I. What is it?

II. Why should we care about it?

III. How to compute it?

▶ What did Adams do in his series of papers?

- $J(X)$ = collection of classes. v.b. E / stable fiber hpty equiv

  $E \sim E'$ iff $E \oplus \mathbb{R}^n$ & $E' \oplus \mathbb{R}^n$ has fiberwise hpty equiv sphere bundle

- FACT $J(S^r) = \operatorname{im} J$. $J: \pi_r SO \longrightarrow \pi_r^S$ = stable stem.

- Thm (Adams)

  $\operatorname{im} J$ = direct summand of π_r^S, cyclic for $r \geq 0$.

  In particular.

  1) $r \equiv 0, 1 \mod 8$, $|\operatorname{im} J| = 2$.

  2) $r \equiv 3, 7 \mod 8$. then $|\operatorname{im} J|$ = denominator of $\frac{B_{2k}}{4k}$

  B_k = Bernouill number

  3) $\operatorname{im} J$ = trivial. else

Recall $\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$

Table:

	0	1	2	3	4	5	6
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$

Main Goal: prove (2).

I. Why care about J-homomorphism?

- $J: \pi_r(SO(n)) \longrightarrow \pi_{n+r}(S^n)$

stable: $J: \pi_r SO \longrightarrow \pi_r^S = \pi_r S^0$

<u>Construction</u> Given $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ orthogonal trans. det $= 1$

$$\downarrow$$

$S^n \longrightarrow S^n$ preserves ∞

$\rightsquigarrow$ map $SO(n) \longrightarrow H(n) =$ htpy self-equiv of S^n.

$$Map_*(S^1, Map_*(S^1, X)) \cong Map_*(S^1 \wedge S^1, X)$$

$$\Rightarrow \quad H(n) = \{ f: S^n \to S^n \}.$$

$$\simeq \Omega^n S^n$$

$\rightsquigarrow J: \pi_r(SO(n)) \longrightarrow \pi_r(H(n)) = \pi_r(\Omega^n S^n) = \pi_{n+r} S^n$

- EHP sequence

comm. diagram

$$\begin{tikzcd} SO(n) \arrow[r] \arrow[d, hook] & H(n) \simeq \Omega^n S^n \arrow[d] \\ SO(n+1) \arrow[r] & H(n+1) \simeq \Omega^{n+1} S^{n+1} \end{tikzcd}$$

fibration: $SO(n) \longrightarrow SO(n+1) \longrightarrow S^n$.

$\Rightarrow$ comm. diagram

$$\begin{array}{ccc} SO(n) & \longrightarrow & \Omega^n S^n \\ \downarrow & & \downarrow E = \text{Einhangjung} : \text{extension} \\ SO(n+1) & \longrightarrow & \Omega^{n+1} S^{n+1} \\ \downarrow & & \downarrow H = \text{Hopf inv.} \\ S^n & \xrightarrow{\alpha} & \Omega^{n+1} S^{2n+1} \end{array}$$

(α, id) adjoint pair. $id : S^n \to S^n$

$\alpha : S^n \to \Omega^{n+1} S^{2n+1}$

(Σ, Ω) · adjoint pair

Hatcher 4._ $\quad S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}$

Recall James construction (X connected CW cpx. $e \in X$ fixed)

$J_m X = X^m / \sim$

$(x_1, x_2, \dots, e, x_i, \dots, x_n) \sim$

$(x_1, x_2, \dots, x_i, e, \dots, x_n)$

$$\begin{array}{ccccc} J_1 X & \subseteq & J_2 X & \subseteq & \cdots \\ \| & & \| & & \\ X & & X \times X / (x,e) \sim (e,x) & & \end{array} \qquad J(X) = \operatorname{colim} J_m X.$$

Prop (Hatcher 4._ . 1) $\quad J(X) \simeq \Omega \Sigma X$

Prop (James splitting) $\quad \Sigma \Omega \Sigma X = \bigvee_{i \geq 1} \Sigma X^{\wedge i}$

$X = S^n \Rightarrow$

$$\begin{array}{ccccc} J(S^n) & \simeq & \Omega \Sigma S^n & = & \Omega S^{n+1} \\ \| & & & & \\ \bigvee_{i \geq 1} \Sigma (S^n)^{\wedge i} & & & & \end{array}$$

For each $i \geq 1$. $\Sigma\Omega S^{n+1} \xrightarrow{\text{prj}} \Sigma(S^n)^{\wedge i} = S^{in+1}$

$$\Big\updownarrow$$

$$\Omega S^{n+1} \longrightarrow \Omega S^{in+1}$$

$i = 2 \Rightarrow \quad S^n \longrightarrow \Omega S^{n+1} \longrightarrow \Omega S^{2n+1}$. EHP sequence

- P = Whitehead product.

1. Hopf inv.

π_* applied to $S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$ + Hurewicz

$$\Rightarrow \quad \begin{array}{ccc} \pi_{2n}(\Omega S^{n+1}) & \xrightarrow{\pi_{2n} \circ H} & \pi_{2n}(\Omega S^{2n+1}) = \mathbb{Z} \\ \downarrow & & \downarrow \cong \\ H_{2n}(\Omega S^{n+1}) & \xrightarrow[\text{l.e.s.}]{\cong} & H_{2n}(\Omega S^{2n+1}) \end{array}$$

$f' : S^{2n} \longrightarrow \Omega S^{n+1} \in \pi_{2n}(\Omega S^{n+1})$

$$\Big\updownarrow$$

$f : S^{2n+1} \longrightarrow S^{n+1}$

$$\Big\downarrow$$

$\alpha \in H_{2n}(\Omega S^{n+1})$

$f' : S^{2n} \longrightarrow \Omega S^{2n+1} \xrightarrow{\Omega f} \Omega S^{n+1}$

iso on H_{2n}.

Prop $(\Omega f)_* : H_{2n}(\Omega S^{2n+1}) \longrightarrow H_{2n}(\Omega S^{n+1})$, $n > 0$

$$\alpha \longmapsto \pm h_f \, \alpha$$

h_f = Hopf inv. of f.

[Hatcher 5.D.1].

2. Connection to v.f. problem.

<u>Result</u> # of v.f. (nowhere zero) on S^n is $\rho(n)-1$

$\rho(n)$ Radon - Hurwitz number.

comm.

$$\begin{tikzcd}
1 & 2 & 3 & 4 \\
\mathbb{RP}^{n-1} \arrow[r] \arrow[d, hook] & SO(n) \arrow[r, "J"] \arrow[d, hook] & \Omega^n S^n \arrow[r] \arrow[d, "E"] & Q\mathbb{RP}^{n-1} \arrow[d] \\
\mathbb{RP}^n \arrow[r] \arrow[d] & SO(n+1) \arrow[r, "J"] \arrow[d] & \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d, "H"] & Q\mathbb{RP}^n \arrow[d] \\
S^n \arrow[r] & S^n \arrow[r] & \Omega^{n+1} S^{2n+1} \arrow[r] & QS^n
\end{tikzcd}$$

$Q = \Omega^\infty \Sigma^\infty$. [Snaith]

<u>Ref</u>. green book 1.5.

Col 1. 2. 3 $\exists$ SS. in $p=2$. gen. $x_k \in E_1^{-k,k+1} = \mathbb{Z}$

first non-trivial differentials of three SSs land where SSs are iso. x_k survives to E_r. $r \le k+1$.

$\Updownarrow$ 1.5. Thm ???

$O(k+1)/O(k+1-r) \to S^k$ admits a cross section

$\Updownarrow$ 1.5 Thm ???

S^k admits $r-1$ linearly indep. v.f. nowhere zero.

<u>Thm</u> by James & Adams. & Snaith. [green book § 1.5].

2. Framed cobordism. Kervaire - Milnor theory.

<u>Recall</u> framed mfd 标架流形. normal bundle.

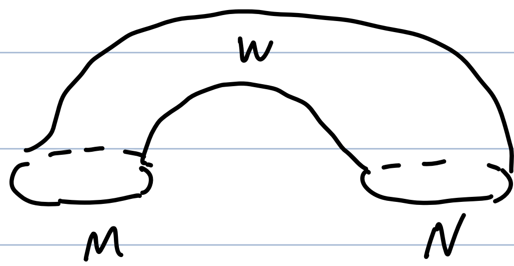
$$M^k \subset \mathbb{R}^{n+k} \quad \text{w/ framing. } f: M^k \times \mathbb{R}^n \longrightarrow N_{\mathbb{R}^{n+k}}/M^k.$$

framed cobordism (W^{k+1}, M^k, N^k) f_N

framed (k+1)-mfd F framed k-mfd closed f_M

$$\partial W^{k+1} = M^k \sqcup N^k$$

$$F|_M = f_M, \quad F|_N = f_N.$$



$\Omega_k^{fr}(\mathbb{R}^{n+k})$ = set of equiv class of framed k-mfds in $\mathbb{R}^{n+k}$.

$M^k \sim N^k$ iff $\exists\, W^{k+1}$ cobordism

$(\Omega_k^{fr}(\mathbb{R}^{n+k}), \sqcup)$ = abelian gp

<u>Thm</u> (Pontryagin - Thom) $k \geq 0$, $n \geq 1$.

$$\Omega_k^{fr}(\mathbb{R}^{n+k}) \cong \pi_{n+k}(S^n).$$

Denote Σ^k = htpy k-sphere. i.e. $\Sigma^k \sim S^k$.

F_1, F_2 framing.

Prop $[\Sigma^k, F_1] - [\Sigma^k, F_2] = [S^k, F]$

F framing on S^k.

$\forall$ framing $F : M^k \times \mathbb{R}^n \longrightarrow N_{\mathbb{R}^{n+k}}/M^k$ can be twisted.

$g : M^k \longrightarrow SO(n)$

$F \circ g : M^k \times \mathbb{R}^n \longrightarrow M^k \times \mathbb{R}^n \longrightarrow N_{\mathbb{R}^{n+k}}/M^k$

$(x, v) \longmapsto (x, g(x) \cdot v)$

$\Rightarrow$ g gives a twisting.

$M^k = S^k$. FACT $[S^k, F]$ non-trivial $\Longleftrightarrow$ F is twisted.

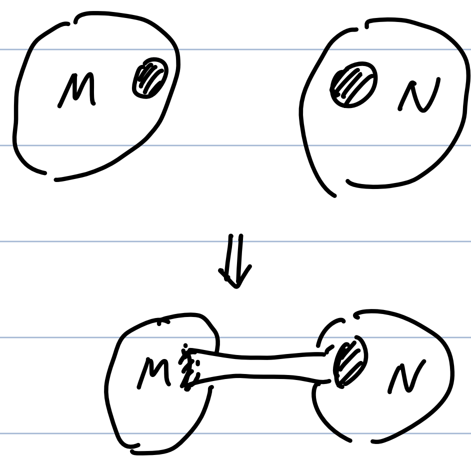
trivial $\Longleftrightarrow$ $\nexists g$. F untwisted.

i.e. det. by an elt in $\pi_k(SO(n))$

$J : \pi_k(SO(n)) \longrightarrow \pi_{n+k} S^n$.

Denote Θ_n = set of htpy n-spheres up to diffeo.

abelian gp, $\#$ = connecting sum



$M \# N$.

$|\Theta_n|$ = # of smooth str. on Σ^n

- $\Theta_n \xrightarrow{F} \pi_n^S / \operatorname{im} J$.

Thm (Kervaire - Milnor)

$\ker F = \Theta_n^{bp}$. If $n \equiv 2 \mod 4$, $\exists$ e.s.

$$0 \to \Theta_n^{bp} \to \Theta_n \xrightarrow{F} \pi_n^S / \operatorname{im} J \xrightarrow{\Phi} \mathbb{Z}/2 \to \Theta_{n-1}^{bp} \to 0$$

Φ = Kervaire inv.

If $n \not\equiv 2 \mod 4$, $\exists$ s.e.s.

$$0 \to \Theta_n^{bp} \to \Theta_n \xrightarrow{F} \pi_n^S / \operatorname{im} J \to 0.$$

And, if $n = 4k-1$, $\Theta_n^{bp} \cong \mathbb{Z}/(2^{2k-2}(2^{2k-1}-1) \cdot c_k)$

c_k = numerator of $\frac{4 B_{2k}}{k}$

if n even, Θ_n^{bp} trivial

if $n = 4k+1$, Open. possible 1 & 2

- Isaksen - Wang - Xu (2023)

$\Rightarrow$ if $|\Theta_n| = 1$. if n odd, $n = 1, 3, 5, 61$

if n even, $n \leq 140$.

$n = 2, 6, 12, 56$

4 : open !

II. How to compute $|\operatorname{im} J|$?

<u>Upshot</u> Bound $|\operatorname{im} J|$ above & below.

- Below : "e - invariant" $\rightsquigarrow$ B_k comes from
- Above : Adams conj.

J(X) - I Thm 1.2. solved by Quillen (1971).

Strategy $\tilde{K}, \widetilde{KO} = [-, BO \times \mathbb{Z}]$

$\tilde{K} = [-, BU \times \mathbb{Z}]$

$\Rightarrow$ principal G-bundle $O \to EO \to BO$

$U \to EU \to BU$

$\Rightarrow \pi_{r-1} O = \pi_r BO = \widetilde{KO}(S^r)$

$\pi_{r-1} U = \pi_r BU = \tilde{K}(S^r)$

FACT $\pi_r U \longrightarrow \pi_r O$ iso $r \equiv 4 \mod 8$

$\cdot 2$ $r \equiv 0 \mod 8$. $n > 1$.

Consider cofiber seq

$$S^{2(k+n)-1} \xrightarrow{f} S^{2k} \longrightarrow C_f = S^{2k} \cup_f CS^{2(k+n)-1}$$
$$\longrightarrow S^{2(k+n)} \xrightarrow{\Sigma f} S^{2k+1} \longrightarrow \cdots \quad (*)$$

Puppe seq after applying $\tilde{K}$. $\tilde{K}(S^{2m}) = \mathbb{Z} = \langle H-1 \rangle$

$\Rightarrow$ s.e.s.

$$0 \to \tilde{K}(S^{2(k+n)}) \longrightarrow \tilde{K}(C_f) \longrightarrow \tilde{K}(S^{2k}) \longrightarrow 0.$$

Can ignore k $\Rightarrow$ only look at

$$\begin{array}{ccccccccc} 0 & \to & \tilde{K}(S^{2n}) & \to & \tilde{K}(C_f) & \to & \tilde{K}(S^0) & \to & 0. \\ & & \downarrow \ {\scriptstyle \cong \mathbb{Z}} & & \downarrow & & \downarrow \ {\scriptstyle \cong \mathbb{Z}} & & (x) \\ 0 & \to & \tilde{H}^{even}(S^{2n};\mathbb{Q}) & \to & \tilde{H}^{even}(C_f;\mathbb{Q}) & \to & \tilde{H}^{even}(S^0;\mathbb{Q}) & \to & 0 \\ & & {\scriptstyle \cong \mathbb{Q}} & & & & {\scriptstyle \cong \mathbb{Q}} & & \end{array}$$

Chern char.

$$\Rightarrow \tilde{K}(C_f) \xrightarrow{ch} \tilde{H}^{even}(C_f; \mathbb{Q})$$
$$a \longmapsto ch(a)$$

$$\nearrow \quad \searrow \qquad\qquad \nearrow \quad \searrow$$
$$t \qquad\quad 1 \qquad\qquad \tilde{e} \qquad\quad 1$$

$\tilde{e} \in \mathbb{Q}/\mathbb{Z}$. $\rightsquigarrow$ e - invariant

So $\forall$ eles $x_{2n} \in \pi_{2n} BU = \tilde{K}(S^{2n})$ $\quad x_{2n} = t$ above

$x_{2n} : E \longrightarrow S^{2n}$

<u>FACT</u> $\tilde{K}(C_f) = \tilde{K}(Th(x_{2n}))$

$f = J(x_{2n})$. $\quad f : S^{2n-1} \longrightarrow S^0$

Let $u =$ Thom class in $\tilde{K}(Th(x_{2n}))$. $\quad ch(u) = \underbrace{\mathcal{X}(x_{2n})}_{\text{generalized cannibalistic class}} \cdot u_H$

To compute $\mathcal{X}(x_{2n})$. Splitting principle $\Rightarrow$ $\mathcal{X}(L)$. $L =$ line bundle.

$\Rightarrow \mathcal{X}(x_{2n}) = 1 + \underbrace{e(f)}_{} \cdot \cdots$

$$e(f) = \tilde{e} \leadsto \text{Bernoulli } \#.$$

$\Rightarrow$ get back to $\quad \pi_{4k} BO \xrightarrow{J} \pi^S_{4k-1} \xrightarrow{e_{\mathbb{R}}} \mathbb{Q}/\mathbb{Z}$

$$\begin{tikzcd}
\operatorname{im} J \arrow[r] \arrow[d] & \mathbb{Z}/\text{denominator } \frac{B_{2k}}{4k}. \\
\pi^S_{4k-1} \arrow[ur] &
\end{tikzcd}$$

$\Rightarrow$ lower bound.

Upper bound $\Rightarrow$ Adams Conj.

- Adams [On $J(X)$ - I $\sim$ IV]

$J(X)$, $X = S^r$, $J(S^r) \simeq \operatorname{im} J$

He define $J'(X)$, $J''(X)$

lower bound   upper bound.

I: general intro to the problem, def $J(X)$, $J'(X)$, $J''(X)$

special case of Adams conj. (Conj. 1.2).

II: § 3. define $J''(X) = \widetilde{KO}(X) / \bigcap_f H_f$

$= \widetilde{KO}(X) / W(X)$.

$f: \mathbb{N} \to \mathbb{N}$

$H_f = \langle k^{f(k)} (\psi^k x - x) \rangle \quad x \in \widetilde{KO}(X)$.

§ 6. define $J'(X) = \widetilde{KO}(X) / V(X)$

$V(X)$ = collection of elems whose cannibalistic classes $\rho^k x = \frac{\psi^k (1+x)}{1+x}$

§ 5.

Example 6.4, 6.5 two computations $\Rightarrow$ lower bound

+ Thm 6.1

Prop 3.1 $\Rightarrow$ upper bond

- lower bound:

$J(X) = \widetilde{KO}(X) / T(X)$, $J'(X) = \widetilde{KO}(X) / V(X)$

if $T(X) \subset V(X)$

$$\Rightarrow \begin{tikzcd} \widetilde{KO}(X) \arrow[rr, "\text{quo}"] \arrow[dr, "\text{quo}"'] & & J'(X) \\ & J(X) \arrow[ur, "\text{epi}"'] & \end{tikzcd}$$

- upper bond: $J''(X) = \widetilde{KO}(X) / W(X)$

if $W(X) \subset T(X)$

$$\Rightarrow \quad \begin{tikzcd} \widetilde{KO}(X) \arrow[rr, "quo"] \arrow[dr, "quo"'] & & J(X) \\ & J''(X) \arrow[ur, "epi"'] & \end{tikzcd}$$

III. $J'(X) \cong J''(X)$.

IV. e-invariant (d-invariant) : § 2.3.

e, d : Toda bracket $\longrightarrow$ Massey product § 5.

§ 4. Massey product

Main results : Thm 7.16, 7.19, 9.5 pf. in § 10.

§ 7. e-inv. property, $e_{\mathbb{C}}$, $e_{\mathbb{R}}$ relation

§ 11. e-inv. of Toda bracket

§ 12. ex. $\Rightarrow$ J-homophism can be used to construct elets in

π_r^S