Notes on Lichtenbaum-Quillen conjecture

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1 Introduction

This notes serves for the short introduction to the Lichtenbaum-Quillen conjecture, along with the Rost-Voevodsky norm residue conjecture.

2 Dedekind zeta function

The Quillen-Lichtenbaum conjecture is classically trying to related the algebraic K-theory to the Dedekind zeta function. Recall that

Definition 1. *Let F be an algebraic number field, then the* ***Dedekind zeta function*** *is defined to be*

$$\zeta_F(s) = \sum_{0 \neq I \trianglelefteq \mathcal{O}_F} ([\mathcal{O}_F : I])^{-s}.$$

where $\mathcal{O}_F$ is the ring of integers. Here we ask $\operatorname{Re} s > 1$.

If $F = \mathbb{Q}$, then $\mathcal{O}_F = \mathbb{Z}$. Since $\mathbb{Z}$ is a PID, the Dedekind zeta function becomes

$$\begin{aligned}\zeta_F(s) &= \sum_{(n):\, n \in \mathbb{Z}} ([\mathbb{Z} : (n)])^{-s} \\ &= \sum_{n \geq 1} \frac{1}{n^s},\end{aligned}$$

which recovers the Riemann zeta function.

As a generalization of Riemann zeta function, Dedekind zeta function also has some analytic properties that we will state without any proof:

Proposition 1.

1. *$\zeta_F(s)$ can be analytically continued to a meromorphic function on $\mathbb{C}$ with a pole at $s = 1$.*
2. *Euler product formula:*
$$\zeta_F(s) = \prod_{o \neq q \in \operatorname{Spec} \mathcal{O}_F} \frac{1}{1 - ([\mathcal{O}_F : q])^{-s}}.$$
3. *Functional equation: assume $[F : \mathbb{Q}] = r_1 + 2r_2 = n$, where r_1 is the number of real embeddings, and r_2 is the number of complex embeddings. Let*
$$\xi_F(s) = \left(\frac{|\triangle_F|}{2^{2r_2} \pi^n} \right)^{s/2} \Gamma(\frac{s}{2})^{r_1} \Gamma(s)^{r_2} \zeta_F(s),$$
where $\triangle_F$ is the discriminant of F. Then
$$\xi_F(1 - s) = \xi_F(s).$$

Recall $\mathrm{K}_0(\mathcal{O}_F) = \mathbb{Z} \oplus \operatorname{Pic} \mathcal{O}_F$ and $\mathrm{K}_1(\mathcal{O}_F) = \mathcal{O}_F^\times$. In general, Quillen showed in [Qui73] that $\mathrm{K}_n(\mathcal{O}_F)$ is finitely generated for all $n \geq 0$, and shortly after that Borel showed in [Bol74] that $\mathrm{K}_n(\mathcal{O}_F)$ is finite for n is even, and if $n = 2m-1$, then $\operatorname{rank}(\mathrm{K}_n(\mathcal{O}_F) \otimes \mathbb{Q}) = d_m$. Here

$$d_m = \begin{cases} r_1 + r_2 - 1 & \text{, if } m = 1; \\ r_1 + r_2 & \text{, if } m \text{ is odd}, m \geq 3; \\ r_2 & \text{, if } m \text{ is even.} \end{cases}$$

for $[F : \mathbb{Q}] = r_1 + 2r_2$. Even shockingly, Hesselholt and Madsen [?] actually computed $\mathrm{K}_i(\mathcal{O}_F/\overline{\mu}^n)$ for $\overline{\mu}$ the uniformizer via the trace method. From that, they ended up proving the Lichtenbaum-Quillen conjecture (1) for local fields. Later, Liu and Wang [LW22] revisited their computation, and gave a even better and efficient computation of $\mathrm{TC}(\mathcal{O}_F/\overline{\mu}^n)$ via the descent spectral sequence with respect to the polynomial rings over the spherical Witt vectors, where they adapted the new definition of TC by Nikolaus-Scholze [NS18]. Recently, Antieau-Krause-Nikolaus [AKN24] proceeded the computation of trace method, and gave the desired result of $\mathrm{K}_i(\mathcal{O}_F/\overline{\mu}^n)$ by computer, via Liu-Wang's method and motivic filtrations on TC by Hahn-Raksit-Wilson [HRW22].

Back to our discussion. In order to prove the structure theorem of $\mathrm{K}_m(\mathcal{O}_F)$, Borel defined a regulator map

$$\rho_{F,m} : \mathrm{K}_{2m-1}(\mathcal{O}_F) \to \mathbb{R}^{d_m},$$

where d_m is defined as above. He showed that for any $m \in \mathbb{Z}_{>0}$, the kernel of this regulator map is finite. So the image of $\rho_{F,m}$ is a lattice in $\mathbb{R}^{d_m}$. We denote the covolume of this lattice (i.e. the determinant of the base vectors) by $R_{F,m}$. This is called the **Borel regulator**. We present a very brief way to define such regulator map as follows:

Starting from the induced map by $F \hookrightarrow \mathbb{C}$,

$$\mathrm{K}_{2m-1}(\mathcal{O}_F) \cong \mathrm{K}_{2m-1}(F) \to \bigoplus_{\operatorname{Hom}(F,\mathbb{C})} \mathrm{K}_{2m-1}(\mathbb{C}) \to \mathbb{Z}[\operatorname{Hom}(F,\mathbb{C})] \otimes \mathbb{R}(m-1), \tag{1}$$

where $\mathbb{R}(m-1) = (2\pi i)^{m-1}\mathbb{R}$, and the map $\mathrm{K}_{2m-1}(\mathbb{C}) \to \mathbb{R}(m-1)$ is given by the composition of b_{m-1} and h_{2m-1}. Here

$$h_{2m-1} : \mathrm{K}_{2m-1}(\mathbb{C}) = \pi_{2m-1}(\mathrm{BGL}(\mathbb{C})^+) \to H_{2m-1}(\mathrm{BGL}(\mathbb{C})^+) \cong H_{2m-1}(\mathrm{GL}(\mathbb{C})^+),$$

and $b_{m-1} \in H^{2m-1}_{\mathrm{cts}}(\mathrm{GL}(\mathbb{C}), \mathbb{R}(m-1))$ is the universal Borel class, see [Gil02]. Here H^{2m-1}_{cts} is the continuous cohomology, which induces a map

$$b_{m-1} : H_{2m-1}(\mathrm{GL}(\mathbb{C})) \to \mathbb{R}(m-1).$$

Thus, the image of (1) is invariant under complex conjugation acting both on $\mathbb{Z}[\operatorname{Hom}(F,\mathbb{C})]$ and $\mathbb{R}(m-1)$. So we get a map

$$\mathrm{K}_{2m-1}(\mathcal{O}_F) \to (\mathbb{Z}[\operatorname{Hom}(F,\mathbb{C})] \otimes \mathbb{R}(m-1))^c,$$

where $(-)^c$ is taking the fixed point under the complex conjugation action. It can be shown that this fixed point is identified with $\mathbb{R}^{d_m}$, as desired. See [Bol74] for a detailed proof.

With the help of Borel regulator map, Borel was able to show that the special value of Dedekind zeta function $\zeta_F(s)$ at $s = 1-m$, denoted $\mathfrak{m} := \lim_{s \to 1-m} (s+m-1)^{-d_m} \zeta_F(s)$, can be interpreted as

$$\mathfrak{m} = C \cdot R_{F,m}$$

for some $C \in \mathbb{Q}$. Lichtenbaum conjectured that, for $m \geq 2$,

$$C \simeq \frac{|\,\mathrm{K}_{2m-2}(\mathcal{O}_F)|}{|\,\mathrm{K}_{2m-1}(\mathcal{O}_F)_{\mathrm{tor}}|}. \tag{2}$$

Here $\simeq$ means equal up to a sign times a power of 2, and $(-)_{\mathrm{rm}}$ means the finite torsion subgroup. This should be some generalization of the Dirichlet class number formula.

3 Étale cohomology

Let S be a scheme over a field k of characteristic p, $S_{\text{ét}}$ be the (small) étale site, whose objects are $X \to S$, i.e. schemes over S. Consider the sheaves of abelian groups on $S_{\text{ét}}$, denoted $\text{Sh}(S_{\text{ét}})$. This category has enough injectives. Consider the functor

$$\mathcal{F} : \text{Sh}(S_{\text{ét}}) \to \text{Ab}$$

sending each sheaf F to $F(S)$. The sheaf $\mathcal{F}$ is left exact, and we define its r-th right derived functor by $H^r(S_{\text{ét}}, -)$. This is the étale cohomology. Explicitly, there is always an injective resolution of any $F \in \text{Sh}(S_{\text{ét}})$

$$0 \to F \to I_0 \to I_1 \to I_2 \to \cdots,$$

and so $H^r(S_{\text{ét}}, F)$ is the r-th cohomology group of the complex

$$I_0(S) \to I_1(S) \to I_2(S) \to \cdots.$$

By standard techniques in homological algebra, one can check that étale cohomology is really a cohomology theory, i.e. satisfies the Eilenberg-Steenrod axioms (homotopy, excision, additivity, exactness, dimension). In general, one can define for any $X \in S_{\text{ét}}$, the étale cohomology of X by taking the right derived functors with respect to the functor $\mathcal{F}_X(-) = \Gamma(X, -)$.

Let μ_m be the étale sheaf on S, given by $\mu_m(X) = \{f \in \Gamma(X, \mathcal{O}_X) : f^m = 1\}$. Let $\mathbb{G}_m$ be the multiplicative group scheme, viewed as an étale sheaf. We have a Kummer sequence

$$0 \to \mu_m \to \mathbb{G}_m \xrightarrow{(-)^m} \mathbb{G}_m \to 0. \tag{3}$$

The **Tate twist** $\mu_m^{\otimes n} =: \mathbb{Z}/m(n)$ is a new étale sheaf satisfies the Poincaré duality:

$$H^i_{\text{ét}}(X, \mathbb{Z}/m(n)) \times H^{2d+1-i}_{\text{ét}}(X, \mathbb{Z}/m(d-n)) \to \mathbb{Z}/m,$$

for X has dimension d. b We can define the ℓ-adic (ℓ is coprime to p and $\ell^{-1} \in k$) Tate twist by taking the limit:

$$\mathbb{Z}_\ell(n) := \lim_r \mathbb{Z}/\ell^r(n).$$

For completeness, We also define $\mathbb{Z}(1) = \mathbb{G}_m[-1]$, $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$, $\mathbb{Z}(0) = \mathbb{Z}$. These are known as the motivic complexes. However, the readers should be warned that, $\mathbb{Z}(n)$ shares no similarities with $\mathbb{Z}_\ell(n)$ defined above since $\mathbb{Z}(n)$ is a sheaf over Zariski site of $\text{Spec}\,\mathbb{Z}$, while $\mathbb{Z}_\ell(n)$ defined above is an étale sheaf.

Let $X = \text{Spec}\,\mathcal{O}_F[1/p]$ and F is a totally real number field for simplicity. By the main conjecture of Iwasawa theory, proved by Mazur-Wiles, we can reformulate the Lichtenbaum conjecture (2) into the cohomological analogy:

$$C \simeq \frac{|H^2_{\text{ét}}(X, \mathbb{Z}_p(m)|}{|H^1_{\text{ét}}(X, \mathbb{Z}_p(m))_{\text{tor}}|}. \tag{4}$$

See Chapter 0 of [Kol02] for a detailed discussion. From (2) and (4), one might expect there is a relationship between the algebraic K-theory and the étale cohomology. Soulé [Sou79] constructed the étale Chern character

$$ch_{i,j} : \text{K}_{2i-j}(\mathcal{O}_F) \otimes \mathbb{Z}_\ell \to H^j_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}_\ell(i)) \tag{5}$$

for $j = 1, 2$ and $i \geq 2$, and proved surjectivity with finite kernel. In general, the Lichtenbaum-Quillen conjecture said:

Conjecture 1 (Lichtenbaum-Quillen)**.** *$ch_{i,j}$ is an isomorphism.*

Lichtenbaum-Quillen conjecture is now proved as a corollary of the norm residue theorem due to Rost and Voevodsky. Before we give the statement of the theorem, we need to construct the norm residue map. Note that the Kummer sequence (3) gives rise to a long exact sequence in étale cohomology

$$0 \to H^0_{\text{ét}}(X, \mu_\ell) \to H^0_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{(-)^\ell} H^0_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{\partial} H^1_{\text{ét}}(X, \mu_\ell) \to H^1_{\text{ét}}(X, \mathbb{G}_m) \to \cdots,$$

where we choose $m = \ell$ and $X = \operatorname{Spec} k$. By Hilbert 90, $H^1_{\text{ét}}(X, \mathbb{G}_m) = \operatorname{Pic}(X) = 0$ because every 1-dimensional k-vector space over k is isomorphic to k itself. Note also $H^0_{\text{ét}}(X, \mathbb{G}_m) = \Gamma(X, \mathbb{G}_m) = \mathbb{G}_m(\operatorname{Spec} k) = k^\times$. Hence ∂ identifies

$$H^0_{\text{ét}}(X, \mathbb{G}_m)/\operatorname{im}(-)^\ell = k^\times/(k^\times)^\ell \xrightarrow{\cong} H^1_{\text{ét}}(X, \mu_\ell),$$

and we define the **first norm residue map** by

$$\partial_1 := \partial : k^\times/(k^\times)^\ell \to H^1_{\text{ét}}(X, \mu_\ell).$$

Abbreviate $k^\times/(k^\times)^\ell$ as $k^\times/\ell$. Taking the tensor power of the source of ∂_1 and the cup product on the target of ∂_1, we obtained the **n-th norm residue map**

$$\partial_n : (k^\times/\ell)^{\otimes n} \to H^1_{\text{ét}}(X, \mu_\ell)^{\otimes n} \xrightarrow{\cup^n} H^n_{\text{ét}}(X, \mu_\ell^{\otimes n}), \tag{6}$$

and $(k^\times/\ell)^{\otimes n} \cong \mathrm{K}^M_n(k)/\ell$ is the n-th Milnor K-theory group. Now, we are ready for the extraordinary Fields Medal-level theorem.

Theorem 1 (Norm residue theorem)**.** *For any $n \geq 1$, ∂_n (6) is an isomorphism for all ℓ (including 2, which is known as the Milnor conjecture).*

Remark 1. *The theorem stated above is sometimes known as the Bloch-Kato conjecture. In some resources, the norm residue theorem is stated as an isomorphism between some motivic cohomology and the étale cohomology. This is classically known as the Beilinson–Lichtenbaum conjecture, which was proved to be equivalent to the Block-Kato conjecture by Voevodsky. For the sake of the proof of Conjecture 1, we write down the relation between motivic cohomology and étale cohomology without any more explanations:*

$$H^i(X, \mathbb{Z}/\ell^r(n)) \cong H^i_{\text{ét}}(X, \mu_{\ell^r}^{\otimes n}), \tag{7}$$

$$H^n(X, \mathbb{Z}(n)) \cong \mathrm{K}^M_n(X), \tag{8}$$

for $0 \leq i \leq n$, and 0 elsewhere and ℓ is a prime that is invertible in k.

Corollary 1. *Lichtenbaum-Quillen conjecture (Conjecture 1) holds.*

Proof. This is basically from the motivic spectral sequence for the algebraic K-theory. Recall that the motivic spectral sequence (see [FS02]) is of the form

$$E_2^{s,t} = H^{s-t}(X, \mathbb{Z}(-t)) \Rightarrow \mathrm{K}_{-s-t}(k)$$

for $s, t \leq 0$. Here the E_2-page is given by the motivic cohomology. By Suslin-Voevodsky theorem [SV95], Theorem 1 and (7),

$$E_2^{0,-t} = H^t(X, \mathbb{Z}(t)) \cong \mathrm{K}^M_t(k).$$

The differentials are torsions and it degenerates modulo groups of finite exponent. See [GS98] and [Kahn99]. We have the edge homomorphism in the spectral sequence

$$\mathrm{K}_{2s-t}(k) \to H^t(X, \mathbb{Z}(s)),$$

given by the composition $e_F^{-1} d_t e_B^{-1}$, where

$$e_B : E_2^{-2s,0} \to E_3^{-2s,0} \to \cdots \to E_\infty^{-2s,0} = \mathrm{K}_{2s}(k)$$
$$e_F : E_2^{0,-t} \to E_3^{0,-t} \to \cdots \to E_\infty^{0,-t} = \mathrm{K}_t(k)$$

and $d_t : E_t^{0,t-1} \to E_t^{t,0}$ is the differential.

We have the isomorphisms

$$\mathrm{K}_{2s-2}(k) \to H^2(X, \mathbb{Z}(s)),$$
$$\mathrm{K}_{2s-1}(k) \to H^1(X, \mathbb{Z}(s)).$$

for all $s \geq 2$ up to 2-torsion. Here we use the fact that, for any global field L, $\mathrm{K}_n(L)$ is finitely generated for n, and torsion when n is even. ☐

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