{ccc} & & \mathrm{Alg}_T(\mathcal{C}) \\ & \overset{\tilde{G}}{\nearrow} & \big\downarrow \text{Forget} \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

The proof may be decomposed in a few steps:

1. Step 1 - Showing $\tilde{F} \dashv \tilde{G}$: We know that we have the following coequalizer in $\mathrm{Alg}_T(\mathcal{C})$:

   $$TTA \underset{\mu_A}{\overset{T\alpha}{\rightrightarrows}} TA \xrightarrow{\alpha} A$$

   We also have that $\tilde{F}(T(A)) = F(A)$ and $\tilde{F}$ preserves colimits.

   In this case, we have the following coequalizer:

   $$F(T(A)) \rightrightarrows F(A) \longrightarrow \tilde{F}(A)$$

2. Step 2: The unit map $id_{Alg_T(\mathcal{C})} \to \tilde{G} \circ \tilde{F}$ is an equivalence.

   The idea for showing the equivalence is that to recall that G preserves reflexive coequalizers. In this case the following is a coequalizer
   $$GF(GF(A)) \rightrightarrows GF(A) \xrightarrow{G\theta} G\tilde{F}(A) \ ,$$
   and we also obtain a coequalizer of the form
   $$GF(GF(A)) \rightrightarrows GF(A) \xrightarrow{\alpha} A$$
   This gives us the following commutative diagram:
   $$\begin{array}{ccc} GFA & \xrightarrow{h\theta} & G\tilde{F}(A,\alpha) \\ & {\scriptstyle\alpha}\searrow & \downarrow \\ & & A \end{array}$$

3. Step 3: The co-unit map is also an equivalence. The proof is similar in this case.

■

Let us come back to prove Theorem 10.3.

Proof of Theorem 10.3. Consider this diagram

$$\begin{array}{ccc} & & \mathrm{Mod}_S \\ & \nearrow & \downarrow{\scriptstyle \text{forget}} \\ \mathrm{Mod}_R & \xrightarrow{M \mapsto \mathrm{Hom}_R(Q,M)} & \mathrm{Mod}_{\mathbb{Z}} \end{array}$$

Here the horizontal map (call it G) is adjoint in the tensor-hom adjunction. Since Q is a generator, G is conservative. The other two conditions shows that it preserves relfexive coequalizers. Thus, this adjunction is monadic, so we have an equivalence between Mod_R and Mod_S. Here Mod_S is equivalent to $\mathrm{Alg}_{T_R}(Ab)$ (recall the example given earlier). ■

10.2 The Story of Monads in Higher Algebra

In the setting of ∞-categories, we would like to do the following modifications. First we will give a quick introduction to the theory of (co)Cartesian fibrations.

Definition 10.14. Let S and T be a simplicial set, with a morphism $F : S \to T$ of simplicial sets. Let $f : x \to y$ be an edge in S.

We say f is a F-**Carteisan edge** if the following lifting problem has a solution

$$\begin{array}{ccc} \Lambda^n_n & \xrightarrow{\sigma_0} & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle F} \\ \Delta^n & \xrightarrow[\overline{\sigma}]{} & Y \end{array}$$

when $n \geq 2$ and the following composite map corresponds to the edge $f : X \to Y$

$$\Delta^1 \simeq N_\bullet(\{n-1 < n\}) \hookrightarrow \Lambda^n_n \to_{\sigma_0} X.$$

We say f is a F**-coCarteisan edge** if the following lifting problem has a solution

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{\sigma_0} & X \\ \downarrow & \nearrow & \downarrow F \\ \Delta^n & \xrightarrow[\sigma]{} & Y \end{array}$$

when $n \geq 2$ and the following composite map corresponds to the edge $f : X \to Y$

$$\Delta^1 \simeq N_\bullet(\{0 < 1\}) \hookrightarrow \Lambda_0^n \to_{\sigma_0} X.$$

Definition 10.15. Let S and T be simplicial sets. A morphism $F : S \to T$ of simplicial sets is a **Cartesian fibration** if it is an inner fibration (recall Definition 3.4) and for an edge $f : x \to y$ in T and every $y' \in S$ such that $F(y') = y$, there exists an F**-Cartesian edge** $f' : x' \to y'$ such that $F(x') = x$.

We say F is a **Cartesian co-fibration** if it is an inner fibration and for an edge $f : x \to y$ in T and every $x' \in S$ such that $F(x') = x$, there exists an F**-co-Cartesian edge** $f' : x' \to y'$ such that $F(y') = y$.

Example 10.16. Let S be a simplicial set an consider the map $F : S \to \Delta^0$ to the zero simplex. Then, S is an ∞-category if and only if F is a Cartesian fibration, if and only if, F is a co-Cartesian fibration.

Definition 10.17 (Monoidal ∞-Category). The idea of monoidal ∞-categories is to look at co-Cartesian fibrations and insist the **Segal condition**. A **monoidal ∞-category** $(\mathcal{C}, \otimes)$ is composed of a simplicial set $\mathcal{C}^\otimes$ and a co-Cartesian fibration $\rho_\otimes : C^\otimes \to N(\Delta)^{op}$. For each $n \in \mathbb{N}$, there is a sequence of induced map $C^\otimes_{[n]} \to C^\otimes_{i,i+1}$ for all $i = 0, ..., n-1$. We also require that the universal property of products gives an equivalence of the following ∞-categories for each n:

$$C^\otimes_n \to C^\otimes_{0,1} \times ... \times C^\otimes_{n-1,n} \simeq (C^\otimes_{[1]})^n.$$

Definition 10.18 (Algebraic Objects). Let $(\mathcal{C}, \otimes)$ be a monoidal ∞-category. This has a co-Cartesian fibration $\rho_\otimes : \mathcal{C}^\otimes \to N(\Delta)^{op}$. An **algebra** of $(\mathcal{C}, \otimes)$ is, roughly speaking, a section $s : N(\Delta)^{op} \to \mathcal{C}^\otimes$.

We also want to obtain an analog of endofunctors.

Definition 10.19. Let $\mathcal{C}$ be an ∞-category and consider the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. Observe that the composition and evaluation maps:

$$\mathrm{Fun}(\mathcal{C}, \mathcal{C}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \text{ and } \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \times \mathcal{C} \to \mathcal{C}$$

gives $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ the structure of a simplicial monoid with a left action on $\mathcal{C}$. Thus, we can regard $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ as a monoidal ∞-category.

Definition 10.20. A monad of an ∞-category $\mathcal{C}$ is an algebraic object of the monoidal ∞-category $\mathrm{End}(\mathcal{C})$. Informally, this should be thought of as the classical monad with endofunctor $T : \mathcal{C} \to \mathcal{C}$ and maps $\mu : T \circ T \to T$, $\eta : 1_C \to T$ satisfying the same diagrams up to coherent homotopy.

We use $\mathrm{Alg}_T(\mathcal{C})$ to denote the ∞-category of (left) T-modules in $\mathcal{C}$. Informally, this is an analog of the algebra over monads in the classical setting for ∞-categories.

Remark 10.21. In Lurie, $\mathrm{Alg}_T(\mathcal{C})$ is denoted as $\mathrm{LMod}_T(\mathcal{C})$. We used the former notation to be consistent with the previous section in the classical setting.

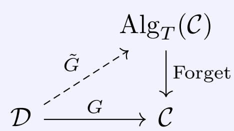
In this case, we still have a way to obtain monads from adjunctions. While we have given one definition of adjunction before, there are many ways to define an adjunction (that are equivalent), the most convenient definition to see how monads arise from adjunctions is the following definition:

Definition 10.22. An adjunction between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $M \to [1] = \{0 \to 1\}$ that is both a coCartesian fibration and a Cartesian fibration. Here we identify the fiber M_0 as $\mathcal{C}$ and M_1 as $\mathcal{D}$ in the usual set-up of an adjunction.

Theorem 10.23 (Barr-Beck-Lurie Monadicity Theorem)**.** Let F, G be adjoints on ∞-categories $\mathcal{C}$ and $\mathcal{D}$. Suppose $\mathcal{D}$ admits **geometric realization** of simplicial objects, then if

- i) G is conservative (if. $G \circ f$ is a equivalence, then f is an equivalence).
- ii) G preserves geometric realizations.

Then $F \dashv G$ is monadic in the sense of ∞-categories. That is, there exists a monad T on $\mathcal{C}$ and an equivalence given in the lift of:



(Here the diagram is up to homotopy)

Why are we doing all the labor to generalize this to the setting of ∞-categories? There are some incredible applications.

Theorem 10.24 (Schwede-Shipley)**.** Let $\mathcal{C}$ be a presentable stable ∞-category with $Q \in \mathcal{C}$ a compact generator, then $\mathcal{C} \cong \mathrm{Alg}_T(\mathrm{Sp}) \cong \mathrm{Mod}(\mathrm{End}(Q^{op}))$. Here T is an appropriately chosen monad coming from the ∞-category analog of tensor-hom adjunction.

Remark 10.25. The original theorem of Schwede-Shipley was done over model categories with a fairly lengthy proof. This proof is shortened and generalized in the language of ∞-categories in Lurie's Higher Algebra.

From this theorem, we obtain the following corollary:

Corollary 10.26. Let R be a ring, then the derived ∞-category $D(R)$ is equivalent to $\mathrm{Mod}(HR)$.

Proof. In the case where $\mathcal{C} = D(R)$, we observe that $\mathrm{End}_{D(R)}(R)$ is concentrated in degree 0 and its degree 0 term is R. Since spectra is determined by maps inducing isomorphisms of homotopy groups, this is exactly HR. Thus, we have that $D(R)$ is equivalent to $\mathrm{Mod}(HR)$. ■

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