Math 545: Harmonic Analysis

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1 Introduction

These notes were taken in University of Illinois, Urbana Champaign (UIUC)'s Math 545 (Harmonic Analysis) class in Spring 2020, taught by Professor Xiaochun Li. Please send questions, comments, complaints, and corrections to `jinghui4@illinois.edu`.

These notes are an overview of harmonic analysis in real methods. Basically we will cover the topics:

- Marcinkiewicz interpolation; Approximation to the identity; Fourier transforms;
- The theory of Calderon-Zygmund singular integrals;
- Littlewood-Paley theory; Multiplies;
- BMO and Carleson measure; $T1$ theroem;
- Besicovitch sets and the unboundedness of the disk multiplier.

The course web page can be founded here: `https://faculty.math.illinois.edu/~xcli/teaching/20math545/math545.html`.

In this note, we will frequently use some abbreviations:

- MCT = Monotone Convergence Theorem;
- DCT = Dominated Convergence Theorem.

2 Marcinkewicz Interpolation Theorem

We will always assume $(X, \mathcal{A}, \mu)$ is a measure space.

Definition 1. A **Weak-L^p norm** is defined to be

$$\|f\|_{p,\infty} = \sup_{\lambda>0} \left[\lambda^p \mu(\{x \in X : |f(x)| > \lambda\})\right]^{\frac{1}{p}}.$$

Denote $L^{p,\infty}(X) = \{f : X \to \mathbb{C} : \|f\|_{p,\infty} < \infty\}$, and set $L^{\infty,\infty}(X) = L^\infty(X)$.

Note 1. We have $L^\infty(X) \subset L^{p,\infty}(X)$.

Theorem 1 (**Riesz-Thorin Interpolation Theorem**)**.** Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, and $p_0, p_1, q_0, q_1 \in [1, \infty]$. If $q_0 = q_1 = \infty$, we further assume ν is σ-finite. If T is a linear operator such that $\|Tf\|_{q_0} \le M_0\|f\|_{p_0}$ for all $f \in L^{p_0}$ (that is, T is strong (p_0, q_0)) and $\|Tf\|_{q_1} \le M_1\|f\|_{p_1}$ for all $f \in L^{p_1}$, then for any $0 < \theta < 1$, we have for any $f \in L^{p_\theta}$,

$$\|Tf\|_{q_\theta} \le M_0^{1-\theta} M_1^\theta \|f\|_{p_\theta}.$$

Here p_θ, q_θ are given by

$$\begin{cases} \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \\ \\ \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \end{cases}$$

For the proof, readers can refer to [1].

Definition 2. T is called **sublinear** if $|T(f_1 + f_2)| \le |Tf_1| + |Tf_2|$ for any $f_1, f_2 \in L^p(X, \mathcal{A}, \mu)$, and

$$|T(\alpha f)| = |\alpha| \cdot |Tf|$$

for any $f \in L^p$ and $\alpha \in C$. If T satisfies $\|Tf\|_{L^{p,\infty}(X)} \le C\|f\|_p$ for any $f \in L^p$, then T is called weak (p, q).

Theorem 2 (**Marcinkiewicz Interpolation Theorem**, or **Real Interpolation Theorem**)**.** Suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, and $p_0, p_1, q_0, q_1 \in [1, \infty]$ s.t. $p_0 \le q_0$, $p_1 \le q_1$, $q_0 \ne q_1$. Let p, q are given by

$$\begin{cases} \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \\ \\ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \end{cases}$$

Then T is a sublinear operator s.t. T is weak (p_0, q_0) and weak (p_1, q_1), then T is weak (p, q).

To prove this theorem, we first need an easy fact.

Lemma 1. $\|f\|_p^p = p\int_0^\infty \lambda^{p-1}\mu(\{x \in X : |f(x)| > \lambda\})d\lambda$, $0 < \lambda < \infty$.

Proof. Suppose μ is σ-finite. Let $E_\lambda = \{x : |f(x)| > \lambda\}$. Then

$$\begin{aligned} p\int_0^\infty \lambda^{p-1}\mu(E_\lambda) &= p\int_0^\infty \lambda^{p-1}\int_X \chi(E_\lambda)d\mu d\lambda \\ &= p\int_X \left(\int_0^\infty \lambda^{p-1}\chi(E_\lambda)d\lambda\right)d\mu \qquad \text{(Fubini's Theorem)} \\ &= p\int_X \left(\int_0^{|f(x)|} \lambda^{p-1}d\lambda\right)d\mu \\ &= \int_X |f(x)|^p d\mu \\ &= \|f\|_p^p. \end{aligned}$$

□

The general case when μ is necessarily σ-finite is left to the readers.

Proof of Theorem 2. We restrict to the simple case when $p_0 = q_0$, $p_1 = q_1$ and $p_0 \neq p_1$.

By assumption, T is weak (p_0, p_0) and weak (p_1, p_1). We show that T is strong (p, q).

$\forall\, \lambda > 0$, write $f = f_0 + f_1$, where $f_0 = f \cdot \chi(\{x : |f(x)| > C\lambda\})$ and $f_1 = f \cdot \chi(\{x : |f(x)| \leq C\lambda\})$. Note

$$\mu(\{x : |Tf(x)| > \lambda\}) \leq \mu(\{x : |T_0 f(x)| > \frac{\lambda}{2}\}) + \mu(\{x : |T_1 f(x)| > \frac{\lambda}{2}\}). \tag{1}$$

1. **Case 1**: $p_1 = \infty$.

   In this case, $\|Tf\|_\infty \leq A_1\|f\|_\infty$. Here we assume, by assumption of weak (p_i, q_i), $\forall\, f$, $\|Tf\|_{p_0,\infty} \leq A_0\|f\|_{p_0}$ and $\|Tf\|_{p_1,\infty} \leq A_1\|f\|_{p_1}$. Thus, $\|Tf_1\|_\infty \leq A_1\|f_1\|_\infty \leq C\lambda A_1$.

   Choose $C = \frac{1}{2A_1}$, we obtain $\|Tf_1\|_\infty \leq \frac{\lambda}{2}$, which implies $\mu(\{x : |T_1 f(x)| > \frac{\lambda}{2}\}) = 0$. Then from Lemma 1, we have

$$\begin{aligned} \|Tf\|_p^p &= p\int_0^\infty \lambda^{p-1}\mu(\{x : |Tf(x)| > \lambda\})d\lambda \\ &\leq p\int_0^\infty \lambda^{p-1}\mu(\{x : |T_0 f(x)| > \frac{\lambda}{2}\})d\lambda \\ &\leq p\int_0^\infty \lambda^{p-1}\frac{(2A_0)^{p_0}\|f_0\|_{p_0}^{p_0}}{\lambda^{p_0}}d\lambda \qquad \text{(Chebyshev's inequality)} \\ &= p(2A_0)^{p_0}\int_0^\infty \lambda^{p-p_0-1}\int_{\{x:|f(x)|>C\lambda\}} |f(x)|^{p_0}d\mu d\lambda \\ &= p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\left(\int_0^{\frac{|f(x)|}{C}} \lambda^{p-p_0-1}d\lambda\right)d\mu \\ &= \frac{p}{p-p_0}\cdot(2A_0)^{p_0}\cdot(2A_1)^{p-p_0}\cdot\|f\|_p^p. \end{aligned}$$

2. **Case 1**: $p_1 < \infty$.

$$\begin{aligned}
\|Tf\|_p^p \leq & p\int_0^\infty \lambda^{p-1}\mu(\{x : |T_0 f(x)| > \frac{\lambda}{2}\})d\lambda + p\int_0^\infty \lambda^{p-1}\mu(\{x : |T_1 f(x)| > \frac{\lambda}{2}\})d\lambda \\
\leq & p\int_0^\infty \lambda^{p-1}\left(\frac{2A_0}{\lambda}\|f_0\|_{p_0}\right)^{p_0} d\lambda + p\int_0^\infty \lambda^{p-1}\left(\frac{2A_1}{\lambda}\|f_0\|_{p_1}\right)^{p_1} d\lambda \\
\leq & p(2A_0)^{p_0}\int_0^\infty \lambda^{p-p_0-1}\int_{\{x:|f(x)|>C\lambda\}} |f(x)|^{p_0} d\mu d\lambda \\
& + p(2A_1)^{p_1}\int_0^\infty \lambda^{p-p_1-1}\int_{\{x:|f(x)|\leq C\lambda\}} |f(x)|^{p_1} d\mu d\lambda \\
= & \left(\frac{p\cdot(2A_0)^{p_0}}{C^{p-p_0}(p-p_0)} + \frac{p\cdot(2A_1)^{p_1}}{C^{p-p_1}(p_1-p)}\right)\cdot\|f\|_p^p.
\end{aligned}$$

□

We have another vision for Theorem 2 in the restricted version, which is stated as follow:

Theorem 3 (**Stein-Weiss Theorem**)**.** Let $p_0, p_1, q_0, q_1 \in [1, \infty]$. Suppose T is linear and for any measurable set E, $\|T\cdot\chi(E)\|_{q_0,\infty} \leq C_0 \cdot |E|^{1/p_0}$ and $\|T\cdot\chi(E)\|_{q_1,\infty} \leq C_1\cdot|E|^{1/p_1}$ Let p, q are given by

$$\begin{cases} \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \\ \\ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \end{cases}$$

where $0 < \theta < 1$. Then T is strong (p, q).

Proof. By the same way we proved Marcinkiewicz Interpolation Theorem, it is obvious that from the assumption in the theorem, T satisfies

$$\|T\cdot\chi_E\|_q \leq C|E|^{1/p}.$$

Call this T **restricted (strong) type** (p, q). Now take $f \in L^{q'}$, where $1/q + 1/q' = 1$, and E measurable with $|E| < \infty$. Define

$$b_f(E) = \int T\cdot\chi_E\cdot f(x)d\mu.$$

By Hölder's inequality,

$$|b_f(E)| \leq \|T\cdot\chi_E\|_q\cdot\|f\|_{q'} \leq C|E|^{1/p}\|f\|_{q'}.$$

Hence $b_f(\cdot)$ is a signed measure, and is absolutely continuous with respect to μ. By Radon-Nikodym Theorem, there exists $h = T^*f \in L^1$ s.t.

$$b_f(E) = \int_E h(x)d\mu.$$

Therefore, for any $f \in L^{q'}$ and measurable E with $|E| < \infty$, there is

$$\int T\cdot\chi_E\cdot f(x)d\mu = \int \chi_E\cdot T^*f(x)d\mu.$$

Because T is linear, for any simple function s, we have

$$\int Ts(x) \cdot f(x) d\mu = \int s(x) \cdot T^* f(x) d\mu.$$

We need the following lemma:

Lemma 2. Suppose T satisfies the conditions stated in the theorem. Then T^* is weak (p, q) type.

Proof. For $\lambda > 0$, denote

$$E^+(\lambda) = \{x : T^* f(x) > \lambda\},$$
$$E^-(\lambda) = \{x : T^* f(x) < -\lambda\}.$$

Then

$$\mu(\{x : |T^* f(x)| > \lambda\}) = \mu(E^+(\lambda)) + \mu(E^-(\lambda)).$$

For $E^+(\lambda)$, we have

$$\begin{aligned}
\mu(E^+(\lambda)) &= \int_X \chi_{E^+(\lambda)}(x) d\mu \\
&\leq \frac{1}{\lambda} \int T^* f(x) \cdot \chi_{E^+(\lambda)}(x) d\mu \\
&= \frac{1}{\lambda} \int f(x) \cdot T\chi_{E^+(\lambda)}(x) d\mu \\
&\leq \frac{1}{\lambda} \|T\chi_{E^+(\lambda)}\|_q \cdot \|\|\|_{q'} \\
&\leq \frac{C}{\lambda} |E^+(\lambda)|^{1/p} \cdot \|\|\|_{q'}.
\end{aligned}$$

Similar we can obtain the estimate for $E^-(\lambda)$. □

Now back to the theorem. Take θ_0, θ_1 s.t. $0 < \theta_0 < \theta < \theta_1 < 1$, then from previous discussion, T is restricted (strong) type $(p_{\theta_0}, q_{\theta_0})$ and restricted (strong) type $(p_{\theta_1}, q_{\theta_1})$. Then from Lemma 2, we know T^* is weak $(q'_{\theta_0}, p'_{\theta_0})$ and weak $(q'_{\theta_1}, p'_{\theta_1})$. From Marcinkiewicz Interpolation Theorem, T^* is strong (q'_θ, p'_θ). In the same way, we know T^{**} is strong (p_θ, q_θ). Now for any simple function s and $f \in L^{q'_\theta}$, we have

$$\int T^* f(x) \cdot s(x) d\mu = \int f(x) \cdot Ts(x) d\mu = \int f(x) \cdot T^{**} s(x) d\mu,$$

hence $T^{**}s = Ts$. By MCT, we know $T = T^*$, which implies that T is strong (p_θ, q_θ), or just strong (p, q). □

3 Maximal Functions

Definition 3. If $\int_K |f(x)|dx < \infty$ for any compact set K, then say $f(x)$ is **locally integrable**. Denote

$$L^1_{loc}(\mathbb{R}^n) \coloneqq \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ locally integrable}\}.$$

Definition 4. For $f \in L^1_{loc}(\mathbb{R}^n)$, we define the **(Hardy-Littlewood) maximal function** to be

$$Mf(x) \coloneqq \sup_B \frac{\chi(B)}{|B|} \int_B |f(y)|dy,$$

for every $x \in \mathbb{R}^n$.

Theorem 4. M is weak (1,1), i.e. $\|Mf\|_{1,\infty} \le C\|f\|_1$, or

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \le \frac{C\|f\|_1}{\lambda},$$

and $\|Mf\|_\infty \le \|f\|_\infty$.

Corollary 1. M is strong (p, p), for $1 < p < \infty$.

Lemma 3 (**Vitali Covering Lemma**). Let $E \subset \mathbb{R}^n$ be Lebesgue measurable set. $E \subset \cup_\alpha B_\alpha$, where B_α are balls in $\mathbb{R}^n$ and $\sup_\alpha r(B_\alpha) < \infty$. Then there exists a disjoint collection $B_{\alpha_1}, B_{\alpha 2}, \cdots$ (at most countably), such that $\sum_{k=1}^\infty m(B_{\alpha_k}) \ge C_n \cdot m(E)$, where the constant depends only on n. (Usually, we choose $C_n = 5^{-n}$.)

Proof. Take B_{α_1} to be a ball in $\{B_\alpha\}$ such that $r(B_\alpha) \ge \frac{1}{2}\sup_\alpha r(B_\alpha)$. Suppose we've chosen $B_{\alpha_1}, \cdots, B_{\alpha_k}$. Now to choose the next ball $B_{\alpha_{k+1}}$ s.t.

1. $B_{\alpha_{k+1}} \cap \left(\cup_{j=1}^k B_{\alpha_j}\right) = \varnothing$.
2. $r(B_{\alpha_{k+1}}) \ge \frac{1}{2}\sup\{r(B_\alpha) : B_\alpha \cap \left(\cup_{j=1}^k B_{\alpha_j}\right) = \varnothing\}$.

Then $B_{\alpha_1}, \cdots, B_{\alpha_k}, \cdots$ are disjoint, and

$$C_n m(E) \le \sum_{k=1}^\infty m(B_{\alpha_k}).$$

We've chosen the sequence $\{B_{\alpha_k}\}$. If RHS $= \infty$, then we have our result. Now assume $\sum_{k=1}^\infty m(B_{\alpha_k}) < \infty$. Let $B^*_{\alpha_k}$ be the ball with same center as B_{α_k}, but 5 times the radius. Claim that $E \subset \cup B^*_{\alpha_k}$. Once we have this, we have

$$m(E) \le m(\cup B^*_{\alpha_k}) \le \sum m(B^*_{\alpha_k}) = 5^n \sum m(B_{\alpha_k}).$$

Suffice to prove the claim. It suffices to show that each $B_\alpha \subset \cup B^*_{\alpha_k}$. Fix α. If $\alpha = \alpha_k$ for some k, we're done. Suppose $\alpha \ne \alpha_k$.

By assumption, $d(B_{\alpha_k}) \to 0$. Let k be the smallest integer s.t. $d(B_{\alpha_{k+1}}) \le \frac{1}{2}d(B_\alpha)$. Then B_α must intersect one of $B_{\alpha_1}, \cdots, B_{\alpha_k}$, or else we may choose it instead of $B_{\alpha_{k+1}}$. Therefore B_α must intersects some B_β for some $\beta \le \alpha_k$. Also $\frac{1}{2}d(B_\alpha) \le d(B_\beta)$. Claim that $B_\alpha \subset B^*_\beta$.

Indeed, let x_β be the center of B_β and y be a point in $B_\beta \cap B_\alpha$. Then $\forall x \in B_\alpha$, we have

$$|x - x_\beta| \le |x - y| + |y - x_\beta| < d(B_\alpha) + \frac{1}{2}d(B_\beta) \le \frac{5}{2}d(B_\beta),$$

or $x \in B_\beta^*$. Therefore we've show $B_\alpha \subset B_\beta^*$ and we're done. ☐

Proof of Theorem 4. We need to show for any $\lambda > 0$,

$$m(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \le \frac{C\|f\|_1}{\lambda}.$$

Let $E_\lambda := \{x : Mf(x) > \lambda\}$. Now for any $x \in E_\lambda$, there exists B_x s.t. $x \in B_x$ and

$$\frac{1}{m(B_x)} \int_{B_x} |f| dy > \lambda. \tag{$*$}$$

Thus $E_\lambda \subset \cup_{x \in E_\lambda} B_x$, $\sup_{x \in E_\lambda} r(B_x) < \infty$. By $(*)$,

$$m(B_x) < \frac{1}{\lambda} \int_{B_x} |f(y)| dy \le \frac{1}{\lambda} \int_{\mathbb{R}^n} |f| = \frac{\|f\|_1}{\lambda} < \infty,$$

and then $\sup_{x \in E_\lambda} m(B_x) \le \frac{\|f\|_1}{\lambda} < \infty$, $m(B_x) = C_n(r(B_x))^n$. By Vitali Covering Lemma (Lemma 3), there exists $B_{x_1}, \cdots, B_{x_k}, \cdots$ s.t. they are disjoint and

$$m(E_\lambda) \le C_n \sum_{k=1}^{\infty} m(B_{x_k}) \overset{(*)}{\le} C_n \sum_{k=1}^{\infty} \frac{1}{\lambda} \int_{B_x} |f| dy = \frac{C_n}{\lambda} \sum_k \int_{B_{x_k}} |f| \le \frac{C_n \|f\|_1}{\lambda}.$$

☐

Theorem 5 (**Lebesgue Differentiation Theorem**)**.** Let $f \in L^p(\mathbb{R}^n)$ $(p \ge 1)$. Then

$$\lim_{\substack{r(B) \to 0 \\ x \in B}} \frac{1}{m(B)} \int_B f(y) dy = f(x) \quad a.e. \ \ x \in \mathbb{R}^n.$$

Proof. If $f \in C_c(\mathbb{R}^n) = \{\text{all continuous functions with compact support}\}$. Then

$$\lim_{r(B) \to 0} \frac{1}{m(B)} \int_B f(y) dy = f(x).$$

Consider for $p = 1$. Define

$$\overline{\lim_{\substack{r(B) \to 0 \\ x \in B}}} \frac{1}{m(B)} \int_B f(y) dy = \inf_{\delta > 0} \sup_{\substack{r(B) < \delta \\ x \in B}} \frac{1}{m(B)} \int_B f(y) dy = \lim_{\delta \to 0} \sup_{\substack{r(B) < \delta \\ x \in B}} \frac{1}{m(B)} \int_B f(y) dy;$$

$$\underline{\lim_{\substack{r(B) \to 0 \\ x \in B}}} \frac{1}{m(B)} \int_B f(y) dy = \sup_{\delta > 0} \inf_{\substack{r(B) < \delta \\ x \in B}} \frac{1}{m(B)} \int_B f(y) dy = \lim_{\delta \to 0} \sup_{\substack{r(B) < \delta \\ x \in B}} \frac{1}{m(B)} \int_B f(y) dy.$$

Define

$$\theta(f)(x) = \overline{\lim_{\substack{r(B) \to 0 \\ x \in B}}} \frac{1}{m(B)} \int_B f(y) dy - \underline{\lim_{\substack{r(B) \to 0 \\ x \in B}}} \frac{1}{m(B)} \int_B f(y) dy,$$

and we will show $\theta(f)(x) = 0$ a.e. Since $C_c(\mathbb{R}^n)$ is dense in L^1, for $f \in L^1$, $\epsilon > 0$, there exists $g \in C_c(\mathbb{R}^n)$ s.t. $\|f - g\|_1 < \epsilon$, $\theta(g)(x) = 0$. Also

$$|\theta(f)(x)| = |\theta(f)(x) - \theta(g)(x)| \le |\theta(|f - g|)(x)| \le 2M(|f - g|)(x).$$

Because $\{x : |\theta(f)(x)| > \lambda\} \subset \{x : M(f-g)(x) > \frac{\lambda}{2}\}$,

$$m(\{x : |\theta(f)(x)| > \lambda\}) \leq m(\{x : M(f-g)(x) > \frac{\lambda}{2}\}) \leq \frac{C}{\lambda}\|f-g\|_1 < \frac{C}{\lambda}\epsilon.$$

Let $\epsilon \to 0$, $m(\{x : |\theta(f)(x)| > \lambda\}) = 0$, which implies $\theta(f)(x) = 0$ a.e.

Let

$$F_B f(x) = \frac{\chi(B)}{m(B)} \int_B f - f(x),$$

we will show $\lim_{\substack{r(B)\to 0\\ x\in B}} F_B f(x) = 0$ a.e. This can be obtained from

$$\left| \lim_{\substack{r(B)\to 0\\ x\in B}} F_B f(x) \right| = \left| \lim_{\substack{r(B)\to 0\\ x\in B}} (F_B f(x) - F_B g(x)) \right| \leq \lim_{\substack{r(B)\to 0\\ x\in B}} 2M(f-g)(x) = 2M(f-g)(x).$$

Hence

$$m(\{x : \left| \lim_{\substack{r(B)\to 0\\ x\in B}} F_B f(x) \right| > \lambda\}) \leq m(\{x : M(f-g)(x) > \frac{\lambda}{2}\}) \leq \frac{C\|f-g\|_2}{\lambda} < \frac{C}{\lambda}\epsilon,$$

which tends to 0 as $\epsilon \to 0$, and thus $\lim_{\substack{r(B)\to 0\\ x\in B}} F_B f(x) = 0$ a.e. $\square$

4 Approximate to the Identity

Definition 5. Let $\phi \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi dx = 1$. Define

$$\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x).$$

Then $\{\phi_\epsilon\}_{\epsilon>0}$ is called an **approximation to the identity**.

Definition 6. **Schwartz space**, denoted by $\mathcal{S}(\mathbb{R}^n)$, is defined to be the space

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty, \alpha, \beta \in \mathbb{N}^n\},$$

where for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$,

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$
$$D^\beta = \partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}.$$

Observation 1. $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.

Lemma 4. Let $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$|D^\beta f(x)| \leq \frac{C_{N,\beta}}{(1+|x|)^N}$$

for any $N \in \mathbb{N}$, $x \in \mathbb{R}^n$ and $\beta \in \mathbb{N}^n$, and $C_{N,\beta}$ is independent of x.

Definition 7. Define the **convolution** of f and g by

$$f * g = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Observation 2. $f * g = g * f$.

Lemma 5. Let $\{\phi_\epsilon\}_{\epsilon>0}$ be an approximation to the identity. Then $\lim_{\epsilon\to0} \phi_\epsilon * f(x) = f(x)$ for any $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. By definition, we have

$$\begin{aligned}\phi_\epsilon * f(x) &= \int_{\mathbb{R}^n} \phi_\epsilon(y) f(x-y)dy \\ &= \epsilon^{-n} \int \phi(\epsilon^{-1}y) f(x-y)dy \\ &= \int \phi(y) f(x-\epsilon y)dy\end{aligned}$$

So $\lim_{\epsilon\to0} \phi_\epsilon * f(x) = \lim_{\epsilon\to0} \int \phi(y) f(x-\epsilon y)dy$. Since $f \in \mathcal{S}$, which implies $f \in L^\infty$, and also

$$|\phi(y) f(x-\epsilon y)| \leq |\phi(y)| \cdot \|f\|_\infty.$$

Then by DCT, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\phi_\epsilon * f(x) &= \lim_{\epsilon\to 0}\int \phi(y)f(x-\epsilon y)dy\\ &= \int \phi(y)\lim_{\epsilon\to 0} f(x-\epsilon y)dy\\ &= \int \phi(y)f(x)dy\\ &= f(x)\int_{\mathbb{R}^n}\phi(y)dy = f(x).\end{aligned}$$

□

Example 1. Let $\phi(x) = e^{-\pi|x|^2}$, then $\{\phi_\epsilon\}_{\epsilon>0}$ is an approximation to the identity.

Lemma 6 (**Minkowski**)**.** For $1 \le p < \infty$,

$$\left[\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x,y)|dy\right)^p dx\right]^{1/p} \le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x,y)|^p dx\right)^{1/p} dy.$$

Remark 1. For $1 \le p < \infty$, we always have

$$\|f\|_p = \sup\left\{\left|\int_{\mathbb{R}^n} fg\right| : g \in L^q(\mathbb{R}^n),\ \|g\|_q = 1,\ \frac{1}{p}+\frac{1}{q}=1\right\}.$$

Proof. By Remark 1, LHS $= \sup\left\{\left|\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x,y)|dy\right)\cdot g(x)dx\right| : g \in L^q(\mathbb{R}^n),\ \|g\|_q = 1,\ \frac{1}{p}+\frac{1}{q}=1\right\}$. Note that

$$\begin{aligned}\left|\int\left(\int|f(x,y)|dy\right)\cdot g(x)dx\right| &\le \int\int|f(x,y)|dy\cdot|g(x)|dx\\ &= \int\left(\int|f(x,y)|\cdot|g(x)|dx\right)dy && \text{(Fubini)}\\ &\le \int\left(\int|f(x,y)|^p dx\right)^{1/p}\cdot\|g\|_q dy && \text{(Hölder)}\\ &= \int\left(\int|f(x,y)|^p dx\right)^{1/p} dy.\end{aligned}$$

Taking exponent $\frac{1}{p}$ to both sides, we obtain our results. □

Theorem 6. Let $1 \le p < \infty$, $\phi \in L^1(\mathbb{R}^n)$, $\int\phi = 1$, and $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$. Then $\forall\, f \in L^p(\mathbb{R}^n)$, we have

$$\lim_{\epsilon\to 0}\|f*\phi_\epsilon - f\|_p = 0.$$

Remark 2. In case for $p = \infty$, the theorem does **NOT** hold generally!

Theorem 7. Let $\phi \in L^1$, $\int\phi = 1$ and $\psi(x) = \sup_{|y|\ge|x|}|\phi(y)|$ (ψ is called the **least decreasing radial majorant of** ϕ). Suppose $\psi \in L^1$, $\int\psi = A$. Then

1. We have $\sup_{\epsilon>0}|f*\phi_\epsilon(x)| \le A\cdot Mf(x)$ for any $f \in L^1_{loc}$.

2. For any $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$,

$$\lim_{\epsilon \to 0} f * \phi_\epsilon(x) = f(x) \;\; a.e.\, x \in \mathbb{R}^n.$$

Proof of Theorem 6. Note that

$$\begin{aligned} f * \phi_\epsilon(x) - f(x) &= \int (f(x-y) - f(x))\, \phi_\epsilon(y) dy \\ &= \int (f(x - \epsilon y) - f(x))\, \phi(y) dy. \end{aligned}$$

By Minkowski's inequality,

$$\|f * \phi_\epsilon - f\|_p \le \int_{\mathbb{R}^n} |\phi(y)| \cdot \|f(\cdot - \epsilon y) - f(\cdot)\|_p dy.$$

Hence

$$\begin{aligned} \lim_{\epsilon \to 0} \|f * \phi_\epsilon - f\|_p &\le \int_{\mathbb{R}^n} |\phi(y)| \cdot \|f(\cdot - \epsilon y) - f(\cdot)\|_p dy \\ &= \int_{\mathbb{R}^n} |\phi(y)| \cdot \lim_{\epsilon \to 0} \|f(\cdot - \epsilon y) - f(\cdot)\|_p dy, \end{aligned} \tag{DCT}$$

and by $\|f(\cdot - \epsilon y) - f(\cdot)\|_p \to 0$, as $\epsilon \to 0$, we obtain the assertion. □

Proof of Theorem 7. Part 2 of the theorem follows from Theorem 1. It suffices to check part 1 of the theorem.

By the translate invariance and dilation invariance, it is suffices to show that $|f * \phi_1(0)| \le A \cdot Mf(0)$. Note that $|\phi(y)| \le |\psi(x)|$ for all x, it suffices to show for any non-negative function f, we have $f * \psi(0) \le A \cdot Mf(0)$.

Recall that

$$\int_{\mathbb{R}^n} f(x) dx = \int_0^\infty \int_{S^{n-1}} f(rx') r^{n-1} dx' dr,$$

where dx' is the surface measure of S^{n-1}. Note

$$\begin{aligned} f * \psi(0) &= \int_{\mathbb{R}^n} f(x)\psi(x) dx \\ &= \int_0^\infty \int_{S^{n-1}} f(rx')\psi(r) r^{n-1} dx' dr \\ &= \int_0^\infty \left(\int_{S^{n-1}} f(rx') dx' \right) \psi(r) r^{n-1} dr. \end{aligned}$$

Set $F(r) = \int_{S^{n-1}} f(rx') dx'$, then $f * \psi(0) = \int_0^\infty F(r)\psi(r) r^{n-1} dr$. Let

$$G(r) = \int_{B(0,r)} f(x) dx = \int_0^r t^{n-1} \int_{S^{n-1}} f(tx') dx' dt = \int_0^r t^{n-1} F(t) dt.$$

Then $G'(r) = r^{n-1}F(r)$, which implies

$$f * \psi(0) = \int_0^\infty G'(r)\psi(r) dr = \psi(r)G(r) \mid_{r=0}^{r=\infty} - \int_0^\infty G(r) d\psi(r).$$

Claim 1. $\lim_{r \to \infty} \psi(r)G(r) = \lim_{r \to 0} \psi(r)G(r) = 0$. Moreover, $\lim_{r \to \infty} \psi(r) r^n = \lim_{r \to 0} \psi(r) r^n = 0$.

Assume Claim 1, we have

$$\begin{aligned}
f * \psi(0) &= \int_0^\infty G(r) d(-\psi(r)) \\
&\leq C_n \cdot Mf(0) \int_0^\infty r^n d(-\psi(r)) \\
&= nC_n \cdot Mf(0) \int_0^\infty \psi(r) r^{n-1} dr \\
&= Mf(0) \int_{\mathbb{R}^n} \psi(x) dx = A \cdot Mf(0).
\end{aligned}$$

Proof of Claim 1. Since $|G(r)| = \left|\int_{B(0,r)} f(x)dx\right| \leq C_n \cdot r^n \cdot Mf(0)$, we have

$$|\psi(r)G(r)| \leq C_n \psi(r) r^n \cdot Mf(0).$$

It remains to show $\lim_{r\to 0} \psi(r)r^n = 0 = \lim_{r\to\infty} \psi(r)r^n$. Note that

$$\begin{aligned}
\psi(r)r^n &= C_n{}' \left(\int_{\frac{r}{2}<|x|\leq r} dx \right) \psi(r) \\
&\leq C_n{}' \left(\int_{\frac{r}{2}<|x|\leq r} \psi(x)dx \right) && (\psi(x) \text{ is monotonically decreasing and } L^1) \\
&\to 0,
\end{aligned}$$

as $r \to 0$ (also when $r \to \infty$). The claim follows. ☐

☐

Lemma 7. Let $\{T_\epsilon\}_{\epsilon>0}$ be a family of linear operators on $L^p(\mathbb{R}^n)$ and define $T^*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|$. If T^* is weak (p,p), then the set $\{f \in L^p(\mathbb{R}^n) : \lim_{\epsilon\to 0} T_\epsilon f(x) = f(x)\ a.e.\}$ is closed in $L^p(\mathbb{R}^n)$.

Proof. Let $f_k \in L^p$ and $\|f_k - f\|_p \to 0$ as $k \to \infty$. Suppose $\lim_{\epsilon\to 0} T_\epsilon f_k(x) = f_k(x)$ a.e. Show that $\lim_{\epsilon\to 0} T_\epsilon f(x) = f(x)$ a.e. This is straightforward, since

$$\begin{aligned}
\left|\left\{x \in X : \limsup_{\epsilon\to 0} |T_\epsilon f(x) - f(x)| > \lambda\right\}\right| &= \left|\left\{x \in X : \limsup_{\epsilon\to 0} |T_\epsilon(f - f_k)(x) - (f - f_k)(x)| > \lambda\right\}\right| \\
&\leq m\left(\left\{x \in X : T^*(f - f_k)(x) > \frac{\lambda}{2}\right\}\right) + m\left(\left\{x \in X : |(f - f_k)(x)| > \frac{\lambda}{2}\right\}\right) \\
&\leq \frac{C_1^p \|f - f_k\|_p^p}{\lambda^p} + \frac{C_2^p \|f - f_k\|_p^p}{\lambda^p} \\
&= \left(\frac{C}{\lambda}\right)^p \|f - f_k\|_p^p
\end{aligned}$$

which converges to 0 as $k \to \infty$. ☐

We end this section with an example, which is fundamental in PDE.

Example 2. Let $\phi(x) = (4\pi)^{-n/2} e^{-|x|^2/4}$ and $\epsilon = \sqrt{t}$ for $t > 0$. Define $\phi_\epsilon(x) = \phi_{\sqrt{t}}(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$.

Now consider the initial value problem:

$$\begin{cases} \Delta_x u = u_t, & (x,t) \in \mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\} \\ u(x,0) = f(x) \in L^p(\mathbb{R}^n), & 1 \leq p < \infty \end{cases} \tag{$*$}$$

Then $u(x,t) = W_t * f(x) = \phi_\epsilon * f(x)$ solves $(*)$, and from Theorem 7, we have $\lim_{t\to 0} u(x,t) = f(x)$.

Remark 3. The function ϕ_ϵ is the function $W_t(x)$, the fundamental solution for $\Delta_x u - u_t = 0$.

5 Fourier Transform

Definition 8. Let $f \in L^1(\mathbb{R}^n)$, define its **Fourier transform** $\hat{f}(\xi)$, $\xi \in \mathbb{R}^n$ to be

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\xi} dx.$$

Theorem 8. Let $f \in L^1(\mathbb{R}^n)$, then

1. $\|\hat{f}\|_\infty \le \|f\|_1$.
2. $\hat{f}$ is uniformly continuous on $\mathbb{R}^n$.
3. $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.
4. If $g \in L^1(\mathbb{R}^n)$, then $\widehat{f * g} = \hat{f} \cdot \hat{g}$.

Proof. Part 1 and 2 are straightforward and left to readers. We only prove the last two parts.

For 3, note

$$\begin{aligned}\hat{f}(\xi) &= \int f(x) e^{-2\pi i x\xi} \left(e^{-2\pi i \xi \cdot \frac{\xi}{2|\xi|^2}}\right) \cdot (-1) dx \\ &= -\int f(x) e^{-2\pi i \xi \left(x + \frac{\xi}{2|\xi|^2}\right)} dx \\ &= -\int f\left(x - \frac{\xi}{2|\xi|^2}\right) e^{-2\pi i \xi x} dx.\end{aligned}$$

Also by definition $\hat{f}(\xi) = \int f(x) e^{-2\pi i x\xi} dx$, so $2\hat{f}(\xi) = \int \left(f(x) - f\left(x - \frac{\xi}{2|\xi|^2}\right)\right) e^{-2\pi i x\xi} dx$, and thus $2|\hat{f}(\xi)| \le \int \left|f(x) - f\left(x - \frac{\xi}{2|\xi|^2}\right)\right| dx$. This implies $\lim_{|\xi|\to\infty} \int \left|f(x) - f\left(x - \frac{\xi}{2|\xi|^2}\right)\right| dx = 0$, since $\frac{\xi}{2|\xi|^2} \to 0$ and by continuity of L^1-norm. Hence we have the result.

For 4, we directly calculate:

$$\begin{aligned}\widehat{f * g}(\xi) &= \int f * g(x) e^{-2\pi i x\xi} \\ &= \int \int f(x-y) g(y) dy \cdot e^{-2\pi i x\xi} dx \\ &= \int g(y) \int f(x) e^{-2\pi i x\xi} dx \cdot e^{-2\pi i y\xi} dy \\ &= \hat{f}(\xi) \cdot \hat{g}(\xi).\end{aligned}$$

$\square$

Theorem 9. Let $f \in L^1$.

1. $\widehat{(T_b f)}(\xi) = e^{-2\pi i \xi b} \hat{f}(\xi)$, where $T_b f(x) = f(x+b)$.
2. $\widehat{(M_h f)}(\xi) = \hat{f}(\xi - h)$, where $M_h f(x) = e^{2\pi i \xi h} f(x)$.
3. $\widehat{(D_t f)}(\xi) = \hat{f}(t\xi)$, where $D_t f(x) = t^{-n} f(t^{-1} x)$.

4. Let ρ be an orthogonal transformation on $\mathbb{R}^n$. (We call a linear transformation $\rho : \mathbb{R}^n \to \mathbb{R}^n$ is **orthogonal** if ρ preserves the inner product, i.e. $\rho(x) \cdot \rho(y) = \rho(x \cdot y)$) Then $\widehat{f \circ \rho}(\xi) = \hat{f} \circ \rho(\xi) = \hat{f}(\rho(\xi))$.

Proof. Parts 1∼3 are left to readers. We only concern part 4. It is also straightforward:

$$\begin{aligned}\widehat{f \circ \rho}(\xi) &= \int f(\rho(x)) e^{-2\pi i x \xi} dx \\ &= \int f(y) e^{-2\pi i \rho^{-1}(y)\xi} dy \\ &= f(y) e^{-2\pi i \rho(\xi) y} dy = \hat{f}(\rho(\xi)).\end{aligned}$$

□

Theorem 10. Let $f \in L^1(\mathbb{R}^n)$, then $\frac{\partial \hat{f}(\xi)}{\partial \xi_k} = (\widehat{-2\pi i x_k f}(x))(\xi)$, if $x_k f(x) \in L^1(\mathbb{R}^n)$, and $\widehat{\left(\frac{\partial f}{\partial x_k}\right)}(\xi)_{(k)} = 2\pi i \xi_k \hat{f}(\xi)$ if $\frac{\partial f}{\partial x_k} \in L^1$.

Proof. Let $h = (0, \cdots, 0, h_k, 0, \cdots, 0)$, then by nasty calculation,

$$\frac{\partial \hat{f}(\xi)}{\partial \xi_k} = \lim_{h_k \to \infty} \frac{\hat{f}(\xi + h) - \hat{f}(\xi)}{h_k} = \cdots = (\widehat{-2\pi i x_k f}(x))(\xi).$$

The other is the same. The calculation is left to readers. □

Theorem 11. Let $D^\alpha \left(= \frac{\partial^\alpha}{\partial x^\alpha}\right) = \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}$, where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \backslash 0$. Let $p(x) = \sum_{\substack{\alpha \\ |\alpha| \le d}} a_\alpha x^\alpha$, where $|\alpha| = \sum_{i=1}^n \alpha_1$. Define $p(D) = \sum_{\substack{\alpha \\ |\alpha| \le d}} a_\alpha D^\alpha$. Then for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$P(D)\hat{f}(\xi) = (P(\widehat{-2\pi i x)} f(x))(\xi),$$

and

$$\widehat{P(D)} f(\xi) = P(2\pi i \xi)\hat{f}(\xi).$$

The proof is left to readers.

Definition 9. The **inverse Fourier Transform** is defined to be

$$\check{g}(x) = \int_{\mathbb{R}^n} g(\xi) e^{2\pi i x \xi} d\xi = \hat{g}(-x).$$

A natural question comes up: whether we can find a relationship of Fourier Transform and inverse Fourier Transform? Or explicitly, does there exist the following relation:

Claim 2. $\check{\hat{f}} = f$?

The answer is yes, provided $f, \hat{f} \in L^1$, which we will state as the following theorem:

Theorem 12. Let $f, \hat{f} \in L^1$, then $\check{\hat{f}} = f$.

We need a few lemmas before we prove the theorem.

Lemma 8. Let $f, g \in L^1$, then $\int \hat{f} g = \int f \hat{g}$.

Proof.

$$\begin{aligned}\text{LHS} &= \int\int e^{-2\pi i\xi x} f(x)dx \cdot g(\xi)d\xi \\ &= \int f(x)\left(\int g(\xi)e^{-2\pi i\xi x}d\xi\right)dx = \text{RHS}.\end{aligned}$$

□

Lemma 9. $\widehat{e^{-\pi|\cdot|^2}}(\xi) = e^{-\pi|\xi|^2}$.

Proof. It suffices to show it in 1-dimensional case, since by Fubini's theorem,

$$\begin{aligned}\text{LHS} &= \int e^{-\pi(x_1^1+\cdots x_n^2)}e^{-2\pi i(x_1\xi_1+\cdots+x_n\xi_n)}d\xi_1\cdots d\xi_n \\ &= \prod_{j=1}^n \int e^{-\pi x_j^2}e^{-2\pi i x_j\xi_j}d\xi_j.\end{aligned}$$

Let $f(x) = e^{-\pi x^2}$, where $x \in \mathbb{R}$. Now to show $f = \hat{f}$. Notice that f is the solution of system

$$\begin{cases} u' + 2\pi x u = 0 \\ u(0) = 1 \end{cases} \tag{$*$}$$

i.e. $(\widehat{u' + 2\pi x u}) = 0$. This implies $\hat{u}' + (\widehat{2\pi x u}) = 0$, and it is $2\pi i\xi\hat{u}(\xi) + i\hat{u}'(\xi) = 0$. So $\hat{u}' + 2\pi\xi\hat{u} = 0$ and $\hat{u}(0) = \int_{\mathbb{R}} u(x)dx = \int_{\mathbb{R}} f = 1$. We observe $\hat{u}$ also satisfies $(*)$. Thus $\hat{f}$ is the solution of $(*)$. By uniqueness of solution, $f = \hat{f}$. □

Corollary 2. $\widehat{e^{-4\pi^2|\cdot|^2}}(\xi) = (4\pi)^{-n/2}e^{-|\xi|^2/4}$.

Example 3 (Gaussian mean)**.** Let $g \in L^1(\mathbb{R}^n)$, then

$$G_\epsilon(g) = \int_{\mathbb{R}^n} g(\xi)e^{-4\pi^2\epsilon^2|\xi|^2}d\xi$$

is called the **Gaussian mean** of $\int_{\mathbb{R}^n} g(\xi)d\xi$.

Observation 3. $\lim_{\epsilon\to 0} G_\epsilon(g) = \int_{\mathbb{R}^n} g(\xi)d\xi$.

Lemma 10. If $f \in L^1(\mathbb{R}^n)$, then when $\epsilon \to 0$,

$$\left\|\int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x\xi}e^{-4\pi^2|\xi|^2\epsilon^2}d\xi - f(x)\right\|_{L^1(\mathbb{R}^n)} \to 0.$$

That is, $\int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x\xi}e^{-4\pi^2|\xi|^2\epsilon^2}d\xi \to f$ in L^1.

Proof.

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x\xi}e^{-4\pi^2|\xi|^2\epsilon^2}d\xi &= \int_{\mathbb{R}^n} f(y)(\widehat{e^{2\pi i x\xi}e^{-4\pi^2|\xi|^2\epsilon^2}})(y)dy \qquad \text{(Lemma 8)} \\ &= \int_{\mathbb{R}^n} f(y)\epsilon^{-n}(\widehat{e^{-4\pi^2|\cdot|^2}})(\epsilon^{-1}(x-y))dy.\end{aligned}$$

Let $\varphi(x) = (\widehat{e^{-4\pi^2|\cdot|^2}})(x) = (4\pi)^{-n/2}e^{-|x|^2/4}$ (by Corollary 2). Note $\int\varphi = 1$, so $\{\varphi_\epsilon\}$ is an approximation to the identity. Thus Lemma 10 holds by Theorem 6. □

Now we're ready to prove our main theorem for the chapter.

Proof of Theorem 12. By Lemma 10, there exists a subsequence $\{\epsilon_k\}$ such that

1. $\epsilon_k \to 0$, as $k \to \infty$.

2. $\lim_{\epsilon\to 0} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\xi} e^{-4\pi^2|\xi|^2\epsilon_k^2} d\xi = f(x)$ a.e. $x \in \mathbb{R}^n$. Then by DCT,

$$\begin{aligned} \text{LHS} &= \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\xi} \lim_{k\to\infty} e^{-4\pi^2|\xi|^2\epsilon^2} d\xi \\ &= \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\xi} \\ &= \check{\hat{f}}. \end{aligned}$$

$\square$

6 Fourier Transform on $L^2(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ $(1 \leq p \leq 2)$

We shall always notice that L^2 is a Hilbert space.

Proposition 1. $f \in \mathcal{S}(\mathbb{R}^n)$ if and only if $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$.

Proof. For implication direction, we need to show if $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\sup_{\xi \in \mathbb{R}^n} \left| (2\pi i \xi)^\alpha D^\beta \hat{f}(\xi) \right| < \infty.$$

Note

$$\begin{aligned}
(2\pi i \xi)^\alpha D^\beta \hat{f}(\xi) &= (2\pi i \xi)^\alpha (\widehat{(-2\pi i x)^\beta f(x)})(\xi) \\
&= [D^\alpha \, (\widehat{(-2\pi i x)^\beta f(x)})](\xi) \\
&= \int D^\alpha \left((-2\pi i x)^\beta f(x) \right) e^{-2\pi i \xi x} dx,
\end{aligned}$$

and Schwartz function implies $|(2\pi i \xi)^\alpha D^\beta \hat{f}(\xi)| \leq \int \frac{C_N}{(1+|x|)^N} dx < \infty$, for any $N \in \mathbb{N}$. For the reverse direction, suppose $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$, then $\check{\hat{f}} \in \mathcal{S}(\mathbb{R}^n)$ by a similar argument as we did for implication direction, since $\check{\hat{f}} = f$ by Theorem 12. $\square$

Proposition 2. Suppose that $f, \hat{f}, h, \hat{h} \in L^1(\mathbb{R}^n)$. Then $\langle f, h \rangle = \left\langle \hat{f}, \hat{h} \right\rangle$, where $\langle f, h \rangle = \int f \overline{h}$. In particular, if $f = h$, $\|f\|_2 = \|\hat{f}\|_2$.

Proof.

$$\left\langle \hat{f}, \hat{h} \right\rangle = \int \hat{f} \overline{\hat{h}} = \int f(x) \hat{\overline{\hat{h}}}. \tag{Fubini}$$

Note

$$\begin{aligned}
\hat{\overline{\hat{h}}} &= \int \overline{\hat{f}}(\xi) e^{-2\pi i \xi x} d\xi \\
&= \overline{\int \hat{h}(\xi) e^{2\pi i \xi x} d\xi} = \overline{\check{\hat{h}}}(x) = \overline{h}(x),
\end{aligned}$$

which implies $\left\langle \hat{f}, \hat{h} \right\rangle = \int f(x) \overline{h}(x) dx = \langle f, h \rangle$. $\square$

Our goal in this section, is to extend our Fourier transform to $L^2(\mathbb{R}^n)$. For any $f \in L^2(\mathbb{R}^n)$, choose $\{f_k\}$ in $\mathcal{S}(\mathbb{R}^n)$ such that $f_k \xrightarrow{L^2} f$. Thus $\hat{f}_k$ is well-defined, $\hat{f}_k \in \mathcal{S}(\mathbb{R}^n)$. Note that Schwartz functions are dense in L^2 space. There are few things to check:

- $\{\hat{f}_k\}$ is Cauchy in $L^2(\mathbb{R}^n)$:

$$\|\hat{f}_k - \hat{f}_j\|_2 = \|\widehat{f_k - f_j}\|_2 = \|f_k - f_j\|_2 \to 0.$$

  By the completeness of $L^2(\mathbb{R}^n)$, there exists $g \in L^2(\mathbb{R}^n)$ such that $\lim_{k\to\infty} \hat{f}_k \overset{L^2}{=} g$, that is, $\lim_{k\to\infty} \|\hat{f}_k - g\|_2 = 0$. Finally, define $\hat{f} = g$.

- This is well-definiteness: Show that g is independent of choice of Cauchy sequence. Take $\tilde{f}_k \xrightarrow{L^2} f$, $\tilde{f}_k \in \mathcal{S}(\mathbb{R}^n)$. Assume that $\hat{\tilde{f}}_k \xrightarrow{L^2} g$. We will show that $\tilde{g} = g$.

  Consider the sequence $\{f_1, \tilde{f}_1, f_2, \tilde{f}_2, \cdots, f_j, \tilde{f}_j, \cdots\} = \{h_k\}_{k=1}^{\infty}$. Then $h_k \xrightarrow{L^2} f$ because $\tilde{f}_k \xrightarrow{L^2} f$ and $f_k \xrightarrow{L^2} f$.

  $\{\hat{h_k}\}$ is Cauchy, hence $\hat{h_k} \xrightarrow{L^2} h$ for some $h \in L^2(\mathbb{R}^n)$. Note $\{\hat{h_k}\} = \{\hat{f}_1, \hat{\tilde{f}}_1, \hat{f}_2, \hat{\tilde{f}}_2, \cdots, \hat{f}_j, \hat{\tilde{f}}_j, \cdots\}$. So $g = \lim_{k\to\infty} \hat{f}_k \overset{L^2}{=} \lim_{k\to\infty} \hat{\tilde{f}}_k = \tilde{g} \overset{L^2}{=} h$. Hence $g = \tilde{g} = h$ in L^2, $g = \tilde{g}$ a.e.

Theorem 13 (**Plancherel**)**.** $f \in L^2$, then $\hat{f} \in L^2$ and $\|\hat{f}\|_2 = \|f\|_2$.

Proof. Let $f_k \xrightarrow{L^2} f$, $f_k \in \mathcal{S}$. Then $\|f\|_2 = \lim_{k\to\infty} \|f_k\|_2$, and $\hat{f} \overset{L^2}{=} \lim_{k\to\infty} \hat{f}_k$. This implies $\lim_{k\to\infty} \|\hat{f}_k\|_2 = \|\hat{f}\|_2$. Notice that $\|f_k\|_2 = \|\hat{f}_k\|_2$, so

$$\|f\|_2 = \lim_{k\to\infty} \|f_k\|_2 = \lim_{k\to\infty} \|\hat{f}_k\|_2 = \|\hat{f}\|_2.$$

□

Theorem 14. Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$. (The unitary operator on H is a linear operator which is an onto isometry.)

Proof. It remains to show Fourier transform is onto. We have the following claim:

Claim 3. The range of Fourier transform is closed.

Proof of Claim 3. Suppose that $\hat{g_k} \xrightarrow{L^2} h$, where $g_k \in L^2$. We show that $h = \hat{g}$ for some $g \in L^2$. $\{g_k\}$ is Cauchy in L^2. By previous theorem, $\{g_k\}$ is Cauchy in L^2. So there exists $g \in L^2$ such that $g \overset{L^2}{=} \lim_{k\to\infty} g_k$, i.e. $\lim_{k\to\infty} \|g_k - g\|_2 = 0$. Suffice to show $h = \hat{g}$. It follows from

$$\begin{aligned} \|\hat{g} - h\|_2 &= \|\hat{g} - \hat{g_k} + \hat{g_k} - h\|_2 \le \|\hat{g} - \hat{g_k}\|_2 + \|\hat{g_k} - h\|_2 \\ &= \|g - g_k\|_2 + \|\hat{g_k} - h\|_2 \to 0. \end{aligned}$$

□

Let $R \coloneqq$ the range of Fourier transform in L^2. R is a closed subspace of L^2. $L^2(\mathbb{R}^n) = R \oplus R^\perp$, where $R^\perp = \{g \in L^2 : \langle f, g\rangle = 0, \forall f \in R\}$. Assume $R \neq L^2(\mathbb{R}^2)$. Then $R^\perp$ contains a non-zero function, say, $h \in R^\perp$ and $h \neq 0$, $\int \hat{f} h = 0$ for all $f \in L^2$.

Exercise 1. Show that $\int \hat{f} g = \int f \hat{g}$ for all $f, g \in L^2$.

Exercise 2. Suppose $f \in L^1$, $g \in L^p$, $1 \le p \le 2$. Show that $\widehat{(f * g)}(\xi) = \hat{f}\hat{g}$ a.e.

By Exercise 1, $\int f\hat{h} = 0$ for all $f \in L^2$, which means $\hat{h} \perp L^2$, thus $\hat{h} = 0$. This implies $\|h\|_2 = \|\hat{h}\|_2 = 0$, and hence $h = 0$ a.e. Contradiction! □

Same as in L^1, we can define inverse Fourier transform:

Definition 10. The **inverse Fourier transform** is defined to be $\check{f}(x) = \int f(\xi) e^{2\pi i x\xi} d\xi$ for $f \in \mathcal{S}(\mathbb{R}^n)$. If $f \in L^2(\mathbb{R}^n)$, we define the inverse Fourier transform $\check{f} \overset{L^2}{=} \lim_{k\to\infty} \check{f}_k$, here $f_k \xrightarrow{L^2} f$ and $f_k \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 15. For any $f \in L^2$, we have $\check{\hat{f}} = f$ a.e.

Proof. Let U be the Fourier transform operator, i.e. $Uf = \hat{f}$. We make the first claim:

Claim 4. The left adjoint operator is given by $U^* f = \check{f}$, i.e. for any $f, g \in L^2$, we have $\langle U^* f, g \rangle = \langle f, Ug \rangle$.

Proof of Claim 4. For $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} \langle U^* f, g \rangle &= \langle f, Ug \rangle = \langle f, \hat{g} \rangle && \text{(Exercise 1)} \\ &= \int f \overline{\hat{g}} \\ &= \int f(x) \overline{\int g(\xi) e^{-2\pi i \xi x} d\xi} dx \\ &= \int \check{f}(\xi) \overline{g(\xi)} d\xi = \left\langle \check{f}, g \right\rangle . \end{aligned}$$

So $U^* f = \check{f}$ for all Schwartz functions f. Now let $f \in L^2(\mathbb{R}^n)$, we need to show the same equality. Let $f_k \in \mathcal{S}(\mathbb{R}^n)$ s.t. $f_k \xrightarrow{L^2} f$, and by definition $\lim_{k\to\infty} \check{f}_k = \check{f}$. For any $g \in L^2$,

$$\begin{aligned} \langle U^* f, g \rangle &= \langle f, Ug \rangle = \langle f, \hat{g} \rangle \\ &= \langle f - f_k, \hat{g} \rangle + \langle f_k, \hat{g} \rangle \\ &= \langle f - f_k, \hat{g} \rangle + \langle U^* f_k, g \rangle , \end{aligned}$$

which implies

$$| \langle U^* f - U^* f_k, g \rangle | = | \langle f - f_k, \hat{g} \rangle | \le \|f - f_k\|_2 \cdot \|\hat{g}\|_2 \to 0.$$

So $U^* f \overset{L^2}{=} \lim_{k\to\infty} U^* f_k \overset{L^2}{=} \lim_{k\to\infty} \check{f}_k \overset{L^2}{=} \check{f}$. □

We have another claim:

Claim 5. U is unitary operator, so $U^* = U^{-1}$.

Proof of Claim 5. Let $x \in L^2$. Then from Claim 4,

$$\langle U^* U x, x \rangle = \langle Ux, Ux \rangle = \|Ux\|_2^2 = \|x\|_2^2 = \langle x, x \rangle .$$

Hence $U^{ast} U x = x$, i.e. $U^* = U^{-1}$. □

Back to our Theorem 15. Combine Claim 4 and Claim 5, we have

$$\check{\hat{f}} = U^* \hat{f} = U^* U f = (U^{-1} U) f = f.$$

□

Now to extend Fourier transform to L^p, $1 \le p \le 2$. We have already done when $p = 1$ and 2. It suffice to consider the $1 < p < 2$ part.

Let $f \in L^p$, then one can write $f = f_1 + f_2$, where $f_1 = f \cdot \chi(\{x : |f(x)| \ge 1\})$ and $f_2 = f \cdot \chi(\{x : |f(x)| < 1\})$, with $f_1 \in L^1$ and $f_2 \in L^2$. Define $\hat{f} \coloneqq \hat{f}_1 + \hat{f}_2$, where $\hat{f}_1$ is Fourier transform in L^1 and $\hat{f}_2$ is Fourier transform in L^2.

This definition is well-defined:

Let $L^1 + L^2 = \{f : f = f_1 + f_2, f_1 \in L^1, f_2 \in L^2\}$, then

- $L^p \subset L^1 + L^2$.

  This is because by define for any $f \in L^p$, let $f = f_1 + f_2$, where $f_1 = f \cdot \chi(\{x : |f(x)| \ge 1\})$ and $f_2 = f \cdot \chi(\{x : |f(x)| < 1\})$, then

$$\int |f_2|^2 \le \int |f_2|^p \le \int |f|^p < \infty,$$
$$\int |f_1| \le \int |f_1|^p \le \int |f|^p < \infty.$$

  This means $f_1 \in L^1$ and $f_2 \in L^2$.

- If $\hat{f} = \hat{f}_1 + \hat{f}_2 = \hat{g}_1 + \hat{g}_2$, where $f_1, g_1 \in L^1$ and $f_2, g_2 \in L^2$. Then $f_1 - g_1 = g_2 - f_2 \in L^1 \cap L^2$. Thus $\hat{f}_1 - \hat{g}_1 = \hat{g}_2 - \hat{f}_2$, which implies $\hat{f}_1 + \hat{f}_2 = \hat{g}_1 + \hat{g}_2$.

Theorem 16 (**Hausdorff-Young**)**.** Let $f \in L^p$, $1 \le p \le 2$. Then $\hat{f} \in L^q$ and $\|\hat{f}\|_q \le \|f\|_p$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Note $\|\hat{f}\|_\infty = \sup_{\mathbb{R}^n} |\hat{f}(\xi)| = \sup_{\mathbb{R}^n} \left|\int f(x) e^{-2\pi i \xi x} dx\right| \le \int |f| = \|f\|_1$ $(f \in L^1)$ and $\|\hat{f}\|_2 \le \|f\|_2$ $(f \in L^2)$. By Riesz-Thorin Interpolation Theorem (Theorem 1), $\|\hat{f}\|_q \le \|f\|_p$. □

Theorem 17 (**Young**)**.** Let $f \in L^p$, $g \in L^q$, $p, q \ge 1$. Then $f * g \in L^r$, where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Moreover,

$$\|f * g\|_r \le \|f\|_p \cdot \|g\|_q.$$

Proof. Fix $f \in L^p$, $\|f * g\|_p \le \|f\|_p \cdot \|g\|_1$ by Minkowski's inequality (Lemma 6). $\|f * g\|_\infty \le \|f\|_p \cdot \|g\|_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, by Hölder's inequality. By Riesz-Thorin Interpolation Theorem (Theorem 1), $\|f * g\|_r \le \|f\|_p \cdot \|g\|_q$ for $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{\infty}$, $\frac{1}{q} = \frac{1-\theta}{1} + \frac{\theta}{p'}$, where $\theta \in (0,1)$, which implies $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. □

Recall the Schwartz functions space $\mathcal{S}(\mathbb{R}^n)$. Now define

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|,$$

and $f_k \to f \in \mathcal{S}(\mathbb{R}^n)$ iff $\lim_{k\to\infty} \|f_k - f\|_{\alpha,\beta} = 0$ for all $\alpha, \beta \in \mathbb{N}^n$.

Definition 11. Let $L : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ be linear. L is called a **continuous linear functional** if $\lim_{k\to\infty} L(f_k) = 0$ whenever $f_k \to 0$ in $\mathcal{S}(\mathbb{R}^n)$.

Definition 12. Let $\mathcal{S}'(\mathbb{R}^n) = \{$all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)\}$, called **the space of tempered distributions**.

Definition 13. Let $T \in \mathcal{S}'(\mathbb{R}^n)$. Define $\hat{T} \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\hat{T}(\varphi) = T(\hat{\varphi}),$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Example 4. f is called **tempered function** if

$$\int \frac{|f(x)|}{(1+|x|)^N} \leq \infty$$

for some $N \geq 1$. Let $J = \{f : f \text{ is tempered}\}$. Then $\forall\, g \in J$, if $\exists\, f$ s.t.

$$\int_{\mathbb{R}^n} g\hat{\varphi} = \int_{\mathbb{R}^n} f\varphi$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then define $\hat{g} \coloneqq f$.

Example 5. Let μ be a finite Borel measure. Define

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\xi} d\mu(x).$$

We define the **Dirac measure** at 0, denoted by δ, that

$$\delta(E) = \begin{cases} 1 & , 0 \in E \\ 0 & , 0 \notin E, \end{cases}$$

where E is Borel set. Consider its Fourier transform as follow:

$$\begin{aligned} \hat{\delta}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i x\xi} d\delta \\ &= \int_{\mathbb{R}^n \setminus \{0\}} e^{-2\pi i x\xi} d\delta + \int_{\{0\}} e^{-2\pi i x\xi} d\delta \\ &= 0 + 1 \cdot \delta(\{0\}) = 1. \end{aligned}$$

Hence a fact is that,

$$\hat{\delta} = 1,$$

or

$$\check{1} = \delta.$$

7 Singular Integrals

Definition 14 (Standard Calderón-Zygmund Kernel)**.** Let $K \in \mathcal{S}^1(\mathbb{R}^n \times \mathbb{R}^n)$. Call K is a **standard Calderón-Zygmund kernel** if K is a $\mathbb{C}$-valued function in $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$, and K satisfies

1. $|K(x, y)| \leq \frac{C}{|x-y|^n}$, for $x \neq y$.
2. $|K(x, y) - K(x, y')| \leq \frac{C|y-y'|^\epsilon}{|x-y|^{n+\epsilon}}$, for $\epsilon > 0$ and $|x - y| > 2|y - y'|$.
3. $|K(x, y) - K(x', y)| \leq \frac{C|x-x'|^\epsilon}{|x-y|^{n+\epsilon}}$, for $\epsilon > 0$ and $|x - y| > 2|x - x'|$.

The above conditions 1∼3 are called **Calderón-Zygmund conditions**, or **C-Z conditions**d

Definition 15. Let $T : \mathcal{S} \to \mathcal{S}'$ be continuous in $\mathcal{S}$ and linear. T is called a **Calderón-Zygmund singular integral operator**, or **C-Z singular integral operator** if T is associated with a standard C-Z kernel, that is,

$$\langle T\varphi, \psi \rangle = \langle K, \varphi \otimes \psi \rangle ,$$

where $\varphi \otimes \psi(x, y) = \varphi(x)\psi(y)$. Indeed, we have

$$\begin{aligned} \langle K, \varphi \otimes \psi \rangle &= \int_{\mathbb{R}^n \times \mathbb{R}^n} K(x, y)\varphi(x)\psi(y)dxdy \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x, y)\varphi(y)dy \right) \psi(x)dx \qquad\qquad (\operatorname{supp}\psi \cap \operatorname{supp}\varphi \neq \varnothing) \\ &= \langle T\varphi, \psi \rangle , \end{aligned}$$

and $T\varphi = \int_{\mathbb{R}^n} K(x, y)\varphi(y)dy$.

Remark 4. One would wonder whether such a singular integral operator T can be extended to a bounded operator on L^2? The answer is yes. One can refer to $T1$ Theorem proved by David-Journé (1984) [2]. Readers can also see in Lecture 16.

Theorem 18. Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a C-Z singular integral operator. If T can be extended to a bounded operator on $L^2(\mathbb{R}^n)$, then T is a weak (1,1) operator, that is,

$$m(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda}\|f\|_1,$$

for any $\lambda > 0$.

Example 6 (Hilbert transform)**.** For $f \in C_c^1(\mathbb{R})$, we define the Hilbert transform $Hf(x)$ to be

$$\begin{aligned} Hf(x) &= \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}dy = \frac{1}{\pi} \text{ p.v. } \int_{\mathbb{R}} \frac{f(y)}{x-y}dy \\ &= \frac{1}{\pi} \text{ p.v. } \int \frac{f(x-y)}{y}dy = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{f(x-y)}{y}dy. \end{aligned}$$

Now write

$$Hf(x) = \lim_{\epsilon \to 0} \int_{\epsilon<|x-y|<1} \frac{f(y)}{x-y}dy + \int_{|x-y|>1} \frac{f(y)}{x-y}dy =: \mathrm{I} + \mathrm{II}.$$

Note that

$$\mathrm{II} \leq \left(\int_{|x-y|>1} \frac{dy}{|x-y|^2} \right)^{1/2} \cdot \left(\int |f(y)|^2 \right) \leq C,$$

also

$$\begin{aligned} |\mathrm{I}| &= \left| \lim_{\epsilon \to 0} \int_{\epsilon<|x-y|<1} \frac{f(y)-f(x)}{x-y} \right| \\ &\leq \lim_{\epsilon \to 0} \int \frac{|f(y)-f(x)| \cdot \chi(\{\epsilon < |x-y| < 1\})}{|x-y|} dy \leq \|f\|_\infty < \infty, \end{aligned}$$

since $|f(y)-f(x)| \leq \|f\|_\infty \cdot |x-y|$.

Example 7 (Riesz transform)**.** Riesz transform is given by

$$R_j f(x) = C_n \cdot \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy = C_n \cdot \text{ p.v. } \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy.$$

Example 8 (Cauchy integral along Lipschitz curve)**.** Let γ be a Lipschitz curve in $\mathbb{C}$, that is, γ is the graph $(x, A(x)) = x + iA(x)$, where $A(x)$ is Lipschitz function. Define the Cauchy integral

$$Cf(z) = \text{ p.v. } \int_{z \in \gamma} \frac{f(\zeta)}{z-\zeta} dz.$$

Substitute z by $x + iA(x)$ and ζ by $y + iA(y)$, we have

$$C\tilde{f}(x) = \text{ p.v. } \int_{\mathbb{R}} \frac{\tilde{f}(y)}{x - y + i(A(x) - A(y))} dy,$$

where $\tilde{f}(y) = (1 + iA'(y)) f(y + iA(y))$.

- One can check the examples above are examples of C-Z singular integral operators.

Lemma 11 (**Calderón-Zygmund Decomposition**)**.** Let $f \in L^1(\mathbb{R}^n)$. Fix $\lambda > 0$. Then there exists non-overlapping family of cubes $\{Q_j\}_{j=1}^\infty$ s.t.

1. $\lambda < |Q_j|^{-1} \int_{Q_j} |f| \leq 2^n \lambda$.

2. $|f| \leq \lambda$ a.e. on $\mathbb{R}^n \backslash (\cup_{j \geq 1} Q_j)$.

3. $\sum_j |Q_j| \leq \lambda^{-1} \|f\|_1$.

Proof. Let $f \in L^1$, $|Q|^{-1} \int_Q |f| \leq |Q|^{-1} \|f\|_1 \to 0$, as $|Q| \to \infty$. Divide $\mathbb{R}^n$ into a union of disjoint cubes Q's with the same size. Let $|Q|$ be big enough such that

$$\frac{1}{|Q|} \int_Q |f| \leq \lambda.$$

We subdivide each Q into 2^n many subcubes Q' with the side length $\frac{1}{2}\ell(Q)$. For each $Q' \subset Q$, Q' satisfies one of the following conditions:

1. $|Q'|^{-1} \int_{Q'} |f| > \lambda$ (called the **good cubes**).

2. $|Q'|^{-1} \int_{Q'} |f| \leq \lambda$ (called the **bad cubes**).

If Q' is good, then

$$\lambda < |Q'|^{-1} \int_{Q'} |f| < \frac{2^n}{|Q|} \int_Q |f| \leq 2^n \lambda.$$

Stop dividing and put Q' as one of the cubes in the collection $\{Q_j\}$. If Q' is bad, then subdivide it into 2^n many subcubes with side length $\frac{1}{2}\ell(Q')$. Repeat the procedure for each subcubes, we obtain $\{Q_j\}$ such that

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} |f| \leq 2^n \lambda,$$

with $Q_j{}'$'s are overlapping. If $x \notin \cup_{j=1}^{\infty} Q_j$, then there exists $\{Q_k\}$ s.t. $|Q_k| \to 0$, $x \in Q_k$ for some k and $|Q_k|^{-1} \int_{Q_k} |f| \leq \lambda$. By Lebesgue Differential Theorem (Theorem 5),

$$\lim_{k\to\infty} \frac{1}{|Q_k|} \int_{Q_k} |f| = |f(x)| \quad a.e.$$

So

$$|f(x)| = \lim_{k\to\infty} \frac{1}{|Q_k|} \int_{Q_k} |f| \leq \lambda.$$

Note $|Q_j|^{-1} \int_{Q_j} |f| > \lambda$ is equivalent to $|Q_j| < \lambda^{-1} \int_{Q_j} |f|$, which implies

$$\sum_j |Q_j| < \frac{1}{\lambda} \sum_j \int_{Q_j} |f| = \frac{1}{\lambda} \int_{\cup_j Q_j} |f| \leq \frac{1}{\lambda} \|f\|_1.$$

□

Lemma 12. Fix $\lambda > 0$. Let $f \in L^1(\mathbb{R}^n)$. Then $f = g + b$ s.t.

1. $g \in L^2(\mathbb{R}^n)$ and $\|g\|_2^2 \leq C\lambda \|f\|_1$.

2. $b(x) = \sum_j b_j(x)$, where b_j is supported in some cube Q_j and Q_j are disjoint.

3. $\sum_j |Q_j| \leq \lambda^{-1} \|f\|_1$ and $\int_{Q_j} b_j = 0$, and also $\sum_j \|b_j\|_1 \leq 2\|f\|_1$.

Proof. For $f \in L^1$ and $\lambda > 0$, let $\{Q_j\}$ be the collection of cubes in Lemma 11. Define

$$b(x) = \sum_j \left(f - \frac{1}{Q_j} \int_{Q_j} f \right) \chi(Q_j) =: \sum_j b_j(x),$$

$$g(x) = f(x) - b(x) = f(x)\chi((\cup_j Q_j)^c) + \sum_j \left(\frac{1}{|Q_j|} \int_{Q_j} f \right) \chi(Q_j).$$

Clearly $f = g + b$, and

$$\begin{aligned} \|g\|_\infty &\leq \|f(x)\chi((\cup_j Q_j)^c)\|_\infty + \sum_j \left(\frac{1}{|Q_j|} \int_{Q_j} |f| \right) \chi(Q_j) \\ &\leq \lambda + 2^n \lambda = C_n \lambda. \end{aligned}$$

Additionally, we have

$$\|g\|_1 \le \|f\|_1 + \|b\|_1 \le \|f\|_1 + \sum_j \|b_j\|_1$$
$$\le \|f\|_1 + 2\sum_j \int_{Q_j} |f| \le 3\|f\|_1.$$

Also $\|g\|_2 \le (3\|f\|_1)^\theta (C_n\lambda)^{1-\theta}$, and $\theta = 1/2$, we have

$$\|g\|_2 \le C' \cdot \lambda^{1/2} \cdot \|f\|_1^{1/2},$$

or $\|g\|_2^2 \le C\lambda\|f\|_1$. $\square$

Proof of Theorem 18. Suppose that C-Z operator T can be extended to a bounded operator L^2. Show that for any $\lambda > 0$, $f \in L^1$,

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \le \frac{C}{\lambda} \cdot \|f\|_1.$$

Write $f = g + b$ as in Lemma 12, then

$$\text{LHS} \le |\{x \in \mathbb{R}^n : |Tg(x)| > \frac{\lambda}{2}\}| + |\{x \in \mathbb{R}^n : |Tb(x)| > \frac{\lambda}{2}\}| =: \text{I} + \text{II}.$$

Note

$$\text{I} \le \frac{C}{\lambda^2}\|Tg\|_2^2 \le \frac{C}{\lambda^2}\|g\|_2^2 \le \frac{C}{\lambda^2} \cdot \lambda \cdot \|f\|_1 = \frac{C\|f\|_1}{\lambda}.$$

It remains to show $\text{II} \le \frac{C}{\lambda}\|f\|_1$. Let $E = \cup_j 5Q_j$. Then

$$\text{II} = |\{x \in E : |Tb(x)| > \frac{\lambda}{2}\}| + |\{x \notin E : |Tb(x)| > \frac{\lambda}{2}\}| \le |E| + |\{x \notin E : |Tb(x)| > \frac{\lambda}{2}\}|.$$

Notice

$$|E| \le \sum_j |5Q_j| \le 5^n \sum_j |Q_j| \le 5^n \cdot \frac{\|f\|_1}{\lambda},$$

so it suffices to prove $|\{x \notin E : |Tb(x)| > \frac{\lambda}{2}\}| \le \frac{C}{\lambda}\|f\|_1$. By Chebyshev's inequality,

$$|\{x \in E^c : |Tb(x)| > \frac{\lambda}{2}\}| \le \frac{2}{\lambda}\int_{E^c} |Tb(x)|dx$$
$$\le \frac{2}{\lambda}\sum_j \int_{E^c} |Tb_j(x)|dx$$
$$\le \frac{2}{\lambda}\sum_j \int_{E^c} \left|\int K(x,y)b_j(y)dy\right| dx.$$

Since $\int b_j(y)dy = 0$ (by Lemma 12 conditions 2 and 3), $\int K(x,y_j)b_j(y) = 0$. Therefore,

$$|\{x \in E^c : |Tb(x)| > \frac{\lambda}{2}\} = \frac{2}{\lambda}\sum_j \int_{E^c} \left|\int (K(x,y) - K(x,y_j))b_j(y)dy\right| dx$$
$$\le \frac{2}{\lambda}\sum_j \int_{E^c}\int_{Q_j} |K(x,y) - K(x,y_j)| \cdot |b_j(y)|dydx$$
$$\le \frac{C}{\lambda}\sum_j \int_{E^c}\int_{Q_j} \frac{|y-y_j|^\epsilon}{|x-y|^{n+\epsilon}} \cdot |b_j(y)|dydx.$$

Note that $|x-y|>2|y-y_j|$, where $x\in E^c$, $y,y_j\in Q_j$, thus $x\in(5Q_j)^c$. Hence by Fubini's Theorem,

$$\begin{aligned}\cdots &= \frac{C}{\lambda}\sum_j\int_{Q_j}|b_j(y)|\cdot\left(\int_{E^c}\frac{|y-y_j|^\epsilon}{|x-y|^{n+\epsilon}}dx\right)dy\\ &\le \frac{C}{\lambda}\sum_j\int_{Q_j}|b_j(y)|\cdot\left(\int_{\{x:|x-y|\ge 2|y-y_j|\}}\frac{|y-y_j|^\epsilon}{|x-y|^{n+\epsilon}}dx\right)dy,\end{aligned}$$

where

$$\int_{\{x:|x-y|\ge 2|y-y_j|\}}\frac{|y-y_j|^\epsilon}{|x-y|^{n+\epsilon}}dx=\int_{\{x:|x|\ge 2|y-y_j|\}}\frac{|y-y_j|^\epsilon}{|x|^{n+\epsilon}}dx=C_n.$$

Hence

$$\cdots=\frac{C_n}{\lambda}\sum_j\int_{Q_j}|b_j(y)|dy=\frac{C_n}{\lambda}\sum_j\|b_j\|_1\le\frac{C_n}{\lambda}\|f\|_1.$$

□

Exercise 3. We can obtain the same argument if condition 2 of C-Z kernel is replaced by the **Hörmander condition**:

$$\int_{\{x:|x-y|\ge 2|y-y_j|\}}|K(x,y)-K(x,y')|dx\le C.$$

8 Hilbert Transform

Definition 16. The **Hilbert transform** for $f \in L^1(\mathbb{R}$ is defined by

$$Hf(x) = \text{ p.v. } \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x-y)}{y} dy = \text{ p.v. } \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y} dy = \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|y|>\epsilon} \frac{f(x-y)}{y} dy.$$

Let $x, t \in \mathbb{R}$ and $t > 0$.

Definition 17. The **Poisson Kernel** is defined by

$$P_t(x) = \frac{1}{\pi}\cdot\frac{t}{t^2+x^2}.$$

Then one can check $\{P_t\}_{t>0}$ is an approximation to identity, and $u(x,t) = P_t * f(x)$ solves (for $\mathbb{R}$-valued $f \in L^1(\mathbb{R})$):

$$\begin{cases} \Delta u = 0, & (x,t) \in \mathbb{R}^2_+ = \{(x,t) \in \mathbb{R}^2 : t > 0\} \\ u(x,0) = \lim_{t\to 0^+} u(x,t) = f(x) \end{cases} \tag{$*$}$$

Let $F(z) = 2\int_0^\infty \hat{f}(\xi)e^{2\pi i\xi z}d\xi$, where $z \in \mathbb{C}$ and $\Im z > 0$. It is easy to see $F(z)$ is analytic in $\mathbb{R}^2_+$ or $\mathbb{H}$. Write $F(z) = u_1(z) + iv(z)$, with $\Re F = u_1$ and $\Im F = v$.

Claim 6. $u_1 = u$, which is defined in equation $(*)$.

In fact, we have

$$\begin{aligned} u_1(z) &= \int_0^\infty \hat{f}(\xi)e^{2\pi i\xi z}d\xi + \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi i\xi\bar{z}}d\xi, \\ iv(z) &= \int_0^\infty \hat{f}(\xi)e^{2\pi i\xi z}d\xi - \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi i\xi\bar{z}}d\xi. \end{aligned}$$

Proof of Claim 6. Easy to see $\Delta u_1 = 0$ (by C-R equation). Note

$$\begin{aligned} u_1(x+i0) = u_1(x) &= \int_0^\infty \hat{f}(\xi)e^{2\pi ix\xi}d\xi + \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi ix\xi}d\xi \\ &= \int_{-\infty}^\infty \hat{f}(\xi)e^{2\pi ix\xi}d\xi = f(x). \end{aligned}$$

Thus u_1 satisfies $(*)$, and by uniqueness we obtain $u = u_1$. $\square$

Definition 18. In the definition above, v is called the **harmonic conjugate**.

Note that

$$\begin{aligned} v(z) &= (-i)\left(\int_0^\infty \hat{f}(\xi)e^{2\pi iz\xi}d\xi - \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi iz\xi}d\xi\right) \\ &= \int_{-\infty}^\infty (-i\,\mathrm{sgn}(\xi))\, e^{2\pi(\Im z)|\xi|}e^{2\pi i(\Re z)\xi}\hat{f}(\xi)d\xi. \end{aligned}$$

Write $z = x + it = (x, t)$, then

$$\begin{aligned} v(z) = v(x,t) &= \int_{-\infty}^{\infty} (-i\,\mathrm{sgn}(\xi))\, e^{-2\pi t|\xi|} e^{2\pi i x\xi} \hat{f}(\xi) d\xi \\ &= \int_{-\infty}^{\infty} \hat{G}(\xi) e^{2\pi i x \xi} d\xi, \end{aligned}$$

where $\hat{G}(\xi) = -i\,\mathrm{sgn}(\xi) e^{-2\pi t|\xi|} \hat{f}(\xi)$.

Claim 7. $F(z) = (P_t + iQ_t) * f(z)$ for $\Re z > 0$, where $Q_t(x) = \frac{1}{\pi} \cdot \frac{x}{t^2+x^2}$.

Proof. Write $z = x + it$. Then

$$\begin{aligned} F(z) &= 2\int_0^\infty \hat{f}(\xi) e^{2\pi i \xi z} d\xi \\ &= 2\int_0^\infty \left(\int f(y) e^{-2\pi i \xi y} dy \right) e^{2\pi i \xi z} d\xi \\ &= 2 \int f(y) \left(\int_0^\infty e^{2\pi i \xi (x+it-y)} d\xi \right) dy && \text{(Fubini)} \\ &= \int f(y) \cdot \frac{i}{\pi(x-y+it)} dy. \end{aligned}$$

On the other hand, $P_t + iQ_t = \frac{i}{\pi z} = \frac{i}{\pi(x+it)}$, which implies

$$(P_t + iQ_t) * f(x) = \int f(y) \cdot \frac{i}{\pi(x-y+it)} dy = F(z).$$

□

Fix t, then

$$\widehat{Q_t * f}(\xi) = -i\,\mathrm{sgn}(\xi) e^{-2\pi t|\xi|} \hat{f}(\xi) = \hat{G}(\xi).$$

It is evident that

$$v(x,t) = Q_t * f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{t^2+y^2} f(x-y) dy.$$

Theorem 19. Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Then

$$\lim_{t \to 0^+} Q_t * f(x) = Hf(x) \ \ a.e.$$

Remark 5. $Hf(x) = \lim_{t\to 0^+} F(x+it)$.

Proof. Let $\psi_t(x) = \frac{1}{\pi} \cdot \frac{1}{x} \cdot \chi(\{|x| > t\})$. Then $Hf(x) = \lim_{t\to 0^+} \psi_t * f(x)$. To prove the theorem, we need to show

$$\lim_{t\to 0^+} (Q_t - \psi_t) * f(x) = 0.$$

Note $(Q_t - \psi_t) * f(x) = \int \frac{1}{\pi} \left(\frac{y}{t^2+y^2} - \frac{1}{y}\chi(\{|y| > t\}) \right) f(x-y) dy$. Let $\Phi(y) = \frac{1}{\pi} \left(\frac{y}{1+y^2} - \frac{1}{y}\chi(\{|y| > 1\}) \right)$, then

$$(Q_t - \psi_t) * f(x) = \Phi_t * f(x),$$

where $\Phi_t(y) = t^{-1}\Phi(t^{-1}y)$. Note that

$$\Phi(y) = \begin{cases} \frac{1}{\pi} \cdot \frac{-1}{y(1+y^2)} & , |y| > 1 \\ \frac{1}{\pi} \cdot \frac{y}{1+y^2} & , |y| \leq 1, \end{cases}$$

also

$$\int \Phi(y)dy = \frac{1}{\pi}\int_{|t|>1} \frac{-1}{y(1+y^2)}dy + \frac{1}{\pi}\int_{-1}^{1} \frac{y}{1+y^2}dy \neq 0.$$

Recall that if $\sup_{|y|\geq|x|} |\Phi(y)| \in L^1(\mathbb{R})$, then $\sup_{t>0} |\Phi_t * f(x)| \leq C \cdot Mf(x)$, and

$$\sup_{|y|\geq|x|} |\Phi(y)| \leq \begin{cases} \frac{1}{\pi} \cdot \frac{1}{|x|(1+x^2)} & , |x| > 1 \\ \frac{1}{2\pi} & , |x| \leq 1, \end{cases}$$

which is in L^1. Hence $\lim_{t\to 0^+} \Phi_t * f(x) = 0$, for $f \in \mathcal{S}(\mathbb{R}^n)$. Since $\sup_{t>0} |\Phi_t * f(x)|$ is weak (p,p),

$$\lim_{t\to 0^+} \Phi_t * f(x) = 0$$

for all $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. □

Theorem 20. $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$, where $f \in L^2$.

Proof. Let $f \in \mathcal{S}(\mathbb{R})$, then $Hf(x) = \lim_{t\to 0^+} Q_t * f(x)$ by Theorem 19.

$$\begin{aligned}
\widehat{Hf}(\xi) &= \int Hf(x)e^{-2\pi i x\xi}dx \\
&= \int \lim_{t\to 0^+} Q_t * f(x)e^{-2\pi i x\xi}dx \\
&= \lim_{t\to 0^+} \int Q_t * f(x)e^{-2\pi i x\xi}dx \qquad \text{(DCT)} \\
&= \lim_{t\to 0^+} \widehat{Q_t * f}(\xi) \\
&= \lim_{t\to 0^+} \left(-i\,\mathrm{sgn}(\xi)e^{-2\pi i t|\xi|}\right)\hat{f}(\xi) \\
&= -\,i\,\mathrm{sgn}(\xi)\hat{f}(\xi).
\end{aligned}$$

Let $f \in L^2$, there exists $f_k \xrightarrow{L^2} f$, $k \to \infty$, where $f_k \in \mathcal{S}(\mathbb{R})$. So

$$\widehat{Hf}(\xi) \overset{L^2}{=} \lim_{k\to\infty} \widehat{Hf_k}(\xi) \overset{L^2}{=} \lim_{k\to\infty} -i\,\mathrm{sgn}(\xi)\hat{f}_k(\xi) \overset{L^2}{=} -i\,\mathrm{sgn}(\xi)\hat{f}(\xi).$$

Therefore, $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$ a.e. □

Corollary 3. $\|Hf\|_2 = \|f\|_2$.

Corollary 4. H is weak (p,p) (by Theorem 18).

Corollary 5. H is of (p,p), $1 \leq p < \infty$, i.e.

$$\|Hf\|_p \leq C_p\|f\|_p.$$

Theorem 21. Let $H^*f(x) = \sup_{\epsilon>0} \left|\frac{1}{\pi}\int_{|y|>\epsilon} f(x-y)\cdot\frac{1}{y}dy\right|$. Then $\|H^*f\|_p \le C_p \cdot \|f\|_p$ for any $f \in L^p$ and any $1 < p < \infty$.

Proof. We will prove

$$H^*f(x) \le M(Hf)(x) + C \cdot Mf(x).$$

Let $\psi_\epsilon(x) = \frac{1}{\pi}\cdot\frac{1}{x}\cdot\chi(\{|x|>\epsilon\})$, then

$$\psi_\epsilon * f(x) = \frac{1}{\pi}\int_{|y|>\epsilon}\frac{f(x-y)}{y}dy.$$

Let $\phi \in \mathcal{S}(\mathbb{R})$ be non-negative, even, decreasing on $(0,\infty)$, supported on $[-1/2,1/2]$ and $\int\psi = 1$. Now $\psi_\epsilon * f(x) = \phi_\epsilon*(Hf)(x) + [\psi_\epsilon*f(x) - \phi_\epsilon*(Hf)(x)]$. By Theorem 7, $\sup_{\epsilon>0}|\phi_\epsilon*(Hf)(x)| \le C\cdot M(Hf)(x)$.

Claim 8. $|\psi_\epsilon * f(x) - \phi_\epsilon * (Hf)(x)| \le C \cdot Mf(x)$.

Proof of Claim 8. Indeed,

$$\text{LHS} = \left|\int\left(\psi_\epsilon(y) - \frac{1}{\pi}\text{ p.v. }\int\frac{\phi_\epsilon(z)}{y-z}dz\right)\cdot f(x-y)dy\right|.$$

Note

$$\left|\psi_\epsilon(y) - \frac{1}{\pi}\text{ p.v. }\int\frac{\phi_\epsilon(z)}{y-z}dz\right| \le \frac{C\epsilon}{\epsilon^2+y^2}. \tag{$**$}$$

Exercise 4. Prove $(**)$.

From the preceding exercise, we have

$$\text{LHS} \le C\cdot\int\frac{\epsilon}{\epsilon^2+y^2}|f(x-y)|dy \le C\cdot Mf(x).$$

□

Hence

$$\begin{aligned}\|H^*f\|_p &\le \|M(Hf)\|_p + C\|Mf\|_p \\ &\le C_p\cdot\|Hf\|_p + C'_p\cdot\|f\|_p \le \tilde{C}_p\cdot\|f\|_p.\end{aligned}$$

□

9 Riesz Transform

Definition 19. Let $1 \le j \le n$. We define the **Riesz transform** of $f \in L^1(\mathbb{R}^n)$ to be

$$\begin{aligned} R_j f(x) &= C_n \cdot \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy \\ &:= C_n \int K_j(x,y) f(y) dy, \end{aligned}$$

where $K_j(x,y) = $ p.v. $\frac{x_j - y_j}{|x-y|^{n+1}}$.

Observation 4. $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi) \cdot \hat{f}(\xi)$ (Theorem 20), which is equivalent to $\widehat{\text{p.v.}\,\frac{1}{t}}(\xi) = -i\,\mathrm{sgn}(\xi)$.

Let $Tf(x) = $ p.v. $\int \frac{\Omega(x-y)}{|x-y|^n} f(y) dy = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy$.

Note 2. Here we urge Ω satisfies:

1. $\Omega(\lambda x) = \Omega(x)$ for any $\lambda > 0, x \in \mathbb{R}^n$ and $n \ge 2$.
2. $\Omega \in L^1(S^{n-1})$.
3. $\int_{S^{n-1}} \Omega(y') d\sigma(y') = 0$.

For Riesz transform, $\Omega(x) = \frac{x_j}{|x|}$, one can easily check it satisfies all the conditions. (Exercise)

Let $K(x) = $ p.v. $\frac{\Omega(x)}{|x|^n} = \lim_{\epsilon \to 0} \frac{\Omega(x)}{|x|^n} \cdot \chi(\{|x| > \epsilon\})$. Then $Tf(x) = K * f(x)$.

Theorem 22. If Ω satisfies 1∼3 above, then

$$\hat{K}(\xi) = \int_{S^{n-1}} \Omega(y') \left[\log \frac{1}{|y' \cdot \xi'|} - i\frac{\pi}{2} \mathrm{sgn}(y' \cdot \xi') \right] d\sigma(y'),$$

where $\xi' = \frac{\xi}{|\xi|} \in S^{n-1}$.

Proof. Let $K_\epsilon(x) = \frac{\Omega(x)}{|x|^n} \cdot \chi(\{\epsilon < |x| < \frac{1}{\epsilon}\}) \in L^1$, since $\Omega \in L^1(S^{n-1})$ by condition 2. Hence

$$\hat{K_\epsilon}(\xi) = \int K_\epsilon(x) \cdot e^{-2\pi i \xi x} dx,$$

and $\hat{K}(\xi) = \lim_{\epsilon \to 0} \hat{K_\epsilon}(\xi)$ in the sense of distribution, i.e. $\left\langle \hat{K}, \varphi \right\rangle = \lim_{\epsilon \to 0} \left\langle \hat{K_\epsilon}, \varphi \right\rangle$, where $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Now

$$\begin{aligned} \hat{K_\epsilon}(\xi) &= \int_{\epsilon < |x| < \frac{1}{\epsilon}} \frac{\Omega(x')}{|x|^n} e^{-2\pi i \xi x} dx \\ &= \int_{S^{n-1}} \int_\epsilon^{\frac{1}{\epsilon}} \frac{\Omega(x')}{r^n} e^{-2\pi i r |\xi| (x' \cdot \xi')} r^{n-1} dr d\sigma(x') \qquad \text{(Let } x = rx' \text{ and } r = |x|) \\ &= \int_{S^{n-1}} \Omega(x') \left(\int_\epsilon^{\frac{1}{\epsilon}} e^{-2\pi i r |\xi| (x' \cdot \xi')} \frac{dr}{r} \right) d\sigma(x') \\ &= \int_{S^{n-1}} \Omega(x') \left(\int_1^{\frac{1}{\epsilon}} e^{-2\pi i r |\xi| (x' \cdot \xi')} \frac{dr}{r} \right) d\sigma(x') + \int_{S^{n-1}} \Omega(x') \left(\int_\epsilon^1 (e^{-2\pi i r |\xi| (x' \cdot \xi')} - 1) \frac{dr}{r} \right) d\sigma(x'), \end{aligned}$$

and the last sentence uses the fact that (by condition 3)

$$\int_{S^{n-1}} \Omega(x') \left(\int_\epsilon^1 \frac{dr}{r} \right) d\sigma(x') = 0.$$

Hence we may continue

$$\begin{aligned}
\cdots = &\int_{S^{n-1}} \Omega(x') \int_\epsilon^1 (\cos(2\pi ir|\xi|(x' \cdot \xi')) - 1)\frac{dr}{r} d\sigma(x') + \int_{S^{n-1}} \Omega(x') \int_1^{\frac{1}{\epsilon}} \cos(-2\pi ir|\xi|(x' \cdot \xi'))\frac{dr}{r} d\sigma(x') \\
&- i \int_{S^{n-1}} \Omega(x') \left(\int_\epsilon^{\frac{1}{\epsilon}} \sin(-2\pi ir|\xi|(x' \cdot \xi'))\frac{dr}{r} \right) d\sigma(x') \\
=: &\Re_\epsilon - i\Im_\epsilon.
\end{aligned}$$

Note

$$\begin{aligned}
\lim_{\epsilon\to 0} \Im_\epsilon &= \int_{S^{n-1}} \Omega(x') \left(\lim_{\epsilon\to 0} \int_\epsilon^{\frac{1}{\epsilon}} \sin(-2\pi ir|\xi|(x' \cdot \xi'))\frac{dr}{r} \right) d\sigma(x') \\
&= \int_{S^{n-1}} \Omega(x') \left(\lim_{\epsilon\to 0} \int_{-2\pi ir|\xi|\cdot|x'\cdot\xi'|\epsilon}^{-2\pi ir|\xi|\cdot|x'\cdot\xi'|\frac{1}{\epsilon}} \sin(s \cdot \operatorname{sgn}(x' \cdot \xi'))\frac{ds}{s} \right) d\sigma(x') \quad (\text{Let } s = -2\pi ir|\xi| \cdot |x' \cdot \xi'|) \\
&= \int_{S^{n-1}} \Omega(x') \operatorname{sgn}(x' \cdot \xi') \left(\int_0^\infty \frac{\sin s ds}{s} \right) d\sigma(x') \\
&= \frac{\pi}{2} \int_{S^{n-1}} \Omega(x') \operatorname{sgn}(x' \cdot \xi') d\sigma(x').
\end{aligned}$$

On the other hand,

$$\lim_{\epsilon\to 0} \Re_\epsilon = \int_{S^{n-1}} \Omega(x') \int_0^{2\pi|\xi|\cdot|x'\cdot\xi'|} \frac{\cos s - 1}{s} ds d\sigma(x') + \int_{S^{n-1}} \Omega(x') \int_{2\pi|\xi|\cdot|x'\cdot\xi'|}^\infty \frac{\cos s}{s} ds d\sigma(x'),$$

and this implies

$$\begin{aligned}
\lim_{\epsilon\to 0} \Re_\epsilon &= \int_{S^{n-1}} \Omega(x') \left(\int_{2\pi|\xi|\cdot|x'\cdot\xi'|}^{2\pi|\xi|} \frac{ds}{s} \right) d\sigma(x') \\
&= \int_{S^{n-1}} \Omega(x') \log \frac{1}{|x' \cdot \xi'|} d\sigma(x').
\end{aligned}$$

Hence we obtain the result. □

If Ω is odd, then

$$\hat{K}(\xi) = -i\frac{\pi}{2} \int_{S^{n-1}} \Omega(x') \operatorname{sgn}(x' \cdot \xi') d\sigma(x').$$

In this case, $\|\hat{K}(\xi)\| \leq \frac{\pi}{2}\|\Omega\|_{L^1(S^{n-1})}$. If Ω is even, then

$$\hat{K}(\xi) = \int_{S^{n-1}} \Omega(x') \log \frac{1}{|x' \cdot \xi'|} d\sigma(x').$$

But note that this might **NOT** be bounded!

Definition 20. Define

$$L \log L(S^{n-1}) := \left\{ g : \int_{S^{n-1}} |g(x')| \log^+ |g(x')| d\sigma(x') < \infty \right\},$$

where $\log^+ t = \max\{0, \log t\}$.

Let $\Omega_e(x) = \frac{1}{2}(\Omega(x) + \Omega(-x))$ and $\Omega_0(x) = \frac{1}{2}(\Omega(x) - \Omega(-x))$. Then $\Omega = \Omega_e(x) + \Omega_0(x)$, which implies

$$\widehat{\text{p.v.}\frac{\Omega(x')}{|x|^n}} = \widehat{\text{p.v.}\frac{\Omega_e(x')}{|x|^n}} + \widehat{\text{p.v.}\frac{\Omega_0(x')}{|x|^n}}.$$

Also,

$$\begin{aligned}\widehat{\text{p.v.}\frac{\Omega_e(x')}{|x|^n}} &= -i\frac{\pi}{2}\int_{S^{n-1}} \Omega_0(x') \cdot \operatorname{sgn}(x' \cdot \xi') d\sigma(x') =: \hat{K}_0; \\ \widehat{\text{p.v.}\frac{\Omega_0(x')}{|x|^n}} &= \int_{S^{n-1}} \Omega_e(x') \log \frac{1}{|x' \cdot \xi'|} d\sigma(x') =: \hat{K}_e.\end{aligned}$$

Theorem 23. Suppose Ω satisfies 1$\sim$3, and $\Omega_0 \in L^1(S^{n-1})$, $\Omega_e \in L\log L(S^{n-1})$. Then

$$\left\| \widehat{\text{p.v.}\frac{\Omega(x')}{|x|^n}} \right\|_\infty \leq C\left(\|\Omega_0\|_{L^1(S^{n-1})} + \|\Omega_e\|_{L\log L(S^{n-1})}\right).$$

Proof. $\|\hat{K}_0\|_\infty \leq \frac{\pi}{2}\int_{S^{n-1}} |\Omega_0| \leq \frac{\pi}{2}\|\Omega\|_{L^1(S^{n-1})}$. It suffices to check the other one. Now let

$$\begin{aligned}\hat{K}_e(\xi) = &\int_{\{x' \in S^{n-1} : |\Omega_e(x')| \leq 1\}} \Omega_e(x') \log \frac{1}{|x' \cdot \xi'|} d\sigma(x') =: I_1 \\ &+ \int_{\{x' \in S^{n-1} : |\Omega_e(x')| > 1\}} \Omega_e(x') \log \frac{1}{|x' \cdot \xi'|} d\sigma(x') =: I_2.\end{aligned}$$

Since

$$\begin{aligned}I_1 \leq &\int_{S^{n-1}} \log \frac{1}{|x' \cdot \xi'|} d\sigma(x') \\ = &\int_0^\pi \log \frac{1}{\cos\theta} \cdot m(\{\text{a sphere in } \mathbb{R}^{n-1} \text{ of radius } \sin\theta\}) d\theta \\ = &C_n \int_0^\pi \log \frac{1}{\cos\theta} \cdot (\sin\theta)^{n-2} d\theta \leq \tilde{C}_n.\end{aligned}$$

Exercise 5. Check the last inequality in the preceding deduction.

Also,

$$\begin{aligned}I_2 = &\sum_{k=0}^\infty \int_{\{x' \in S^{n-1} : 2^k < |\Omega_e(x')| \leq 2^{k+1}\}} |\Omega_e(x')| \log \frac{1}{|x' \cdot \xi'|} d\sigma(x') \\ \leq &\sum_{k=0}^\infty 2^{k+1} \int_{\{x' \in S^{n-1} : 2^k < |\Omega_e(x')| \leq 2^{k+1}\}} \log \frac{1}{|x' \cdot \xi'|} d\sigma(x') \\ \leq &\sum_{k=0}^\infty 2^{k+1} \int_{\{x' \in S^{n-1} : 2^k < |\Omega_e(x')| \leq 2^{k+1}, |x' \cdot \xi'| > 2^{-2k}\}} \log \frac{1}{|x' \cdot \xi'|} d\sigma(x') =: I_{21} \\ &+ \sum_{k=0}^\infty 2^{k+1} \int_{\{x' \in S^{n-1} : 2^k < |\Omega_e(x')| \leq 2^{k+1}, |x' \cdot \xi'| \leq 2^{-2k}\}} \log \frac{1}{|x' \cdot \xi'|} d\sigma(x') =: I_{22}\end{aligned}$$

Note

$$
\begin{aligned}
I_{21} &\leq C\sum_{k=0}^{\infty} k\cdot 2^{k+1}\sigma(\{x'\in S^{n-1}:2^k<|\Omega_e(x')|\leq 2^{k+1}\})\\
&\leq C\sum_{k=0}^{\infty}\int_{\{x'\in S^{n-1}:2^k<|\Omega_e(x')|\leq 2^{k+1}\}}|\Omega_e(x')|\log^+|\Omega_e(x')|d\sigma\\
&\leq C\cdot\|\Omega_e\|_{L\log L(S^{n-1})},
\end{aligned}
$$

and

$$
\begin{aligned}
I_{22} &\leq \sum_{k=0}^{\infty}2^{k+1}\int_{\{x'\in S^{n-1}:|x'\cdot\xi'|\leq 2^{-2k}\}}\log\frac{1}{|x'\cdot\xi'|}d\sigma(x')\\
&\leq C\sum_{k=0}^{\infty}2^{k+1}\int_{\frac{\pi}{2}-\epsilon\cdot 2^{-2k}}^{\frac{\pi}{2}+\epsilon\cdot 2^{-2k}}\log\frac{1}{|\frac{\pi}{2}-\theta|}d\theta\\
&= C\sum_{k=0}^{\infty}2^{k+1}\int_0^{\epsilon\cdot 2^{-2k}}\log\frac{1}{\theta}d\theta\\
&\leq C\sum_{k=0}^{\infty}2^{k+1}\int_0^{\epsilon\cdot 2^{-2k}}\frac{1}{\theta^\delta}d\theta\\
&\leq C\sum_{k=0}^{\infty}2^{k+1}\cdot 2^{-2k}\cdot 2^{\delta k}\leq C.
\end{aligned}
$$

Combine all the estimates above, we get the desired result. □

Corollary 6. Let $T_nf(x) = \text{p.v.}\int\frac{\Omega(x-y)}{|x-y|^n}f(y)dy$. Suppose that Ω satisfies 1∼3, and $\Omega_0\in L^1(S^{n-1})$, $\Omega_e\in L\log L(S^{n-1})$. Then T_n is bounded in $L^2(\mathbb{R}^n)$.

Corollary 7. Riesz transform R_j is bounded in L^p for $1<p<\infty$.

Proof. R_j is a C-Z singular integral operator, which is weak (1,1). □

Remark 6.

1. $K(x,y) = \text{p.v.}\frac{\Omega(x-y)}{|x-y|^n}$ is **NOT** a C-Z kernel unless Ω is "smooth" enough.
2. If $\Omega\in L\log L(S^{n-1})$ and Ω satisfies 1∼3, then T_n is weak (1,1).
3. Let $\Omega\in L^1(S^{n-1})$ and Ω be odd (it satisfies 1∼3). We have the following open question: Does T_n define a weak (1,1) operator?

10 Methods of Rotation

Let T_n, Ω defined as in Chapter 9, namely Ω is characterized by Note 2, and

$$T_n f(x) = \text{ p.v. } \int \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$

Then by Corollary 6, T_n is bounded in $L^2(\mathbb{R}^n)$.

Definition 21. For $y \in S^{n-1}$, define

$$H_y f(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|r|>\epsilon} f(x-ry) \frac{dr}{r}$$

to be the **directional Hilbert transform**.

Exercise 6. Prove that $\|H_y f\|_p \leq C_p \|f\|_p$ for any $f \in L^p$, $1 < p < \infty$.

Theorem 24. Suppose Ω is odd, then T_n is bounded in L^p, $1 < p < \infty$, that is, for any $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\|T_n f\|_p \leq C_p \|f\|_p.$$

Proof. Take $f \in \mathcal{S}(\mathbb{R}^n)$. By definition,

$$\begin{aligned}
T_n f(x) &= \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{\Omega(y)}{|y|^n} f(x-y) dy \\
&= \lim_{\epsilon \to 0} \int_{S^{n-1}} \Omega(y') \int_{\epsilon}^{\infty} f(x-ry') \cdot \frac{dr}{r} d\sigma(y') \qquad \text{(Substitute } y = ry'\text{, where } y' \in S^{n-1}\text{)} \\
&= \frac{1}{2} \lim_{\epsilon \to 0} \int_{S^{n-1}} \Omega(y') \int_{|r|>\epsilon} f(x-ry') \cdot \frac{dr}{r} d\sigma(y') \qquad (\Omega \text{ is odd}) \\
&= \frac{1}{2} \lim_{\epsilon \to 0} \int_{S^{n-1}} \Omega(y') \int_{\epsilon<|r|<1} (f(x-ry') - f(x)) \cdot \frac{dr}{r} d\sigma(y') + \frac{1}{2} \int_{S^{n-1}} \Omega(y') \int_{|r|>1} f(x-ry') \cdot \frac{dr}{r} d\sigma(y') \\
&= \frac{1}{2} \int_{S^{n-1}} \Omega(y') \lim_{\epsilon \to 0} \int_{\epsilon<|r|<1} (f(x-ry') - f(x)) \cdot \frac{dr}{r} d\sigma(y') + \frac{1}{2} \int_{S^{n-1}} \Omega(y') \int_{|r|>1} f(x-ry') \cdot \frac{dr}{r} d\sigma(y') \\
&= \frac{1}{2} \int_{S^{n-1}} \Omega(y') \lim_{\epsilon \to 0} \int_{|r|>\epsilon} f(x-ry') \frac{dr}{r} d\sigma(y').
\end{aligned}$$

Thus by definition of directional Hilbert transform, we have

$$T_n f(x) = \frac{\pi}{2} \int_{S^{n-1}} \Omega(y') H_{y'} f(x) d\sigma(y'),$$

and

$$\|T_n f\|_p \leq \frac{\pi}{2} \int_{S^{n-1}} |\Omega(y')| \cdot \|H_{y'} f\|_p d\sigma(y')$$

by Minkowski's inequality (Lemma 6). From Exercise 6,

$$\|T_n f\|_p \leq C_p \int_{S^{n-1}} |\Omega(y')| \cdot \|f\|_p d\sigma(y') \leq C_p \|\Omega\|_{L^1(S^{n-1})} \cdot \|f\|_p.$$

$\square$

Recall the Riesz transform

$$R_j f(x) = \frac{1}{C_n}\,\text{p.v.}\int \frac{y_j}{|y|^{n+1}} f(x-y)dy.$$

Theorem 25. $\widehat{\text{p.v.}\frac{x_j}{|x|^{n+1}}}(\xi) = -iC_n \cdot \frac{\xi_i}{|\xi|}$.

Proof. We start by a claim.

Claim 9. $\widehat{\frac{1}{|x|^{n-1}}}(\xi) = \frac{C_n}{|\xi|}$.

Proof of Claim 9. LHS is radial because $\frac{1}{|x|^{n-1}}$ is radial. Also, it is homogeneous of degree -1, i.e.

$$\widehat{\frac{1}{|x|^{n-1}}}(\lambda\xi) = \frac{1}{\lambda}\widehat{\frac{1}{|x|^{n-1}}}(\xi).$$

Indeed,

$$\widehat{\frac{1}{|x|^{n-1}}}(\lambda\xi) = \lambda^{-n}\widehat{\frac{1}{|\frac{x}{\lambda}|^{n+1}}}(\xi) = \frac{1}{\lambda}\widehat{\frac{1}{|x|^{n-1}}}(\xi).$$

Since for $g(\lambda\xi) = \lambda^{-1}g(\xi)$, or $\lambda\xi g(\lambda\xi) = \xi g(\xi)$. Let $G(\xi) = \xi g(\xi)$, which implies $G(\lambda\xi) = G(\xi)$ and thus $G(|\xi|) = G(1)$. Hence $|\xi| g(|\xi|) = C$. Take $g = \widehat{\frac{1}{|x|^{n-1}}}$ and we're done. $\square$

Note that $\frac{x_j}{|x|^{n+1}} = \frac{1}{1-n}\frac{\partial}{\partial x_j}\left(\frac{1}{|x|^{n-1}}\right)$, we therefore have

$$\begin{aligned}
\widehat{\text{p.v.}\frac{x_j}{|x|^{n+1}}}(\xi) &= \frac{1}{1-n}\left(\frac{\partial}{\partial x_j}\widehat{\left(\frac{1}{|x|^{n-1}}\right)}\right)(\xi)\\
&= \frac{1}{1-n}\cdot 2\pi i\xi_j \cdot \widehat{\frac{1}{|x|^{n-1}}}(\xi)\\
&= -iC_n{}' \cdot \frac{\xi_j}{|\xi|}.
\end{aligned}$$

$\square$

Corollary 8. 1. $\widehat{R_j f}(\xi) = \frac{1}{C_n}\cdot(-iC_n)\cdot\frac{\xi_j}{|\xi|}\hat{f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$.

2. $\sum_{j=1}^n R_j^2 = -I$, the identity operator.

From now on, we will use the notation "$\lesssim$", and define $A \lesssim B$ if $A \le CB$ for some constant C.

Corollary 9. Let Δ be the Laplacian, $1 < p < \infty$. For any j, k, we have

$$\|\frac{\partial^2}{\partial x_j \partial x_k} u\|_p =: \|\partial x_j \partial x_k u\|_p \lesssim \|\Delta u\|_p.$$

Proof. Observe that $\partial x_j \partial x_k u = -R_j R_k(\Delta u) = -R_j(R_k(\Delta u))$. Note $\|\partial x_j \partial x_k u\|_p = \|R_j R_k(\Delta u)\|_p \lesssim \|\Delta\|_p$ under the observation. Thus

$$\begin{aligned}
\widehat{\partial x_j \partial x_k} u(\xi) &= (2\pi i\xi_j)\widehat{\partial x_k u}(\xi)\\
&= (2\pi i\xi_j)\cdot(2\pi i\xi_k)\hat{u}(\xi)\\
&= -4\pi^2\xi_j\xi_k\hat{u}(\xi).
\end{aligned}$$

Now from

$$\begin{aligned}
\widehat{\partial x_j \partial x_k u}(\xi) &= -\widehat{R_j R_k(\Delta u)} = \left(-i\frac{\xi_j}{|\xi|}\right)\left(-i\frac{\xi_k}{|\xi|}\right)(\widehat{\Delta u})(\xi) \\
&= \left(-i\frac{\xi_j}{|\xi|}\right)\left(-i\frac{\xi_k}{|\xi|}\right)\cdot(4\pi^2 i^2)|\xi|^2\hat{u}(\xi) \\
&= -4\pi^2\xi_j\xi_k\hat{u}(\xi),
\end{aligned}$$

we conclude our observation is true. □

Now let T_n, Ω be given as the beginning of this chapter, and Ω satisfies the conditions stated in Note 2 with the second of them replaced by $\Omega \in L^q(S^{n-1})$, where $q > 1$. The main part of the chapter will be the next theorem:

Theorem 26. Let Ω be characterized as the revised conditions as above, and be a even function. Then for any $f \in L^p$, $1 < p < \infty$, we have $\|T_n f\|_p \lesssim \|f\|_p$.

Proof. From Corollary 8, $T_n f = I(T_n f) = -\sum_{i=1}^n R_j^2(T_n f) = -\sum_{i=1}^n R_j(R_j(T_n f))$. To prove the theorem, we need to show

$$\|(R_j T_n)f\|_p \leq C_p\|f\|_p.$$

Note $T_n f(x) = K * f(x)$, where $K(x) = \text{p.v.}\ \frac{\Omega(x)}{|x|^{n+1}}$. This implies

$$\begin{aligned}
\widehat{R_j(T_n f)}(\xi) &= -i\frac{\xi_j}{|\xi|}\widehat{T_n f}(\xi) \\
&= -i\frac{\xi_j}{|\xi|}\hat{K}(\xi)\cdot\hat{f}(\xi) \\
&= \widehat{R_j K}(\xi)\hat{f}(\xi).
\end{aligned}$$

Claim 10. $R_j K$ is an odd kernel and it is homogeneous of degree $-n$.

Proof of Claim 9. Note that K is even since Ω is even, then

$$\begin{aligned}
R_j K(-x) &= C_n\,\text{p.v.}\int \frac{y_j}{|y|^{n+1}}K(-x-y)dy \\
&= C_n\,\text{p.v.}\int \frac{y_j}{|y|^{n+1}}K(x+y)dy \\
&= C_n\,\text{p.v.}\int \frac{-y_j}{|y|^{n+1}}K(x-y)dy \qquad (y \to -y) \\
&= -R_j K(x).
\end{aligned}$$

Similarly, we can show $R_j K(\lambda x) = \lambda^{-n} R_j K(x)$ for any $\lambda > 0$. □

Apply the method of rotation which we deal with $T_n f$ in Theorem 24,

$$(R_j K) * f(x) = \frac{1}{2}\int_{S^{n-1}} R_j K(y') H_{y'} f(x) d\sigma(y'),$$

and so

$$\|(R_j K) * f\|_p \leq C_p\|f\|_p \int_{S^{n-1}} |R_j K(y')| d\sigma(y').$$

Claim 11. $\int_{S^{n-1}} |R_jK(y')|d\sigma(y') \leq C_q\|\Omega\|_{L^q(S^{n-1})}$.

Proof of Claim 11. We've showed in Claim 10 that $R_jK(\lambda x) = \lambda^{-n}R_jK(x)$. Now by substitution,

$$\int_{S^{n-1}} |R_jK(y')|d\sigma(y') = C\int_{1<|x|<2} |R_jK(x)|dx.$$

Let $K_\epsilon(x) = \frac{\Omega(x')}{|x|^n} \cdot \chi(\{|x| > \epsilon\})$. Then

$$\int_{1<|x|<2} |R_jK(x)|dx \leq \int_{1<|x|<2} |R_jK(x) - R_jK_{1/2}(x)|dx + \int_{1<|x|<2} |R_jK_{1/2}(x)|dx =: \mathrm{I} + \mathrm{II}.$$

Note by Hölder's inequality,

$$\begin{aligned}
\mathrm{II} &\leq C_q\|R_jK_{1/2}\|_q \\
&\leq \|K_{1/2}\|_q && \text{(By Corollary 8)} \\
&= C_q\left[\int_{S^{n-1}} |\Omega(y')|^q \left(\int_{\frac{1}{2}}^\infty \frac{r^{n-1}}{r^{qn}}dr\right) d\sigma(y')\right]^{1/q} \\
&\leq C_{q,n}\|\Omega\|_{L^q(S^{n-1})}.
\end{aligned}$$

So it remains to show the part. Note

$$\begin{aligned}
R_jK(x) - R_jK_{1/2}(x) &= C_n \,\text{p.v.} \int_{|x-y|>\epsilon} \frac{x_j - y_j}{|x-y|^{n+1}} K(y)(1 - \chi(\{|y| > \frac{1}{2}\}))dy \\
&= C_n \,\text{p.v.} \int_{|x-y|>\epsilon} \frac{x_j - y_j}{|x-y|^{n+1}} K(y)\chi(\{|x| \leq \frac{1}{2}\})dy \\
&= C_n \,\text{p.v.} \int_{\substack{|y|\leq\frac{1}{2} \\ |x-y|>\epsilon}} \frac{x_j - y_j}{|x-y|^{n+1}} \frac{\Omega(y)}{|y|^n}dy,
\end{aligned}$$

and the last step implies $|x-y| \geq |x| - |y| > \frac{1}{2}$ since $1 < |x| < 2$ and $|y| \leq \frac{1}{2}$. Since by assumption of Ω, we can subtract $\frac{x_j}{|x|^{n+1}} \cdot \frac{\Omega(y)}{|y|^n}$ in the integral without affecting anything. Hence from $\left|\frac{|x_j - y_j|}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}}\right| \leq \frac{C|y|}{|x|^{n+1}}$, $1 \leq |x| < 2$, $|y| \leq \frac{1}{2}$, we have

$$\begin{aligned}
|R_jK(x) - R_jK_{1/2}(x)| &\leq \frac{C_n}{|x|^{n+1}} \int_{|y|\leq\frac{1}{2}} \frac{|\Omega(y)|}{|y|^{n-1}}dy \\
&= \frac{C_n}{|x|^{n+1}} \int_{S^{n-1}} |\Omega(y')| \left(\int_0^{\frac{1}{2}} dr\right) d\sigma(y') \\
&\leq C\|\Omega\|_{L^1(S^{n-1})} \leq C_q\|\Omega\|_{L^q(S^{n-1})},
\end{aligned}$$

since $q > 1$. Combine I and II, we finish the proof. $\square$

Combine Claim 10 and Claim 11, we conclude our proof. $\square$

11 Littlewood-Paley Theorem

Let $\Delta_j = \{x \in \mathbb{R} : 2^j \le |x| < 2^{j+1}\}$. Define S_j by $\widehat{(S_j f)}(\xi) = \chi(\Delta_j)\hat{f}(\xi)$.

Theorem 27 (**Littlewood-Paley Theorem**)**.** Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Then there exists positive constants C_1, C_2 such that for any $f \in L^p$,

$$C_1 \cdot \|f\|_p \le \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \le C_2 \cdot \|f\|_p,$$

or equivalently, $\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \sim \|f\|_p$.

Let $\psi \in \mathcal{S}(\mathbb{R})$ be non-negative, $\operatorname{supp}\psi \subset \{1/2 \le |\xi| \le 4\}$ and $\psi(\xi) = 1$ if $1 \le |\xi| \le 2$. Let $\psi_j(\xi) = \psi(2^{-j}\xi)$, and $\widehat{S'_j f}(\xi) = \hat{f}(\xi)\psi_j(\xi)$.

Theorem 28. For any $f \in L^p$, $1 < p < \infty$, $\left\| \left(\sum_j |S'_j f|^2 \right)^{1/2} \right\|_p \le C\|f\|_p$.

Proof. Let $\overrightarrow{T} f(x) = \{S'_j f\}_j = \{S'_1 f, S'_2 f, \cdots\}$, then LHS $= \|\overrightarrow{T} f\|_{l^2}$. To show $\|\overrightarrow{T} f\|_{L^p(l^2)} = \| \ \|\overrightarrow{T} f\|_{l^2}\|_p \le C\|f\|_p$. For $p = 2$ case,

$$\begin{aligned}
\|\overrightarrow{T} f\|_{l^2}^2 &= \int \left(\sum_j |S'_j f(x)|^2 \right) dx \\
&= \sum_j \int |S'_j f(x)|^2 dx \\
&= \sum_j \int |\hat{f}(\xi)|^2 |\psi_j(\xi)|^2 d\xi \\
&= \int |\hat{f}(\xi)|^2 \sum_j |\psi_j(\xi)|^2 d\xi \\
&\le C\|\hat{f}\|_2^2 = C\|f\|_2^2,
\end{aligned}$$

where we use the fact that $\sum_i |\psi(\xi)|^2 \le 3$ by definition. $\square$

Theorem 29 (**Calderón-Zygmund Theorem**)**.** Let $\overrightarrow{T} f(x) = \overrightarrow{K} * f(x)$, where $\overrightarrow{K} = (K_1, K_2, \cdots)$, $x \in \mathbb{R}^n$. Suppose that

1. $\|\overrightarrow{K}(x-y) - \overrightarrow{K}(x-y')\|_{l^2} \le \frac{C|y-y'|^\epsilon}{|x-y|^{n+\epsilon}}$, if $|x-y| > 2|y-y'|$ and $\epsilon > 0$.
2. $\|\overrightarrow{T} f\|_{L^2(l^2)} \le C\|f\|_2$.

Then $\left|\{x \in \mathbb{R}^n : \|\overrightarrow{K} * f(x)\|_{l^2} > \lambda\}\right| \le \frac{C}{\lambda}\|f\|_1$.

The proof is left to readers. Now suppose $K_j = \check{\psi}_j$, and $\overrightarrow{K}$ is defined similarly as above.

Claim 12. $\|\overrightarrow{K}(x-y) - \overrightarrow{K}(x-y')\|_{l^2} \le \frac{C|y-y'|^\epsilon}{|x-y|^{1+\epsilon}}$, if $|x-y| > 2|y-y'|$ and $\epsilon > 0$.

Proof. By definition, we know

$$\begin{aligned}|K_j(x-y)-K_j(x-y')| &= |\check{\psi}_j(x-y)-\check{\psi}_j(x-y')| \\ &\le |\check{\psi}_j'(\eta)|\cdot|y-y'|,\end{aligned}$$

where $\eta=\theta(x-y)+(1-\theta)(x-y')$ for some $\theta\in[0,1]$. Note

$$\begin{aligned}|\eta| &= |(x-y')-\theta(y-y')| = |(x-y)-(1-\theta)(y-y')| \\ &\ge |x-y|-|y-y'| \ge \frac{1}{2}|x-y|.\end{aligned}$$

Note also

$$\begin{aligned}\check{\psi}_j = K_j &= \int \psi_j(\xi)e^{2\pi i\xi\cdot x}d\xi \\ &= \int \psi(2^{-j}\xi)e^{2\pi i\xi\cdot x}d\xi = 2^j\check{\psi}(2^jx).\end{aligned}$$

This implies

$$\begin{aligned}|\check{\psi}_j'(\eta)| &= |(2^j\check{\psi})'(2^j\eta)| = |2^{2j}(\check{\psi})'(2^j\eta)| \\ &\le \frac{C_N\cdot 2^{2j}}{(1+|2^j\eta|)^N} \qquad \text{(By Lemma 4)} \\ &\le \frac{C_N\cdot 2^{2j}}{(1+2^j|x-y|)^N}.\end{aligned}$$

Hence we have

$$\begin{aligned}\text{LHS} &= \left(\sum_j |\check{\psi}_j(x-y)-\check{\psi}_j(x-y')|^2\right)^{1/2} \le C_N\left[\sum_j \frac{2^{4j}|y-y'|^2}{(1+2^j|x-y|)^{2N}}\right]^{1/2} \\ &\lesssim \left[\sum_{\substack{j \\ 2^j|x-y|<1}} \frac{2^{4j}|y-y'|^2}{(1+2^j|x-y|)^{2N}}\right]^{1/2} + \left[\sum_{\substack{j \\ 2^j|x-y|\ge 1}} \frac{2^{4j}|y-y'|^2}{(1+2^j|x-y|)^{2N}}\right]^{1/2} \\ &\lesssim \sum_{\substack{j \\ 2^j|x-y|<1}} 2^{2j}|y-y'| + \sum_{\substack{j \\ 2^j|x-y|\ge 1}} \frac{2^{2j}|y-y'|}{(2^j|x-y|)^N}.\end{aligned}$$

Here we have used the inequality

$$\left(\sum |a_i|\right)^r \le \sum |a_i|^r,$$

where $0\le r\le 1$. Thus

$$\begin{aligned}\text{LHS} &\lesssim \frac{|y-y'|}{|x-y|^2} + \sum_{\substack{j \\ 2^j|x-y|\ge 1}} \frac{1}{2^{(N-2)j}}\cdot\frac{|y-y'|}{|x-y|^N} \\ &\lesssim \frac{|y-y'|}{|x-y|^2}.\end{aligned}$$

$\square$

Now we go back to probability theory to induce a useful theorem we'll need later. First recall that for a random variable X on the probability space Ω with probability measure P, the expectation of X is defined by

$$\mathbb{E}(X) = \int_\Omega X dP.$$

We have a famous result:

Lemma 13 (**Khinchin's Inequality**)**.** Let $\{\omega_n\}_{n=1}^N$ be sequence of independent random variables taking values ± 1 with equal probability, i.e. $P(\{t : \omega_n(t) = 1\}) = P(\{t : \omega_n(t) = -1\}) = \frac{1}{2}$. Then

$$\mathbb{E}\left(\left|\sum_{n=1}^N a_n\omega_n\right|^p\right) \sim \left(\sum_{n=1}^N |a_n|^2\right)^{p/2},$$

where $A \sim B$ if $\exists\, c_1, c_2 > 0$, $c_1 A \leq B \leq c_2 A$, and the constants are independent of N (but might depend on p).

Proof. Let $\mu > 0$. Clearly by independence,

$$\mathbb{E}\left(e^{\mu \sum_n a_n\omega_n}\right) = \mathbb{E}\left(\prod_n e^{\mu a_n \omega_n}\right) = \prod_n \mathbb{E}\left(e^{\mu a_n\omega_n}\right).$$

Note by definition,

$$\begin{aligned}\int_\Omega e^{\mu a_n\omega_n} dP &= \int_{\{\omega_n = 1\}} e^{\mu a_n} dP + \int_{\{\omega_n=-1\}} e^{-\mu a_n} dP \\ &= \frac{1}{2}\left(e^{\mu a_n} + e^{-\mu a_n}\right) \leq e^{\frac{1}{2}(\mu a_n)^2},\end{aligned}$$

which implies

$$\mathbb{E}\left(e^{\mu \sum_n a_n\omega_n}\right) = \prod_n \frac{1}{2}\left(e^{\mu a_n} + e^{-\mu a_n}\right) \leq \prod_n e^{\frac{1}{2}(\mu a_n)^2} = e^{\frac{\mu^2}{2}\sum_n a_n^2}.$$

Let $E_\lambda = \{t : \sum_n a_n\omega_n(t) \geq \lambda\} \subset \Omega$,

$$P(E_\lambda)e^{\mu\lambda} \leq \int_\Omega e^{\mu\sum_n a_n\omega_n(t)} dP \leq e^{\frac{\mu^2}{2}\sum_n |a_n|^2},$$

which gives

$$P(E_\lambda) \leq e^{-\mu\lambda} e^{\frac{\mu^2}{2}\sum_n |a_n|^2}.$$

Let $\mu = \frac{\lambda}{\sum_n |a_n|^2}$, then $P(E_\lambda) \leq e^{-\frac{\lambda^2}{2\sum_n |a_n|^2}}$. Similarly, $P(\{t : \sum_n a_n\omega_n(t) \leq -\lambda\}) \leq e^{-\frac{\lambda^2}{2\sum_n |a_n|^2}}$,

$P(\{|\sum_n a_n\omega_n| > \lambda\}) \le 2e^{-\frac{\lambda^2}{2\sum_n |a_n|^2}}$. Now

$$\begin{aligned}
\mathbb{E}\left(\left|\sum_n a_n\omega_n\right|^p\right) &= \int_\Omega \left|\sum_n a_n\omega_n\right|^p dP \\
&= p\int_0^\infty \lambda^{p-1} P\left(\left\{\left|\sum_n a_n\omega_n\right| > \lambda\right\}\right) d\lambda \\
&\le 2p\int_0^\infty \lambda^{p-1} e^{-\frac{\lambda^2}{2\sum_n |a_n|^2}} d\lambda \\
&= 2p\left(\sum_n |a_n|^2\right)^{p/2} \left(\int_0^\infty \lambda^{p-1} e^{-\lambda^2/2} d\lambda\right) \qquad (\text{let } \lambda \to (\textstyle\sum_n |a_n|^2)^{1/2}\lambda) \\
&= C_p \left(\sum_n |a_n|^2\right)^{p/2}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_n |a_n|^2 = \mathbb{E}\left(\left|\sum_n a_n\omega_n\right|^2\right) &= \int_\Omega \left|\sum_n a_n\omega_n\right| \cdot \left|\sum_n a_n\omega_n\right| dP \\
&\le \mathbb{E}\left(\left|\sum_n a_n\omega_n\right|^p\right)^{1/p} \cdot \mathbb{E}\left(\left|\sum_n a_n\omega_n\right|^q\right)^{1/q}, \qquad (\text{Hölder})
\end{aligned}$$

we have $\sum_n |a_n|^2 \le \mathbb{E}(|\sum_n a_n\omega_n|^p)^{1/p} \cdot C_q(|\sum_n a_n\omega_n|^2)^{1/2}$, and therefore

$$\left(\sum_n |a_n|^2\right)^{1/2} \le C_p \mathbb{E}\left(\left|\sum_n a_n\omega_n\right|^p\right)^{1/p}.$$

□

Theorem 30. Suppose T is a linear operator s.t. for any $f \in L^p$, $1 < p < \infty$,

$$\|Tf\|_p \le C_p \cdot \|f\|_p,$$

then

$$\left\|\left(\sum_j |Tf_j|^2\right)^{1/2}\right\|_p \le C_p' \left\|\left(\sum_j |f_j|^2\right)^{1/2}\right\|_p$$

Proof. From Khinchin's inequality (Lemma 13), we know

$$\left(\sum_j |Tf_j|^2\right)^{p/2} \sim \mathbb{E}\left(\left|\sum_j Tf_j\omega_j\right|^p\right),$$

where $\{\omega_j\}$ is a sequence of random variables taking values ± 1 with equal probability. Therefore,

$$\begin{aligned}
\left\|\left(\sum_j |Tf_j|^2\right)^{1/2}\right\|_p^p &= \int_X \left(\sum_j |Tf_j|^2\right)^{p/2} d\mu \\
&\sim \int_X \int_\Omega \left|\sum_j Tf_j(x)\omega_j\right|^p dP d\mu \\
&= \int_X \int_\Omega \left|T(\sum_j f_j\omega_j)\right|^p dP d\mu && (\mu \text{ is } \sigma\text{-finite}) \\
&= \int_\Omega \left(\int_X \left|T(\sum_j f_j\omega_j)\right|^p\right) dP, && \text{(Fubini)}
\end{aligned}$$

hence we obtain

$$\left\|\left(\sum_j |Tf_j|^2\right)^{1/2}\right\|_p^p \lesssim \int_\Omega \left(\int_X \left|T(\sum_j f_j\omega_j)\right|^p\right) dP \lesssim \left\|\left(\sum_j |f_j|^2\right)^{1/2}\right\|_p^p.$$

□

Lemma 14. Define $\widehat{S_{[a,b)}f}(\xi) = \hat{f}(\xi)\cdot\chi(\{\xi : a \leq \xi < b\})$. Recall $M_a f(x) = e^{2\pi i a x} f(x)$, then

$$S_{[a,b)} = \frac{i}{2}(M_a H M_{-a} - M_b H M_{-b}),$$

where H is Hilbert transform.

Proof. Note $\frac{i}{2}\widehat{(M_a H M_{-a} - M_b H M_{-b})f}(\xi) = \frac{i}{2}(M_a \widehat{HM_{-a}f}(\xi) - M_b \widehat{HM_{-b}f}(\xi))$, by the fact $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$, and we're done. □

Proof of Theorem 27. Recall that $\widehat{(S_j f)}(\xi) = \chi(\Delta_j)\hat{f}(\xi)$, and $\psi_j\chi(\Delta_j) = \chi(\Delta_j)\psi_j$, we have $\widehat{S_j(S_j' f)} = \widehat{S_j'(S_j f)}$. From Lemma 14, we know

$$S_j = \frac{i}{2}(M_{2^j} H M_{-2^j} - M_{2^{j+1}} H M_{-2^{j+1}}) + \frac{i}{2}(M_{-2^{j+1}} H M_{2^{j+1}} - M_{-2^j} H M_{2^j}).$$

It suffices to show

$$\left\|\left(\sum_j |M_{2^j} H M_{-2^j} S_j' f|^2\right)^{1/2}\right\|_p \lesssim \|f\|_p.$$

Notice that

$$\begin{aligned}
\text{LHS} &= \left\|\left(\sum_j |H(M_{-2^j} S_j' f)|^2\right)^{1/2}\right\|_p \\
&\lesssim \left\|\left(\sum_j |M_{-2^j} S_j' f|^2\right)^{1/2}\right\|_p = \left\|\left(\sum_j |S_j' f|^2\right)^{1/2}\right\|_p \\
&\lesssim \|f\|_p. && \text{(Theorem 28)}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_X \sum_j S_j f \overline{S_j g} dx &= \sum_j \langle S_j f, S_j g\rangle \\
&= \sum_j \left\langle \widehat{S_j f}, \widehat{S_j g} \right\rangle \\
&= \sum_j \int_{\Delta_j} \hat{f}(\xi)\overline{\hat{g}}(\xi) d\xi \\
&= \int_{\mathbb{R}} \hat{f}\overline{\hat{g}} = \left\langle \hat{f}, \hat{g} \right\rangle = \langle f, g\rangle .
\end{aligned}$$

Since $\|f\|_p = \sup_{\substack{g\in L^q \\ \|g\|_q=1}} = |\langle f, g\rangle|$, where $1/p + 1/q = 1$, we have

$$\begin{aligned}
\|f\|_p &= \sup_{\substack{g\in L^q \\ \|g\|_q=1}} \left| \int_{\mathbb{R}} \sum_j S_j f \overline{S_j g} \right| \\
&\leq \sup_{\substack{g\in L^q \\ \|g\|_q=1}} \int \left(\sum_j |S_j f|^2 \right)^{1/2} \left(\sum_j |S_j g|^2 \right)^{1/2} && \text{(Cauchy-Schwarz)} \\
&\leq \sup_{\substack{g\in L^q \\ \|g\|_q=1}} \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \cdot \left\| \left(\sum_j |S_j g|^2 \right)^{1/2} \right\|_q && \text{(Hölder)} \\
&\lesssim \sup_{\substack{g\in L^q \\ \|g\|_q=1}} \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \cdot \|g\|_q \\
&\lesssim \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p .
\end{aligned}$$

□

12 Littlewood-Paley Theorem in $\mathbb{R}^n$

We've learnt Littlewood-Paley Theorem in Lecture 11, it is natural to ask for higher dimensional generalization. In this lecture, we will give a generalization to $\mathbb{R}^n$.

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\psi(0) = 0$. For $j \in \mathbb{Z}$, we define S_j by

$$\widehat{S_j f}(\xi) = \psi(2^{-j}\xi)\hat{f}(\xi),$$

where $\xi \in \mathbb{R}^n$ and $f \in L^p$ for $1 < p < \infty$. This operator can be extended over L^p by usual limit process as we did for Fourier transform.

Theorem 31 (**Littlewood-Paley Theorem**)**.** For $1 < p < \infty$, there is a constant C_p which depends on p and ψ, such that for any $f \in L^p$,

$$\left\| (\sum_{j\in\mathbb{Z}} |S_j f|^2)^{1/2} \right\|_p \leq C_p \|f\|_p.$$

Moreover, suppose $C = \sum_{j\in\mathbb{Z}} |\psi(2^{-j}\xi)|^2 \geq 0$ for all $\xi \neq 0$, then

$$\|f\|_p \leq C_p \left\| (\sum_{j\in\mathbb{Z}} |S_j f|^2)^{1/2} \right\|_p .$$

Remark 7. Such ψ in the special case of the theorem does exist. Indeed, let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be non-negative and radial, and $\phi(\xi) = 1$ for $|\xi| \leq 1/2$ and $\phi(\xi) = 0$ for $|\xi| \geq 1$. Define $(\psi(\xi))^2 = \phi(\xi/2) - \phi(\xi)$ for any $\xi \in \mathbb{R}^n$. Easy to see RHS is non-negative, and

$$\sum_{j\in\mathbb{Z}} |\psi(2^{-j}\xi)|^2 = \sum_{j\in\mathbb{Z}} |\phi(2^{-j-1}\xi) - \phi(2^{-j}\xi)|^2 = \phi(0) = 1.$$

From definition of S_j, one can represent $S_j f(x) = K_j * f(x)$, where

$$K_j(x) = \int \psi(2^{-j}\xi) e^{2\pi i \xi x} d\xi = 2^{nj}\hat{\psi}(-2^j x).$$

We define the vector-valued operator $\overrightarrow{T}$ by

$$\overrightarrow{T} f(x) = \{S_j f\}_{j\in\mathbb{Z}} = \overrightarrow{K} * f(x),$$

where $\overrightarrow{K} = \{K_j\}_{j\in\mathbb{Z}}$. Hence first part of the theorem 31 becomes $\|\overrightarrow{T} f\|_{L^p(l^2)} := \left\| \|\overrightarrow{T} f\|_{l^2} \right\|_p \leq C_p \|f\|_p$.

Proof of Theorem 31. First to estimate for $p = 2$, i.e. $\|\overrightarrow{T} f\|_{L^2(l^2)} \lesssim \|f\|_2$. By Plancherel Theorem (Theorem 13),

$$\|\overrightarrow{T} f\|_{L^2(l^2)}^2 = \sum_j \int |S_j f|^2 = \int |\hat{f}(\xi)|^2 \sum_j |\psi(2^{-j}\xi)|^2 d\xi,$$

which reduces to the verification of

$$\sum_j |\psi(2^{-j}\xi)|^2 \leq C.$$

Note that

$$|\psi(2^{-j}\xi)|^2 = |\psi(2^{-j}\xi) - \psi(0)|^2 \leq |\nabla\psi(\eta)|^2 |2^{-j}\xi|^2 \leq C_1 2^{-2j}|\xi|^2,$$

where the constant C_1 depends on ψ. On the other hand, since ψ is a Schwartz function, we see that

$$|\psi(2^{-j}\xi)| \leq \frac{C_N}{(1+|2^{-j}\xi|)^N}$$

for any non-negative integer N, and C_N depends on ψ. It follows that

$$\sum_j |\psi(2^{-j}\xi)|^2 \lesssim \sum_j \min\{\frac{1}{(1+|2^{-j}\xi|)^N}, 2^{-2j}|\xi|^2\},$$

and one can check RHS is controlled by a constant.

To obtain the L^p estimate, we need to verify $\overrightarrow{K}$ is a C-Z kernel.

Exercise 7. Check that $\overrightarrow{K}$ is a C-Z kernel, and in particular for $|x-y| > 2|y-y'|$,

$$\|\overrightarrow{K}(x-y) - \overrightarrow{K}(x-y')\|_{l^2} \leq \frac{C|y-y'|}{|x-y|^{n+1}}.$$

After we check this, it follows immediately that $\|\overrightarrow{T}f\|_{L^p(l^2)} \leq C_p\|f\|_p$. Now it suffices to check the case when $C = \sum_{j\in\mathbb{Z}} |\psi(2^{-j}\xi)|^2$. This is from the part we proved and a standard duality argument. Note

$$\int \sum_j S_j f S_j g = \int \sum_j |\psi(2^{-j}\xi)|^2 \hat{f}(\xi)\bar{\hat{g}}(\xi) = C\langle f, g\rangle.$$

By Cauchy-Schwartz inequality,

$$|\langle f,g\rangle| \lesssim \int (\sum_j |S_j f|^2)^{1/2} (\sum_j |S_j g|^2)^{1/2},$$

which is bounded by

$$\left\|(\sum_j |S_j f|^2)^{1/2}\right\|_p \cdot \left\|(\sum_j |S_j g|^2)^{1/2}\right\|_{p'} \lesssim \left\|(\sum_j |S_j f|^2)^{1/2}\right\|_p \cdot \|g\|_{p'},$$

by Hölder inequality and the part proved. By duality, we're done the proof. □

Remark 8. $\operatorname{supp}\psi$ may not be compact. Theorem 31 is still valid when $\psi(2^{-j}\xi)$ is replaced by non-smooth cut-off, say the indicate function of the annulus $\{\xi \in \mathbb{R}^n : 2^j \leq |\xi| < 2^{j+1}\}$, when $n=1$. However, it is not true for $n \geq 2$ unless $p=2$, because of Fefferman's shocking result, which tells us L^p-unboundedness ($p \neq 2$) of S_j' if $n \geq 2$, where $\widehat{S_j' f}(\xi) = \chi(\{\xi \in \mathbb{R}^n : 2^j \leq |\xi| < 2^{j+1}\}) \cdot \hat{f}(\xi)$.

However in the Remark 8, if we replace the annuli $\{\xi \in \mathbb{R}^n : 2^j \leq |\xi| < 2^{j+1}\}$ by disjoint dyadic rectangular boxes, one can have the non-smooth cut-off version of Littlewood-Paley Theorem. We'll state the two-dimensional case, which can be extended to higher dimension by induction.

Recall $\Delta_j = \{x \in \mathbb{R} : 2^j \leq |x| < 2^{j+1}\}$, we define S_j^1 by

$$\widehat{S_j^1 f}(\xi_1,\xi_2) = \chi_{\Delta_j}(\xi_1)\hat{f}(\xi_1,\xi_2),$$

where $(\xi_1,\xi_2) \in \mathbb{R}^2$, and define S_j^2 by

$$\widehat{S_j^1 f}(\xi_1,\xi_2) = \chi_{\Delta_j}(\xi_2)\hat{f}(\xi_1,\xi_2).$$

Theorem 32. There exist positive constants c_p and C_p such that

$$c_p\|f\|_p \leq \left\| \left(\sum_{j,k} |S_j^1 S_k^2 f|^2 \right)^{1/2} \right\|_p \leq C_p\|f\|_p.$$

13 Multipliers

Let m be a measurable function on $\mathbb{R}^n$. Define T by

$$\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi),$$

for any $\xi \in \mathbb{R}^n$ and any $f \in L^p \cap L^2$. Here p is a given number in $[1, \infty]$.

Definition 22. The function m is called a **multiplier** and T is called a **multiplier operator**. The multiplier m is called a **L^p-multiplier** if T is bounded on L^p, i.e. $\|Tf\|_p \le C_p \|f\|_p$ for all $f \in L^p \cap L^2$.

Remark 9. When $1 \le p < \infty$ holds, T can be extended uniquely to an operator which is bounded on L^p. We will abuse the same notation T is denote the extension operator.

Remark 10. There are two questions arise naturally from the definition:

1. Given m, does T define a bounded operator on L^p?
2. How to characterize L^p-multiplier?

Both of them can be answered in L^2. The first question is easy, and is dependent on m. For example, if m is a smooth bump function on a unit cube, then the multiplier operator T is bounded on L^p for any $p \in [1, \infty]$. The second one is extremely challenging for $p \neq 2$. However, we can do it for $p = 2$, and moreover the Hörmander multiplier theorem, which characterize L^p-multiplier for "smooth-enough" multiplier. We start with a easy characterization.

Theorem 33. m is an L^2-multiplier iff $m \in L^\infty$.

Proof. By Plancherel's Theorem (Theorem 13), we get the "if" part:

$$\|Tf\|_2 = \|\widehat{Tf}\|_p = \|m\hat{f}\|_2 \le \|m\|_\infty \|\hat{f}\|_2 = \|m\|_\infty \|f\|_2.$$

Now assume m is an L^2-multiplier. Define the norm of T to be

$$\|T\| = \sup_{\substack{f \in L^2 \\ f \neq 0}} \frac{\|Tf\|_2}{\|f\|_2}.$$

WLOG, suppose $\|T\| \neq 0$. We will show $\|m\|_\infty \le 2\|T\|$ under the condition $\|T\| > 0$. For any $k \in \mathbb{Z}$, set $E_k = \{\xi \in \mathbb{R}^n : 2^k \le |\xi| < 2^{k+1}, |m(\xi)| > 2\|T\|\}$. Denote $|E|$ to be the Lebesgue measure of E. It is clear that

$$4\|T\|^2 |E_k| \le \int |m(\xi)|^2 |\chi_{E_k}(\xi)|^2 d\xi = \|m\chi_{E_k}\|_2^2,$$

by Chebyshev's inequality. On the other hand, by Plancherel's Theorem,

$$\|m\chi_{E_k}\|_2^2 = \|T\check{\chi}_{E_k}\|_2^2 \le \|T\|^2 \cdot \|\chi_{E_k}\|_2^2 = \|T\|^2 \cdot |E_k|.$$

It follows that

$$4\|T\|^2 |E_k| \le \|T\|^2 \cdot |E_k|,$$

which implies

$$4|E_k| \le |E_k|,$$

since $\|T\| \neq 0$. So $|E_k| = 0$, which gives $|m(\xi)| \le 2\|T\|$ a.e. $\square$

Definition 23. Let α be a real number. Define the **first Sobolev spaces**

$$L^2_\alpha(\mathbb{R}^n) = \{f : (1+|\xi|^2)^{\alpha/2}\hat{f}(\xi) \in L^2\}$$

and the **Sobolev norm** by

$$\|f\|_{L^2_\alpha} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^\alpha |\hat{f}(\xi)|^2 d\xi\right)^{1/2}.$$

Note 3. For $\alpha \geq 0$, the Sobolev space L^2_α is a collection of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ obeying that both f and $D^\alpha f$ lie in L^2, where the differential operator D^α can be defined in terms of Fourier transform by $\widehat{D^\alpha f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi)$. One can check it is indeed a multiplier.

Lemma 15. Let $\alpha > n/2$ and $f \in L^2_\alpha(\mathbb{R}^n)$. Then $\hat{f} \in L^1(\mathbb{R}^n)$. In particular, f is continuous and bounded.

Proof. By Cauchy-Schwartz inequality, we get

$$\|\hat{f}\|_1 \leq \left(\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^\alpha} d\xi\right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^\alpha |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \leq C_{\alpha,n}\|f\|_{L^2_\alpha} < \infty.$$

By inverse Fourier theorem, we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x\xi} d\xi,$$

since $\hat{f} \in L^1$ and $f \in L^2$. Hence f is continuous (uniformly) and bounded. □

Theorem 34. Let $m \in L^2_\alpha$ with $\alpha > n/2$. Then m is an L^p-multiplier for any $p \in [1,\infty]$.

Proof. Let T be operator associated to m. Let $K(x) = \hat{m}(-x)$, then

$$\hat{K}(\xi) = \int \hat{m}(-x) e^{-2\pi i x\xi} dx,$$

because $\hat{m} \in L^1$ by Lemma 15. Changing the variable $-x \to x$, we have

$$\hat{K}(\xi) = \int \hat{m}(x) e^{2\pi i x\xi} dx = \check{\hat{m}}(\xi) = m(\xi).$$

Hence we can represent $Tf(x) = K * f(x)$, since $\widehat{Tf} = \hat{K}\hat{f} = m\hat{f}$. Moreover, we see that the kernel $K \in L^1$ because $\hat{m} \in L^1$. Then

$$\|Tf\|_1 = \|K * f\|_1 \leq \int\int |K(y)f(x-y)| dx dy \leq \|K\|_1 \cdot \|f\|_1,$$

and

$$\|Tf\|_\infty \leq \|K\|_1 \cdot \|f\|_\infty.$$

By interpolation theorem we're done. □

Lemma 16. Let $m \in L^2_\alpha(\mathbb{R}^n)$ with $\alpha > n/2$ and $\lambda > 0$. Define T_λ by

$$\widehat{T_\lambda f}(\xi) = m(\lambda\xi)\hat{f}(\xi).$$

Then

$$\int_{\mathbb{R}^n} |T_\lambda f(x)|^2 u(x)dx \le C \int_{\mathbb{R}^n} |f(x)|^2 Mu(x)dx,$$

where u is any non-negative measurable function, and C is a constant independent of u, λ and f, M is Hardy-Littlewood maximal function.

Proof. Let $K(x) = \hat{m}(-x)$. Since $\hat{m} \in L^1$ by Theorem 15, we see in proof of Theorem 34 that $T_1 = K * f$. It is clear that

$$\left(\int |K(x)|^2(1+|x|^2)^\alpha dx\right)^{1/2} = \|m\|_{L^2_\alpha}.$$

By dilation, we see easily that $T_\lambda f(x) = K_\lambda * f(x)$, where $K_\lambda(x) = \lambda^{-n}K(\lambda^{-1}x)$. We have

$$\text{LHS} = \int |K_\lambda * f(x)|^2 u(x)dx = \int \left|\int \frac{K_\lambda(x-y)(1+|\lambda^{-1}(x-y)^2|)^{\alpha/2}}{(1+|\lambda^{-1}(x-y)^2|)^{\alpha/2}} f(y)dy\right|^2 u(x)dx.$$

By Cauchy-Schwartz inequality,

$$\begin{aligned}
\text{LHS} &\le \int \left(\int |K(\lambda^{-1}(x-y))|^2(1+|\lambda^{-1}(x-y)^2|)^\alpha dy\right)\left(\int \frac{\lambda^{-2n}|f(y)|^2}{(1+|\lambda^{-1}(x-y)^2|)^\alpha}dy\right)u(x)dx \\
&= \|m\|_{L^2_\alpha} \int\int \frac{\lambda^{-n}|f(y)|^2}{(1+|\lambda^{-1}(x-y)^2|)^\alpha}dy \cdot u(x)dx \\
&= \|m\|_{L^2_\alpha} \int |f(y)|^2 \left(\int \frac{\lambda^{-n}u(x)}{(1+|\lambda^{-1}(x-y)^2|)^\alpha}dx\right)dy. && \text{(Fubini)}
\end{aligned}$$

The second factor in the integrand can be controlled by Mu (up to constant) since $\alpha > n/2$. Hence we're done. ☐

We're ready to state the main theorem for the lecture:

Theorem 35 (Hörmander). Let $\psi \in C^\infty$ be a radial function supported on $1/2 \le |\xi| \le 2$ such that

$$\sum_{j\in\mathbb{Z}} |\psi(2^{-j}\xi)|^2 = 1$$

for all $\xi \neq 0 \in \mathbb{R}^n$. Suppose that $m \in L^\infty$ is a measurable function obeying, for some $\alpha > n/2$,

$$\sup_{j\in\mathbb{Z}} \|m(2^j\cdot)\psi\|_{L^2_\alpha} < \infty.$$

Then m is an L^p-multiplier for any $p \in (1,\infty)$.

Proof. We only need to prove m is an L^p-multiplier for $p > 2$, while the other part follows from the duality. We will build up L^p estimate for the operator T.

Define S_j by $\widehat{S_j f}(\xi) = \psi(2^{-j}\xi)\hat{f}(\xi)$. By Littlewood-Paley Theorem in $\mathbb{R}^n$ (Theorem 31), we have

$$\left\|\left(\sum_{j\in\mathbb{Z}} |S_j f|^2\right)^{\frac{1}{2}}\right\|_p \sim \|f\|_p.$$

Let $\psi' \in \mathcal{S}(\mathbb{R}^n)$ with $\psi'(\xi) = 1$ when $1/2 \le |\xi| \le 2$ and it is supported in annulus $\{\xi \in \mathbb{R}^n : 1/4 \le |\xi| \le 4\}$. We define S_j' by

$$\widehat{S_j' f}(\xi) = \psi'(2^{-j}\xi)\hat{f}(\xi),$$

then by Fourier transform,

$$S_j T = S_j T S_j'.$$

Apply Littlewood-Paley Theorem, with f replaced by Tf, we get

$$\|Tf\|_p \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |S_j T S_j' f|^2 \right)^{\frac{1}{2}} \right\|_p .$$

From the definitions of T and S_j, we obtain for any $f \in L^p$, $\widehat{S_j T f}(\xi) = \psi(2^{-j}\xi)m(\xi)\hat{f}(\xi)$. So from the condition of m, we use the Lemma 16 to obtain

$$\int_{\mathbb{R}^n} |S_j T f(x)|^2 u(x) dx \le C \int_{\mathbb{R}^n} |f(x)|^2 M u(x) dx, \tag{1}$$

where C depends on $\sup_{j\in\mathbb{Z}} \|m(2^j\cdot)\psi\|_{L^2_\alpha}$, but is independent of u, f, S_j and T. Denote $S_j' f = g_j$. Then by duality,

$$\left\| \left(\sum_{j\in\mathbb{Z}} |S_j T g_j|^2 \right)^{\frac{1}{2}} \right\|_p = \left[\int \left(\sum_{j\in\mathbb{Z}} |S_j T g_j|^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} = \left(\int \sum_j |S_j T g_j|^2 h \right)^{\frac{1}{2}},$$

for some h with $\|h\|_{(p/2)'} = 1$. Use (1) with $f = g_j$ and $u = |h|$, we obtain

$$\left\| \left(\sum_{j\in\mathbb{Z}} |S_j T g_j|^2 \right)^{\frac{1}{2}} \right\|_p \lesssim \left(\int \sum_j |g_j(x)|^2 M h(x) dx \right)^{\frac{1}{2}} .$$

By Hölder's inequality, we have

$$\text{RHS} \le C \left\| \left(\sum_j |g_j|^2 \right)^{\frac{1}{2}} \right\|_p \cdot \|Mh\|_{(p/2)'} \lesssim \|f\|_p \cdot \|h\|_{(p/2)'} \lesssim \|f\|_p.$$

Here we use Littlewood-Paley Theorem and the $L^{(p/2)'}$ boundedness of Hardy-Littlewood maximal function M, which implies the L^p estimate of T combining the results before. $\square$

Theorem 36. Denote $\mathbb{N}_0$ to be the set of non-negative integers. Let $m : \mathbb{R}^n \to \mathbb{C}$ be an L^∞ function which lies in $\mathbb{C}^k$ away from the origin for $k = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Suppose that m satisfies

$$\sup_{R>0} R^{|\beta|} \left(\frac{1}{R^n} \int_{R<|\xi|<2R} |D^\beta m(\xi)|^2 d\xi \right)^{1/2} < \infty \tag{2}$$

for any multi-index $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}_0^n$ with $|\beta| = \beta_1 + \cdots \beta_n \le k$ and $D^\beta = \partial_{\xi_1}^{\beta_1} \cdots \partial_{\xi_n}^{\beta_n}$. Then m is an L^p-multiplier for $1 < p < \infty$.

Proof. Suffice to verify

$$\sup_{j\in\mathbb{Z}} \|m(2^j\cdot)\psi\|_{L^2_k} < \infty. \tag{3}$$

Change the variable $\xi \to \mathbb{R}\xi$, (2) becomes

$$\sup_{R>0} \left(\int_{1<|\xi|<2} |D^\beta m_R(\xi)|^2 d\xi \right)^{1/2} < \infty, \tag{4}$$

where $m_R(\xi) = m(R\xi)$. Let ψ be the smooth cut-off function given as in Theorem 35.

Now to check (3). For any two multi-index $\beta = (\beta_1, \cdots, \beta_n)$ and $\gamma = (\gamma_1, \cdots, \gamma_n)$, both in $\mathbb{N}_0^n$, we say $\gamma \le \beta$ if $\gamma_j \le \beta_j$ for all $j = 1, \cdots, n$. Furthermore, we define for $\gamma \le \beta$,

$$\binom{\beta}{\gamma} = \prod_{j=1}^n \binom{\beta_j}{\gamma_j}.$$

For any multi-index β, we denote $\beta! \coloneqq \prod_{j=1}^n \beta_j!$. By this new notation, we can represent

$$\binom{\beta}{\gamma} = \frac{\beta!}{\gamma!(\beta-\gamma)!}.$$

By Leibniz's law, we have

$$D^\beta(m(2^j\xi)\psi(\xi)) = \sum_{\substack{\gamma\in\mathbb{N}_0^n \\ \gamma\le\beta}} \binom{\beta}{\gamma} D^\gamma m_{2^j}(\xi) D^{\beta-\gamma}\psi(\xi),$$

where $m_{2^j}(\xi) = m(2^j\xi)$.

Exercise 8. For any integer $\alpha \ge 0$,

$$\|f\|_{L^2_\alpha} \le C_\alpha \sum_{\substack{\beta\in\mathbb{N}_0^n \\ |\beta|\le\alpha}} \|D^\beta f\|_2,$$

where C_α is a constant only depending on α. In fact, the inequality can be reversed.

From the above Exercise, there is a constant C_k (depending only on k) s.t.

$$\begin{aligned}
\sup_{j\in\mathbb{Z}} \|m(2^j\cdot)\psi\|_{L^2_k} &\le C_k \sum_{|\beta|\le k} \|D^\beta(m(2^j\cdot)\psi)\|_2 \\
&\le C_k \sum_{\substack{\gamma\le\beta \\ |\beta|\le k}} \binom{\beta}{\gamma} \left(\int |D^\gamma m_{2^j}(\xi)|^2 \cdot |D^{\beta-\gamma}\psi(\xi)|^2 \right)^{1/2}.
\end{aligned}$$

Since ψ is a nice function supported in $\{1/2 \le |\xi| \le 2\}$, we have

$$|D^{\beta-\gamma}\psi(\xi)| \le C(k),$$

for any (β, γ) with $\gamma \le \beta$ and $|\beta| \le k$, and $C(k)$ is a constant depending on k and ψ. We also are able to control those $\binom{\beta}{\gamma}$'s by a constant depending on k. Therefore, we obtain

$$\sup_{j\in\mathbb{Z}} \|m(2^j\cdot)\psi\|_{L^2_k} \lesssim \max_{|\gamma|\le k} \sup_j \left(\int_{1<|\xi|<2} |D^\gamma m_{2^j}(\xi)|^2 d\xi \right)^{1/2} < \infty,$$

from (4), where the implicit constant hidden in $\lesssim$ is

$$C_k C(k) \sum_{\substack{\gamma \leq \beta \\ |\beta| \leq k}} \binom{\beta}{\gamma},$$

which is a constant relying on k. $\square$

Corollary 10 (**Mikhlin Multiplier Theorem**)**.** Let $k = \lfloor \frac{n}{2} \rfloor + 1$. Suppose that $m \in L^\infty(\mathbb{R}^n)$ satisfies that there is a constant C s.t.

$$|D^\beta m(\xi)| \leq C|\xi|^{-|\beta|}$$

for any $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq k$, and for all $\xi \neq 0 \in \mathbb{R}^n$. Then m is an L^p-multiplier for $1 < p < \infty$.

Exercise 9. Prove Mikhlin Multiplier Theorem.

14 Fractional Integrals

For any $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\widehat{(-\Delta f)}(\xi) = 4\pi^2|\xi|^2\hat{f}(\xi),$$

where Δ is Laplacian. One can extend the definition of $-\Delta$ to $(-\Delta)^{\alpha/2}$ for $\alpha \in \mathbb{R}$. In fact, one can define the operator $(-\Delta)^{\alpha/2}$ in $\mathcal{S}(\mathbb{R}^n)$ first by

$$(\widehat{(-\Delta)^{\alpha/2}}f)(\xi) = (2\pi|\xi|)^{\alpha}\hat{f}(\xi).$$

Then we can extend this operator to more general case. We abuse the same notation to denote the extension. When $\alpha \geq 0$, the operator $(-\Delta)^{\alpha/2}$ is essentially α-order differential operator. When $\alpha < 0$, we use $I_{-\alpha}$ to denote $(-\Delta)^{\alpha/2}$, which can be viewed as $(-\alpha)$-order integration operator.

One can define for $\alpha > 0$,

$$\widehat{I_\alpha f}(\xi) = (2\pi|\xi|)^{-\alpha}\hat{f}(\xi),$$

and one can view it as a α-order integration operator.

Definition 24. When $0 < \alpha < n$, I_α is called a **fractional integral**.

Remark 11. I_α is a multiplier operator, so it can be represented by $I_\alpha f = K * f$ for some kernel K. Next lemma is devoted to find such K.

Proposition 3. We have the following properties:

1. $I_\alpha I_\beta = I_{\alpha+\beta}$, for $0 < \alpha, \beta < n$ and $\alpha + \beta < n$.
2. $\Delta I_\alpha = -I_{\alpha-2}$ for $2 < \alpha < n$.
3. $(-\Delta)^{\beta/2} I_\alpha = I_{\alpha-\beta}$ for $0 < \alpha < \beta < n$.
4. I_2 is the fundamental solution of $-\Delta$, that is, $u = -I_2 f$ is the solution of $\Delta u = f$.

Exercise 10. Check these properties.

Lemma 17. Let $0 < \alpha < n$,

$$C_{n,\alpha} = \pi^{\frac{n}{2}-\alpha}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})},$$

then

$$\widehat{|x|^{\alpha-n}}(\xi) = C_{n,\alpha}|\xi|^{-\alpha}$$

in the sense of distribution, i.e. for all $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |x|^{\alpha-n}\hat{\psi}(x)dx = C_{n,\alpha}\int_{\mathbb{R}^n} |\xi|^{-\alpha}\psi(\xi)d\xi.$$

Proof. Recall that $\widehat{e^{-\pi\delta|\cdot|^2}}(\xi) = \delta^{-n/2}e^{-\pi|\xi|^2/\delta}$. For any $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int e^{-\pi\delta|x|^2}\hat{\psi}(x)dx = \int \widehat{e^{-\pi\delta|\cdot|^2}}(\xi)\psi(\xi)d\xi = \delta^{-n/2}\int e^{-\pi|\xi|^2/\delta}\psi(\xi)d\xi,$$

by Exercise 1. Let $\beta = \frac{n-\alpha}{2}$, multiplying both sides by $\delta^{\beta-1}$ and then integrating in δ from 0 to ∞, revealing

$$\int_0^\infty \delta^{\beta-1} \int e^{-\pi\delta|x|^2} \hat{\psi}(x)dxd\delta = \int \psi(\xi) \left(\int_0^\infty \delta^{\beta-1-\frac{n}{2}} e^{-\pi|\xi|^2/\delta} d\delta \right) d\xi. \tag{1}$$

Note that the left hand side of equation (1) is

$$\text{LHS} = \int \hat{\psi}(x) \left(\int_0^\infty \delta^{\beta-1} e^{-\pi\delta|x|^2} d\delta \right) dx = \pi^{-\beta}\Gamma(\beta) \int \hat{\psi}(x)|x|^{\alpha-n}dx. \tag{2}$$

On the other hand, the right hand of equation (1) is

$$\text{RHS} = \int \psi(\xi) \left(\int_0^\infty \delta^{-\frac{\alpha}{2}} e^{-\pi|\xi|^2/\delta} \frac{d\delta}{\delta} \right) d\xi = \int \psi(\xi) \left(\int_0^\infty \rho^{\frac{\alpha}{2}} e^{-\rho} \frac{d\rho}{\rho} \right) (\pi|\xi|^2)^{-\frac{\alpha}{2}} d\xi,$$

by letting $\rho = \pi|\xi|^2/\delta$. Hence we have

$$\text{RHS} = \Gamma(\frac{\alpha}{2})\pi^{-\frac{\alpha}{2}} \int \psi(\xi)|\xi|^{-\alpha}d\xi. \tag{3}$$

Comparing equations (2) and (3), we obtain the desired result. □

Corollary 11. For any $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$I_\alpha f(x) = C(n,\alpha) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y)dy,$$

where $C(n,\alpha) = 2^{-\alpha}\pi^{n/2}\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}$. Hence $I_a f(x) = K * f(x)$, where $K(x) = C(n,\alpha)|x|^{\alpha-n}$.

This can be extended to $f \in L^p$ with $1 \le p < \frac{n}{\alpha}$. To see that, we partition $K(x)$ into $K_1(x) = K(x) \cdot \chi(\{|x| \le 1\})$ and $K_2(x) = K(x) \cdot \chi(\{|x| > 1\})$. Notice that K_1 is an integrable function because $0 < \alpha < n$, and thus $\|K_1 * f\|_p \le \|K_1\|_1 \cdot \|f\|_p$ by Young's inequality. Hence $K_1 * f$ converges absolutely a.e. since it belongs to L^p. By Hölder's inequality, $|K_2 * f| \le \|K_2\|_{p'}\|f\|_p$. When p satisfies the condition, $K_2 \in L^{p'}$ since $(n-\alpha)p' > n$. Hence $I_a f = K * f$ for $f \in L^p$ is well-defined.

Our next goal is to find for what pairs of p and q, we have the inequality

$$\|I_\alpha f\|_q \le C\|f\|_p \tag{4}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Now for any $\delta > 0$, let

$$f_\delta(x) = \delta^{n/p} f(\delta x).$$

From the Corollary 11, we have $I_\alpha f_\delta(x/\delta) = \delta^{-\alpha+\frac{n}{p}} I_\alpha f(x)$. On the other hand,

$$\|I_\alpha f_\delta\|_q = \frac{1}{\delta^{n/q}} \|I_\alpha f_\delta(\frac{\cdot}{\delta})\|_q.$$

So using equation 4 for $f = f_\delta$, we see that

$$\|I_\alpha f_\delta\|_q = \delta^{-\alpha+\frac{n}{p}-\frac{n}{q}} \|I_\alpha f\|_q \le C\|f_\delta\|_p = C\|f\|_p.$$

This can be true only if

$$-\alpha + \frac{n}{p} - \frac{n}{q} = 0. \tag{5}$$

Two special cases arise when $(p,q) = (1, \frac{n}{n-\alpha})$ and $(p,q) = (\frac{n}{\alpha}, \infty)$.

Lemma 18. In either case, the presumed inequality (4) does not hold.

Proof. It suffices to show for the case $(p,q) = (1, \frac{n}{n-\alpha})$. The other is dual to this case. If in this case the result is valid, we can replace f by a sequence $\{f_k\}$ of positive integrable functions whose common integral is 1 and whose supports converge to the origin, or an approximation to the identity. Denote $B_\epsilon = \{x \in \mathbb{R}^n : \epsilon \leq |x| \leq 1/\epsilon\}$. For n sufficiently large and $x \in B_\epsilon$, we have

$$I_\alpha f_k(x) = C(n,\alpha) \int |x-y|^{\alpha-n} \chi_{B_{\epsilon/2}}(x-y) f_k(y)dy,$$

because f_k is supported in $\{y : |y| < \epsilon/4\}$ when n is large. Let $K_\epsilon = K \cdot \chi(B_{\epsilon/2})$, which is integrable. From $\|I_\alpha f\|_{\frac{n}{n-\alpha}} \leq C\|f\|_1$, it follows that

$$\|K_\epsilon * f_k\|_{L^{\frac{n}{n-\alpha}}(B_\epsilon)} = \|I_\alpha f_k\|_{L^{\frac{n}{n-\alpha}}(B_\epsilon)} \leq C\|f_k\|_1 = C.$$

We know

$$\lim_{k\to\infty} \|K_\epsilon * f_k\|_{L^{\frac{n}{n-\alpha}}(B_\epsilon)} = \|K_\epsilon\|_{L^{\frac{n}{n-\alpha}}(B_\epsilon)} = C(n,\alpha) \left\| |\cdot|^{-n+\alpha} \right\|_{L^{\frac{n}{n-\alpha}}(B_\epsilon)} \leq C,$$

which implies by letting $\epsilon \to 0$,

$$\int_{\mathbb{R}^n} |x|^{-n} dx < \infty,$$

and this leads to a contradiction! $\square$

Example 9. We give a counterexample to demonstrate (4) is not valid for $(p,q) = (n/\alpha, \infty)$. Let $0 < \alpha < n$, and $\varepsilon > 0$ small. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{|x|^\alpha}\left(\log\frac{1}{|x|}\right)^{-\frac{\alpha}{n}\cdot(1+\varepsilon)} & , |x| \leq \frac{1}{2} \\ 0 \ , |x| > \frac{1}{2}. \end{cases}$$

One can check that $f \in L^{n/\alpha}(\mathbb{R}^n)$ and $I_\alpha f \notin L^\infty$ as long as $\alpha/n(1+\varepsilon) \leq 1$. (Exercise)

It turns out that after removing the two special cases, the equation (5) is also a sufficient condition of (p,q) to make (4) holds for all $f \in L^p$. We can therefore formulate Hardy-Littlewood-Sobolev Theorem of fractional integrals.

Theorem 37 (Hardy-Littlewood-Sobolev Theorem). Let $0 < \alpha < n$, $1 \leq p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then

1. If $p > 1$, then there exists a constant C such that for any $f \in L^p$, $\|I_\alpha f\|_q \leq C\|f\|_p$.
2. There exists a constant C such that for any $\lambda > 0$ and any $f \in L^1$,

$$|\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}| \leq \frac{C\|f\|_1^q}{\lambda^q}.$$

Proof. Notice p satisfies $1 \leq p < n/\alpha$, and thus Corollary 11 is valid for representing I_α for $f \in L^p$. We can show first that, for any real number $R > 0$,

$$|I_\alpha f(x)| \lesssim R^\alpha Mf(x) + R^{-n/q}\|f\|_p. \tag{6}$$

Split the kernel K into $K_1(x) = K(x)\cdot\chi(\{|x|\le R\})$ and $K_2(x) = K(x)\cdot\chi(\{|x|>R\})$. The first part K_1 can be divided further into $\sum_{k=0}^{\infty} K\cdot\chi(\{2^{-k-1}R<|x|\le 2^{-k}R\})$. Then it is not difficult to prove

$$|K_1 * f(x)| \lesssim R^{\alpha} Mf(x).$$

On the other hand, apply the Hölder's inequality, we have

$$|K_2 * f(x)| \lesssim R^{-n/q}\|f\|_p.$$

Now (6) follows. Choosing $R^{-n/p} = \frac{Mf(x)}{\|f\|_p}$, we then obtain

$$|I_\alpha f(x)| \lesssim \|f\|_p^{\frac{\alpha p}{n}}\cdot(Mf(x))^{p/q}. \tag{7}$$

For $p>1$, we see that

$$\|I_\alpha f\|_q \lesssim \|f\|_p^{\frac{\alpha p}{n}}\cdot\|Mf(x)\|_p^{\frac{p}{q}} \lesssim \|f\|_p^{\frac{\alpha p}{n}+\frac{p}{q}} \lesssim \|f\|_p,$$

since M is bounded on L^p and $\frac{\alpha p}{n}+\frac{p}{q}=1$. For the case $p=1$, it follows from (7) and weak (1,1)-estimate of M that

$$|\{x\in\mathbb{R}^n : |I_\alpha f(x)|>\lambda\}| \le |\{x : Mf(x)\ge C_{\alpha,n}\|f\|_1^{-\frac{\alpha q}{n}}\lambda^q\}| \lesssim \frac{\|f\|_1^q}{\lambda^q}.$$

□

15 Littlewood-Paley Theorem in Continuous Version

Let ψ be a radial and real-valued Schwartz function, obeying $\int_{\mathbb{R}^n} \psi(x)dx = 0$. It is not difficult to show $\hat{\psi}$ is real-valued and radial too. Since $\hat{\psi}$ is radial, for $\xi \in \mathbb{R}^n$, $\hat{\psi}(\xi) = \hat{\psi}(|\xi|)$ and it gives a function defined on $\mathbb{R}$.

Let

$$\psi_t(x) = t^{-n}\psi(t^{-1}x),$$

for any $t > 0$, and

$$Q_t f(x) = \psi_t * f(x).$$

We shall notice that

$$\int_0^\infty (\hat{\psi}(t))^2 \frac{dt}{t} = \int_0^\infty |\hat{\psi}(t)|^2 \frac{dt}{t} < \infty. \tag{1}$$

To see why this is true, we split the integral in the left side into

$$\int_0^1 |\hat{\psi}(t)|^2 \frac{dt}{t} + \int_1^\infty |\hat{\psi}(t)|^2 \frac{dt}{t}.$$

The first term is clearly finite because $\hat{\psi} \in \mathcal{S}(\mathbb{R}^n)$. The second term equals to

$$\int_1^\infty |\hat{\psi}(t) - \hat{\psi}(0)|^2 \frac{dt}{t} \le \int_0^1 |\nabla\hat{\psi}(\eta)|^2 t^2 \frac{dt}{t} \le C_\psi.$$

Here we used $\hat{\psi}(0) = \int \psi = 0$ and the mean value theorem. Hence we obtain (1).

We can normalize ψ so that

$$\int_0^\infty (\hat{\psi}(t))^2 \frac{dt}{t} = 1. \tag{2}$$

We now state the Calderón reproducing formula, which allows us to represent $f \in L^2$ in terms of the operator Q_t.

Lemma 19. Suppose that ψ satisfies (2). Then for any $f \in L^2$,

$$\int_0^\infty Q_t^2 f(x) \frac{dt}{t} = f(x)$$

in L^2 dense, that is,

$$\left\| \int_\epsilon^R Q_t^2 f \frac{dt}{t} - f \right\|_2 \to 0 \tag{3}$$

as $\epsilon \to 0$, $R \to \infty$. Here Q_t^2 is given by $Q_t^2 f = Q_t(Q_t f) = \psi_t * \psi_t * f$.

Proof. For $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned} \left\| \int_\epsilon^R Q_t^2 f \frac{dt}{t} - f \right\|_2 &= \left\| \int_\epsilon^R \widehat{Q_t^2 f} \frac{dt}{t} - \hat{f} \right\|_2 && \text{(Plancherel)} \\ &= \left\| \hat{f} \left(\int_{\epsilon|\cdot|}^{R|\cdot|} (\hat{\psi}(t))^2 \frac{dt}{t} - 1 \right) \right\|_2 && \text{(Definition of } Q_t) \end{aligned}$$

By DCT, we see that the very right side tends to 0 as $\epsilon \to 0$ and $R \to \infty$ due to (2).

Now let

$$T_{\epsilon,R}f(x) = \int_\epsilon^R Q_t^2 f(x)\frac{dt}{t}.$$

For any $f, g \in L^2$, we see that

$$\langle T_{\epsilon,R}f, g\rangle = \int_\epsilon^R \int Q_t^2 f(x)\overline{g(x)}dx\frac{dt}{t} = \int_\epsilon^R \int (\hat{\psi}(t|\xi|))^2(x)\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi\frac{dt}{t},$$

which is bounded by, after using Fubini theorem and changing of variables,

$$\int |\hat{f}(\xi)\hat{g}(\xi)| \int_\epsilon^R |\hat{\psi}(t)|^2\frac{dt}{t}d\xi \leq \|f\|_2 \cdot \|g\|_2.$$

By duality, we can conclude that for any $f \in L^2$,

$$\|T_{\epsilon,R}f\|_2 \leq \|f\|_2.$$

We are able to use the uniform L^2 boundedness of $T_{\epsilon,R}$ to extend the identity (3) from $\mathcal{S}$ to L^2. (This is a standard trick in analysis.) Indeed, because $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, for any $f \in L^2$, there is a Schwartz function φ such that $\|f - \varphi\| < \epsilon$. Then by triangle inequality and result above, we see

$$\|T_{\epsilon,R}f - f\|_2 \leq \|T_{\epsilon,R}f - T_{\epsilon,R}\varphi\|_2 + \|T_{\epsilon,R}\varphi - \varphi\|_2 + \|\varphi - f\|_2 \leq 2\|\varphi - f\|_2 + \|T_{\epsilon,R}\varphi - \varphi\|_2,$$

which is bounded by

$$2\epsilon + \|T_{\epsilon,R}\varphi - \varphi\|_2.$$

Letting $\epsilon \to 0$ and $R \to \infty$, we end up with 0, and so

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \|T_{\epsilon,R}f - f\|_2 = 0.$$

$\square$

Definition 25. We define **Littlewood-Paley g-function** by

$$g(f)(x) = \left(\int_0^\infty |Q_t f(x)|^2\frac{dt}{t}\right)^{1/2}.$$

We now aim to build up Littlewood-Paley theorem for the g-function, say, $\|g(f)\|_p \sim \|f\|_p$ for $p \in (1, \infty)$. It is not difficult to make L^2 theory.

Theorem 38. Let $f \in L^2$. Then

$$\|g(f)\|_2 = \|f\|_2.$$

Proof. By Plancherel Theorem (Theorem 13), we get for $f \in L^2$,

$$\|g(f)\|_2^2 = \int\int_0^\infty |Q_t f(x)|^2\frac{dt}{t} = \int |\hat{f}(\xi)|^2\left(\int_0^\infty |\hat{\psi}(t|\xi|)|^2\frac{dt}{t}\right)d\xi = \|f\|_2^2.$$

$\square$

In some old book, one may see the definition of g-function via Poisson kernel. We now summarize the old but very classical versions of g-function here. Let $t > 0$, then the **Poisson kernel** P_t on $\mathbb{R}^n$ is defined by (for $x \in \mathbb{R}^n$)

$$P_t(x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} e^{-2\pi t|\xi|} d\xi.$$

In other words, P_t is Fourier transform of $e^{-2\pi t|\cdot|}$. It is well-known that

$$P_t(x) = \frac{c_n t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}, \tag{4}$$

where $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$.

Exercise 11. This exercise aims to prove (4). We shall follow the steps:

1. Show that for $\delta > 0$,
$$\int_{\mathbb{R}^n} e^{-\pi\delta|\xi|^2} e^{-2\pi i x\cdot\xi} d\xi = \delta^{-n/2} e^{-\pi|x|^2/\delta}.$$

2. For any $\gamma > 0$,
$$e^{-\gamma} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\gamma^2/4u} du.$$
Hint: Write $e^{-\gamma} = \frac{1}{\pi}\int_{-\infty}^\infty \frac{e^{i\gamma x}}{1+x^2}dx$, and express the factor $\frac{1}{1+x^2}$ as $\int_0^\infty e^{-(1+x^2)u}du$, then evaluate the integral.

3. Use 1 and 2 to prove (4).

The **Poisson integral** is defined by

$$u(x,t) = P_t * f(x)$$

for $f \in L^2$, which in terms of Fourier transform, can be represented as

$$u(x,t) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-2\pi i x\cdot\xi} e^{-2\pi t|\xi|} d\xi.$$

The function $e^{-2\pi t|\xi|}$ is rapidly decreasing in $|\xi|$ since $t > 0$. This yields the absolute convergence of the integral. For the same reason, the integral above can be differentiated with respect to x and t any number of times by carrying out the operation for the integral. Hence we see that for $t > 0$,

$$\Delta u = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial t^2} = 0.$$

Therefore, the Poisson kernel P_t can be used to describe the fundamental solution of the Laplacian on the upper half plane.

The classical g-function is given by

$$g^*(f)(x) = \left(\int_0^\infty |\nabla u(x,t)|^2 t dt\right)^{1/2},$$

where $\nabla u = (\partial_{x_1} u, \cdots, \partial_{x_n} u, \partial_t u)$. Moreover, we define

$$g_1^*(f)(x) = \left(\int_0^\infty \left| \frac{\partial u}{\partial t}(x,t) \right|^2 t dt \right)^{1/2}$$

and

$$g_x^*(f)(x) = \left(\int_0^\infty |\nabla_x(x,t)|^2 t dt \right)^{1/2},$$

where $\nabla_x u = (\partial_{x_1} u, \cdots, \partial_{x_n} u)$. It follows immediately from the definitions that

$$g^*(f) = \sqrt{(g_1^*(f))^2 + (g_x^*(f))^2}.$$

One can check that Littlewood-Paley theorem holds for g^*, g_1^* and g_x^*. Now let us state the L^2 result as follows:

Theorem 39. We have

$$\|g^*(f)\|_2 = \frac{1}{\sqrt{2}} \|f\|_2$$

and

$$\|g_1^*(f)\|_2 = \|g_x^*(f)\|_2 = \frac{1}{2} \|f\|_2.$$

Exercise 12. Prove Theorem 39. Hint: Plancherel Theorem.

Now we're ready to state Littlewood-Paley theory for g-function. For the time being, we assume ψ to be radial and $\mathbb{R}$-valued Schwartz function, satisfying $\int_0^\infty (\hat{\psi}(t))^2 \frac{dt}{t} = 1$.

Theorem 40 (Littlewood-Paley Theorem). For any $1 < p < \infty$, there are constants c_p and C_p such that

$$c_p \|f\|_p \le \|g(f)\|_p \le C_p \|f\|_p,$$

for any $f \in L^p$.

The proof relies on vector-valued Calderón-Zygmund theory. For any $t > 0$, let $K_t \in \mathcal{S}'(\mathbb{R}^n)$. K_t is a $\mathbb{C}$-valued function on $\{\in \mathbb{R}^n : x \neq 0\}$. When x is fixed, $K_t(x)$ can be considered as a function of $t > 0$, denoted by h_x. We assume the function h_x belongs to $\mathbb{H}$ when x is fixed. We use the $\|\cdot\|$ to denote the norm generated by the inner product of $\mathbb{H}$. We now define an $\mathbb{H}$-valued function $\mathbf{K}$ by

$$\mathbf{K}(x) = \{K_t(x)\}_{t>0},$$

and its norm, for any given x, by

$$\|\mathbf{K}(x)\|_{\mathbb{H}} = \|h_x\| (= \|K_t(x)\|).$$

An $\mathbb{H}$-valued kernel $\mathbf{K}$ is called a **Calderón-Zygmund kernel** if it satisfies for some $\epsilon \in (0,1]$,

$$\|\mathbf{K}(x)\|_{\mathbb{H}} \le \frac{C}{|x|^n} \tag{5}$$

for any $x \in \mathbb{R}^n$ with $x \neq 0$, and

$$\|\mathbf{K}(x) - \mathbf{K}(x')\|_{\mathbb{H}} \leq \frac{C|x-x'|^{\epsilon}}{|x|^{n+\epsilon}}, \tag{6}$$

if $|x| > 2|x-x'|$.

We define T_t by

$$T_t f(x) = K_t * f(x),$$

and a vector-valued operator $\mathbf{T}f = \mathbf{K} * f = \{K_t * f\}_{t>0}$. In addition, we set

$$\|\mathbf{T}f\|_{L^p(\mathbb{H})} = \left(\int_{\mathbb{R}^n} \|\mathbf{T}f\|_{\mathbb{H}}^p\right)^{1/p},$$

where $\|\mathbf{T}f\|_{\mathbb{H}} = \|K_t * f(x)\|$, exactly as how we define $\|\mathbf{K}(x)\|_{\mathbb{H}}$ above. We will employ the vector-valued Calderón-Zygmund theory for convolution type operator, which is stated as follows:

Theorem 41. Let $\mathbf{K}$ be a vector-valued Calderón-Zygmund kernel satisfying (5) and (6). Suppose that

$$\|\mathbf{T}f\|_{L^2(\mathbb{H})} \lesssim C\|f\|_2,$$

for any $f \in L^2$. Then for any $1 < p < \infty$,

$$\|\mathbf{T}f\|_{L^p(\mathbb{H})} \leq C\|f\|_p,$$

for any $f \in L^p$.

The proof can be prove by Calderón-Zygmund decomposition as we did in Lecture 7. So we will omit the proof of the theorem. We shall now find a suitable Hilbert space for us to represent the g-function. The Hilbert space we need to select is $L^2(\mathbb{R}_+, \frac{dt}{t})$, which is a collection of $h : \mathbb{R}_+ \to \mathbb{C}$ such that

$$\|h\|_{L^2(\mathbb{R}_+, \frac{dt}{t})} = \left(\int_0^{\infty} |h(t)|^2 \frac{dt}{t}\right)^{1/2} < \infty.$$

Now the vector-valued kernel $\mathbf{K}$ is given by

$$\mathbf{K}(x) = \{\psi_t(x)\}_{t>0}.$$

Then to prove $\|g(f)\|_p \lesssim \|f\|_p$, it is equivalent to show

$$\|\mathbf{K} * f\|_{L^p\left(L^2(\mathbb{R}_+, \frac{dt}{t})\right)} \lesssim \|f\|_p, \tag{7}$$

because $\|g(f)\|_p = \|\mathbf{K} * f\|_{L^p\left(L^2(\mathbb{R}_+, \frac{dt}{t})\right)}$. From Theorem 38, it is sufficient to verify the kernel $\mathbf{K}$ is a Calderón-Zygmund kernel satisfying (5) and (6).

Proof of Theorem 40. First, we have

$$\|\mathbf{K}(x)\|_{L^2(\mathbb{R}_+, \frac{dt}{t})} = \left(\int_0^{\infty} |\psi_t(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which is bounded by

$$\left(\int_0^\infty \frac{C_N t^{-2n}}{(1+|x/t|)^N}\frac{dt}{t}\right)^{1/2} \lesssim \left(\int_{t>|x|}\frac{1}{t^{2n+1}}dt\right)^{1/2} + \left(\int_{0<t\le|x|}\frac{t^N}{t^{2n+1}|x|^N}dt\right)^{1/2} \lesssim \frac{1}{|x|^n}.$$

Here we use the fact that $\psi \in \mathcal{S}$ and the number N can be chosen to be greater than $2n+1$. Hence we obtain (5) for the kernel $\mathbf{K}$.

On the other hand, note that

$$\|\mathbf{K}(x)-\mathbf{K}(x')\|_{L^2(\mathbb{R}_+,\frac{dt}{t})} = \left(\int_0^\infty |\psi_t(x)-\psi_t(x')|^2\frac{dt}{t}\right)^{1/2},$$

which equals to

$$\left(\int_0^\infty t^{-2n}|\psi(x/t)-\psi(x'/t)|^2\frac{dt}{t}\right)^{1/2}.$$

Using the mean value theorem for ψ, we get

$$\|\mathbf{K}(x)-\mathbf{K}(x')\|_{L^2(\mathbb{R}_+,\frac{dt}{t})} \le \left(\int_0^\infty t^{-2n}\left|\nabla\psi\left(\frac{\eta}{t}\right)\right|^2\cdot\left|\frac{x-x'}{t}\right|^2\frac{dt}{t}\right)^{1/2},$$

where $\eta=(1-\theta)x+\theta x' = x-\theta(x-x')$ for some $\theta\in[0,1]$. Since x,x' satisfy $|x|>2|x-x'|$, we get

$$|\eta| \ge |x|-|x-x'| > \frac{|x|}{2},$$

from which it follows that

$$\begin{aligned}\|\mathbf{K}(x)-\mathbf{K}(x')\|_{L^2(\mathbb{R}_+,\frac{dt}{t})} &\lesssim \left(\int_0^\infty t^{-2n}\frac{1}{(1+|x|/t)^N}\cdot\left|\frac{x-x'}{t}\right|^2\frac{dt}{t}\right)^{1/2}\\ &\lesssim \left(\int_{t>|x|}\frac{1}{t^{2n+3}}dt\right)^{1/2}\cdot|x-x'| + \left(\int_{t\le|x|}\frac{t^{N-2n-3}}{|x|^N}dt\right)^{1/2}\cdot|x-x'|\\ &\lesssim \frac{|x-x'|}{|x|^{n+1}}.\end{aligned}$$

Henceforth, we get (6). As a consequence of Theorem 41, (7) follows. Therefore we see g-function is bounded on L^p.

Finally the reverse inequality can be prove by duality.

Exercise 13. For any $f\in L^2\cap L^p$, we have

$$\|f\|_p = \sup_{\substack{h\in\mathcal{S}(\mathbb{R}^n)\\ \|h\|_{p'}\le 1}}\left|\int f(x)h(x)dx\right|.$$

Use the Exercise above, with Lemma 19, we can represent, for $f\in L^2\cap L^p$ and $h\in\mathcal{S}(\mathbb{R}^n)$,

$$\left|\int f(x)h(x)dx\right| = \left|\int\int_0^\infty Q_t^2 f(x)\frac{dt}{t}h(x)dx\right|. \tag{8}$$

Interchanging integrals by Fubini's Theorem, we see that the right side of (8) can be written as

$$\int_0^\infty\int Q_t^2 f(x)h(x)dx\frac{dt}{t} = \int_0^\infty\int Q_tf(x)Q_th(x)dx\frac{dt}{t} = \int_0^\infty\int Q_tf(x)Q_th(x)\frac{dt}{t}dx,$$

which is controlled by Cauchy-Schwarz inequality and then Hölder inequality,

$$\int g(f)(x)g(h)(x)dx \leq \|g(f)\|_p \cdot \|g(h)\|_{p'} \lesssim \|g(f)\|_p.$$

Here in the last step, we applied that g-function is bounded on $L^{p'}$. Again we shall rely on that $L^p \cap L^2$ is dense in L^p. Indeed, for any $f \in L^p$, we can find a sequence $\{f_k\}$ in $L^p \cap L^2$ such that $\lim_{k\to\infty} \|f_k - f\|_p = 0$. Then

$$\begin{aligned} \|f\|_p = \lim_{k\to\infty} \|f_k\|_p &\lesssim \lim_{k\to\infty} \|g(f_k)\|_p \lesssim \lim_{k\to\infty} \|g(f_k - f)\|_p + \|g(f)\|_p \\ &\lesssim \lim_{k\to\infty} \|f_k - f\|_p + \|g(f)\|_p \lesssim \|g(f)\|_p. \end{aligned}$$

Hence, the reverse inequality $\|f\|_p \lesssim \|g(f)\|_p$ holds for any $f \in L^p$. Now we finish the proof of the theorem. □

The g-function in the proof is modern. The classical way to define the g-function is based on Poisson integrals. For instance, g^*, g_1^*, g_x^* defined before. Littlewood-Paley estimates for those classical functions can be established too. More precisely, we have

Theorem 42. For $1 < p < \infty$ and any function $f \in L^p$,

$$\|g^*(f)\|_p \sim \|g_1^*(f)\|_p \sim \|g_x^*(f)\|_p \sim \|f\|_p.$$

Exercise 14. Prove Theorem 42. This can be done in a similar way as Theorem 40, via a use of Calderón-Zygmund theory.

16 $T1$ Theorem in a Simple Version

In this lecture, we aim to solve the question arose in Lecture 7 Remark 4. For now we will only present a simple version to the question, or known as T_1 Theorem. We will prove the whole part after we learn the BMO space.

Recall that in Lecture 7, we have the following definition: (Here we let $K \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ and K is a $\mathbb{C}$-valued function in $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$)

Definition 26. Let $T : \mathcal{S} \to \mathcal{S}'$ be continuous in $\mathcal{S}$ and linear. T is called a **singular integral operator**, or **SIO**, if T is associated with K, that is,

$$\langle T\varphi, \psi \rangle = \langle K, \varphi \otimes \psi \rangle ,$$

where $\varphi \otimes \psi(x,y) = \varphi(x)\psi(y)$ for any $\mathbb{R}^n \times \mathbb{R}^n$, $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and $T\varphi(x) = \int_{\mathbb{R}^n} K(x,y)\varphi(y)dy$.

Definition 27. For T defined above, say T satisfies **weak boundedness property**, or **WBP**, if

$$|\langle \varphi, T\psi \rangle| \leq CR^n \left(\|\varphi\|_\infty + R\|\nabla\varphi\|_\infty\right)\left(\|\psi\|_\infty + R\|\nabla\psi\|_\infty\right)$$

for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ that are supported in a ball in $\mathbb{R}^n$ of radius R, and C is a constant independent of φ, ψ and R.

Lemma 20. Suppose that T can be extended to a bounded operator on $L^2(\mathbb{R}^n)$, then T satisfies WBP.

Proof. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ be supported in a ball in $\mathbb{R}^n$ of radius R. By Schwartz inequality and L^2 boundedness, we have

$$|\langle \varphi, T\psi \rangle| \leq \|\varphi\|_2 \|T\psi\|_2 \lesssim \|\varphi\|_2 \|\psi\|_2 \lesssim R^n \|\varphi\|_\infty \|\psi\|_\infty,$$

since both of them are supported in a ball. $\square$

Definition 28. The **adjoint operator** of T is defined to be a SIO associated to the kernel $K^*(x,y) = \overline{K(y,x)}$ for $x \neq y$. That is,

$$\langle T^*\varphi, \psi \rangle = \langle K^*, \varphi \otimes \psi \rangle .$$

Note 4. We have

$$\langle T\varphi, \psi \rangle = \langle \varphi, T^*\psi \rangle .$$

Denote $\mathcal{S}_0(\mathbb{R}^n) = \{\phi \in C_c^\infty(\mathbb{R}^n) : \int \phi = 0\}$. Let T be a Calderón-Zygmund SIO.

Definition 29. We define a linear functional $T1$ on $\mathcal{S}_0(\mathbb{R}^n)$ as follow: for any $\phi \in \mathcal{S}_0(\mathbb{R}^n)$, there is a ball B in $\mathbb{R}^n$ such that ϕ takes value 0 outside B and $\eta \in \mathbb{C}_c^\infty(\mathbb{R}^n)$ with $\eta(x) = 1$ for $x \in 3B$. Define

$$\langle T1, \phi \rangle = \langle T\eta, \phi \rangle + \langle 1 - \eta, T^*\phi \rangle .$$

Note 5. We need to verify the well-definedness. $\langle T\eta, \phi\rangle$ is well-defined since both η and ϕ are Schwartz functions. But $1-\eta$ is not a Schwartz function, so we need to check the well-definedness of $\langle 1-\eta, T^*\phi\rangle$. Express

$$\langle 1-\eta, T^*\phi\rangle = \int (1-\eta)(x) \left(\int \overline{K^*(x,y)\phi(y)} dy \right) dx.$$

Denote $r(B)$ to be the radius of B and x_0 to be the center of B. When $x \in \operatorname{supp}(1-\eta)$ and $y \in B$, (note $\operatorname{supp}(1-\eta) \cap \operatorname{supp}\phi = \varnothing$)

$$|x-y| \geq |x-x_0| - |y-x_0| > 3r(B) - r(B) = 2r(B) \geq 2|y-x_0|.$$

Hence for $0 < \epsilon \leq 1$, K Calderón-Zygmund kernel,

$$|K(y,x) - K(x_0,x)| \leq \frac{|y-x_0|^\epsilon}{|x-y|^{n+\epsilon}}.$$

On the other hand, since $\int \phi = 0$, we have

$$\left| \int \overline{K^*(x,y)\phi(y)} dy \right| = \left| \int (K(y,x) - K(x_0,x)) \overline{\phi(y)} dy \right| \lesssim \|\phi\|_\infty \int_B \frac{|y-x_0|^\epsilon}{|x-y|^{n+\epsilon}} dy,$$

hence using boundedness of η,

$$|\langle 1-\eta, T^*\phi\rangle| \lesssim \|\phi\|_\infty \int_{(3B)^c} \int_B \frac{|y-x_0|^\epsilon}{|x-y|^{n+\epsilon}} dy dx \lesssim \|\phi\|_\infty \cdot r(B)^n < \infty,$$

where we use a fact left as an exercise:

Exercise 15. Let $c(B)$ be the center of the ball B, then

$$\int_{(3B)^c} \int_B \frac{|y-c(B)|^\epsilon}{|x-y|^{n+\epsilon}} dy dx \lesssim r(B)^n.$$

One can also easily check the independence of choice of η. Hence $T1$ is well-defined.

Theorem 43 (**$T1$ Theorem, simple version**)**.** Let T be a SIO associated with a Calderón-Zygmund kernel K. Suppose that T satisfies WBP and $T1 = T^*1 = 0$, where 0 stands for the zero functional in $\mathcal{S}_0(\mathbb{R}^n)$. Then T extended to a bounded operator on $L^2(\mathbb{R}^n)$.

Before we prove this theorem, we need some convolution type operator. Choose $\phi \in C_c^\infty(\mathbb{R}^n)$ radial and $\int_{\mathbb{R}^n} \phi = 1$. From $\partial_j \hat{\phi}(\xi) = -2\pi i \int \phi(x) x_j e^{-2\pi i x\xi} dx$ for any $1 \leq j \leq n$ with $x = (x_1, \cdots, x_n)$, we have

$$\nabla \hat{\phi}(0) = 0.$$

Define for any $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathbf{P}_t f(x) = \phi_t * f(x),$$

where $\phi_t(x) = t^{-n}\phi(t^{-1}x)$ for any $x \in \mathbb{R}^n$.

Lemma 21. Suppose T satisfies WBP. For any $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle T\varphi, \psi\rangle = \lim_{t\to 0} \left\langle \mathbf{P}_t^2 T \mathbf{P}_t^2 \varphi, \psi \right\rangle,$$

where $\mathbf{P}_t^2 = \mathbf{P}_t \mathbf{P}_t$.

Proof. Suppose $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi$ are both in a ball of radius R. We can take t small enough so that $\mathbf{P}_t^2\varphi$ and $\mathbf{P}_t^2\psi$ are supported in a ball of radius $2R$. For any function $f \in C^1$ supported in a ball of radius $2R$, we set

$$\|f\| = \|f\|_\infty + R\|\nabla f\|_\infty.$$

Observe that

$$\left|\left\langle \mathbf{P}_t^2 T\mathbf{P}_t^2\varphi, \psi\right\rangle - \langle T\varphi, \psi\rangle\right| = \left|\left\langle T\mathbf{P}_t^2\varphi, \mathbf{P}_t^2\psi\right\rangle - \langle T\varphi, \psi\rangle\right|,$$

which is bounded by

$$\left|\left\langle T(\mathbf{P}_t^2\varphi - \varphi), \mathbf{P}_t^2\psi\right\rangle\right| + \left|\left\langle T\varphi, \mathbf{P}_t^2\psi - \psi\right\rangle\right| \lesssim R^n\left(\|\mathbf{P}_t^2\varphi - \varphi\| \cdot \|\mathbf{P}_t^2\psi\| + \|\varphi\| \cdot \|\mathbf{P}_t^2\psi - \psi\|\right).$$

By definition, we can see $\|\mathbf{P}_t^2\psi\| \le \|\psi\|$ since $\int \phi_t = \int \phi = 1$. On the other hand, by Hausdorff-Young inequality, we have

$$\|\mathbf{P}_t^2\varphi - \varphi\| \le \|\widehat{\mathbf{P}_t^2\varphi - \varphi}\|_1 + R\sum_{j=1}^n \|\xi_j(\widehat{\mathbf{P}_t^2\varphi - \varphi})(\xi)\|_1.$$

Note that

$$(\widehat{\mathbf{P}_t^2\varphi - \varphi})(\xi) = \left((\hat{\phi}(t\xi))^2 - 1\right)\hat{\varphi}(\xi),$$

from which, we see

$$\lim_{t\to 0}\|\widehat{\mathbf{P}_t^2\varphi - \varphi}\|_1 = \int_{\mathbb{R}^n}\lim_{t\to 0}\left|\left((\hat{\phi}(t\xi))^2 - 1\right)\right| \cdot |\hat{\varphi}(\xi)|d\xi = 0,$$

by DCT and $\hat{\phi}(0) = 1$. Similarly,

$$\lim_{t\to 0}\sum_{j=1}^n \|\xi_j(\widehat{\mathbf{P}_t^2\varphi - \varphi})(\xi)\|_1 = 0.$$

Hence, it follows that

$$\lim_{t\to 0}\|\mathbf{P}_t^2\varphi - \varphi\|_1\| = 0,$$

and this holds if we replace φ by ψ. So we've shown

$$\lim_{t\to 0}\left|\langle T\varphi, \psi\rangle - \lim_{t\to 0}\left\langle \mathbf{P}_t^2 T\mathbf{P}_t^2\varphi, \psi\right\rangle\right| = 0,$$

which conclude the proof. □

Lemma 22. Suppose T satisfies WBP. For any $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$,

$$\lim_{t\to\infty}\left\langle \mathbf{P}_t^2 T\mathbf{P}_t^2\varphi, \psi\right\rangle = 0.$$

The proof is similar to the previous one, so we will omit the details here. Now back to proof of Theorem 43. From Lemma 21 and Lemma 22, we see for any $\varphi, \psi \in C_c^\infty$,

$$\langle T\varphi, \psi\rangle = \lim_{\epsilon\to 0}\left\langle (\mathbf{P}_\epsilon^2 T\mathbf{P}_\epsilon^2\varphi - \mathbf{P}_{1/\epsilon}^2 T\mathbf{P}_{1/\epsilon}^2\varphi), \psi\right\rangle.$$

To prove T extends to a bounded operator on L^2, it is sufficient to prove

$$\lim_{\epsilon\to 0}\|\mathbf{P}_\epsilon^2 T\mathbf{P}_\epsilon^2\varphi - \mathbf{P}_{1/\epsilon}^2 T\mathbf{P}_{1/\epsilon}^2\varphi\|_2 \lesssim \|\varphi\|_2$$

for any $\varphi \in C_c^\infty$. By fundamental theorem of calculus, we can write

$$\mathbf{P}_\epsilon^2 T\mathbf{P}_\epsilon^2\varphi - \mathbf{P}_{1/\epsilon}^2 T\mathbf{P}_{1/\epsilon}^2\varphi = -\int_\epsilon^{1/\epsilon} \partial_t(\mathbf{P}_t^2 T\mathbf{P}_t^2\varphi)dt.$$

Define $\partial_t(\mathbf{P}_t^2)$ by

$$\partial_t(\mathbf{P}_t^2)f(x) = \partial_t(\phi_t * \phi_t) * f(x)$$

for any $f \in L^2$. By product rule,

$$\partial_t(\mathbf{P}_t^2 T\mathbf{P}_t^2\varphi) = \partial_t(\mathbf{P}_t^2)T\mathbf{P}_t^2\varphi + \mathbf{P}_t^2\partial_t(T\mathbf{P}_t^2)\varphi. \tag{1}$$

One can check two operators appear in the right side of (1) are adjoint to each other. Thus we only need to check the first term. It remains to prove for any $\varphi \in C_c^\infty$,

$$\lim_{\epsilon\to 0}\left\|\int_\epsilon^{1/\epsilon} \partial_t(\mathbf{P}_t^2)T\mathbf{P}_t^2\varphi dt\right\|_2 \lesssim \|\varphi\|_2. \tag{2}$$

Define for any $f \in L^2$,

$$\mathbf{Q}_t f(x) = t\partial_t(\mathbf{Q}_t^2)f(x).$$

Using such definition, (2) becomes

$$\lim_{\epsilon\to 0}\left\|\int_\epsilon^{1/\epsilon} \mathbf{Q}_t^2 T\mathbf{P}_t^2\varphi \frac{dt}{t}\right\|_2 \lesssim \|\varphi\|_2 \tag{3}$$

for any $\varphi \in C_c^\infty$. By Fourier transform, we have for $\xi \in \mathbb{R}^n$,

$$\widehat{\mathbf{Q}_t f}(\xi) = t\partial_t(\widehat{(\mathbf{P}_t^2) f}(\xi)) = t\partial_t(\hat{\phi}^2(\xi)\hat{f}(\xi)) = 2t\hat{\phi}(t\xi)\xi\cdot(\nabla\hat{\phi})(t\xi)\cdot\hat{f}(\xi).$$

Define the vector-valued functions

$$\Psi_t^{(1)}(x) = \frac{i}{\pi}t^{-n}(\nabla\phi)(t^{-1}x),$$
$$\Psi_t^{(2)}(x) = -2\pi i t^{-n}\phi(t^{-1}x)\cdot\frac{x}{t}.$$

For any vector-valued function $F = (f_1, \cdots, f_n)$, we define its Fourier transform by

$$\hat{F} = (\hat{f}_1, \cdots, \hat{f}_n),$$

then with this definition, we have

$$\widehat{\Psi_t^{(1)}}(\xi) = 2t\hat{\phi}(t\xi)\xi,$$
$$\widehat{\Psi_t^{(2)}}(\xi) = (\nabla\hat{\phi})(t\xi).$$

It is clear $\widehat{\mathbf{Q}_t f}(\xi) = \widehat{\Psi_t^{(1)}}(\xi)\widehat{\Psi_t^{(2)}}(\xi)\hat{f}(\xi)$. Denote

$$\mathbf{Q}_t^{(1)}f(x) = \Psi_t^{(1)} * f(x),$$
$$\mathbf{Q}_t^{(2)}f(x) = \Psi_t^{(2)} * f(x).$$

Recall that we have

$$\int \nabla\phi = \int x\phi(x) = 0,$$

then by Littlewood-Paley Theorem in continuous version, we have for any $f \in L^2$,

$$\left\| \left(\int_0^\infty |\mathbf{Q}_t^{(j)} f|^2 \frac{dt}{t} \right)^{1/2} \right\|_2 \lesssim \|f\|_2,$$

since LHS is a vector-valued Littlewood-Paley g-function, $j = 1$ or 2.

For two vector-valued functions F and G, we define

$$\langle F, G \rangle = \sum_{j=1}^n \langle f_j, g_j \rangle .$$

Then for $f, g \in L^2$, one can check that

$$\langle \mathbf{Q}_t f, g \rangle = \left\langle \mathbf{Q}_t^{(2)} f, \mathbf{Q}_t^{(1)} g \right\rangle .$$

To estimate (3), we consider for any $\varphi, \psi \in C_c^\infty$,

$$\left\langle \int_\epsilon^{1/\epsilon} \mathbf{Q}_t^2 T \mathbf{P}_t^2 \varphi \frac{dt}{t}, \psi \right\rangle = \int_\epsilon^{1/\epsilon} \left\langle \mathbf{Q}_t^2 T \mathbf{P}_t^2 \varphi, \psi \right\rangle \frac{dt}{t}.$$

using Fubini's Theorem since T satisfies WBP. Therefore,

$$\left\langle \int_\epsilon^{1/\epsilon} \mathbf{Q}_t^2 T \mathbf{P}_t^2 \varphi \frac{dt}{t}, \psi \right\rangle = \int_\epsilon^{1/\epsilon} \left\langle \mathbf{Q}_t^{(2)} T \mathbf{P}_t^2 \varphi, \mathbf{Q}_t^{(1)} \psi \right\rangle \frac{dt}{t},$$

and by Cauchy-Schwartz inequality, we can control it by

$$\begin{aligned} \cdots &\le \left\| \int_\epsilon^{1/\epsilon} |\mathbf{Q}_t^{(2)} T \mathbf{P}_t^2 \varphi|^2 \frac{dt}{t} \right\|_2 \cdot \left\| \int_\epsilon^{1/\epsilon} |\mathbf{Q}_t^{(1)} \psi|^2 \frac{dt}{t} \right\|_2 \\ &\le \left\| \int_\epsilon^{1/\epsilon} |\mathbf{Q}_t^{(2)} T \mathbf{P}_t^2 \varphi|^2 \frac{dt}{t} \right\|_2 \cdot \|\psi\|_2. \end{aligned}$$

Henceforth, we can reduce $T1$ theorem to the following lemma:

Lemma 23. There is a constant C independent of choice of ϵ such that

$$\int_{\mathbb{R}^n} \int_\epsilon^{1/\epsilon} |\mathbf{Q}_t^{(2)} T \mathbf{P}_t^2 \varphi|^2 \frac{dt}{t} dx \le C \|\varphi\|_2^2,$$

for any $\varphi \in C_c^\infty(\mathbb{R}^n)$.

In order to do this, we need some technical tools. Define $\mathbf{L}_t = \mathbf{Q}_t^{(2)} T \mathbf{P}_t$.

Lemma 24. The operator $\mathbf{L}_t$ is a SIO associated to a kernel L_t, which is a vector-valued function.

Proof. We need to find such L_t. Introduce the notation

$$\langle F, g\rangle = (\langle f_1, g\rangle, \cdots, \langle f_n, g\rangle),$$

where $F = (f_1, \cdots, f_n)$ is a vector-valued function and g is any function. Then one can express for any $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle \mathbf{L}_t \varphi, \psi\rangle = \int_{\mathbb{R}^n\times\mathbb{R}^n} L_t(x,y)\varphi(y)\psi(x)dxdy.$$

On the other hand,

$$\langle \mathbf{L}_t\varphi, \psi\rangle = \left\langle \mathbf{Q}_t^{(2)} T\mathbf{P}_t\varphi, \psi\right\rangle = \left\langle T\mathbf{P}_t\varphi, \mathbf{Q}_t^{(2)}\psi\right\rangle = \int_{\mathbb{R}^n\times\mathbb{R}^n} K(x,y)\mathbf{P}_t\varphi(y)\mathbf{Q}_t^{(2)}\psi(x)dxdy. \tag{4}$$

Define the notations

$$\begin{aligned} \phi_t^y(z) &= \phi_t(z-y), \\ \Psi_t^{(2),x}(z) &= \Psi_t^{(2)}(z-x), \end{aligned}$$

then one can check we can represent (4) by

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \left\langle T\phi_t^y(z), \Psi_t^{(2),x}(z)\right\rangle \varphi(y)\psi(x)dxdy,$$

via changing of variables and Fubini's Theorem. Hence we end up with a representation of L_t by

$$L_t(x,y) = \left\langle T\phi_t^y(z), \Psi_t^{(2),x}(z)\right\rangle.$$

□

Now we will show that this kernel L_t is bounded function for any $t \in (\epsilon, 1/\epsilon)$.

Lemma 25. There is a real number $\sigma \in (0,1]$ such that for any $x, y \in \mathbb{R}^n$,

$$|L_t(x,y)| \le \frac{Ct^\sigma}{(t+|x-y|)^{n+\sigma}},$$

where C is a constant independent of x, y and t.

Proof. WLOG, we assume ϕ is supported in a unit ball, centered at the origin. Then by definition, ϕ_t and $\Psi_t^{(2)}$ are supported in a ball, centered at the origin with radius t. We consider the two cases $|x-y| < 10t$ and $|x-y| \ge 10t$. In the first case, or $|x-y| < 10t$, we apply WBP to obtain

$$\begin{aligned} |L_t(x,y)| &= \left|\left\langle T\phi_t^y(z), \Psi_t^{(2),x}(z)\right\rangle\right| \\ &\lesssim t^n \left(\|\phi_t^y\|_\infty + R\|\nabla\phi_t^y\|_\infty\right)\left(\|\Psi_t^{(2),x}\|_\infty + R\|\nabla\Psi_t^{(2),x}\|_\infty\right). \end{aligned}$$

The translation factors x, y won't affect the norms, thus we can remove them. On the other hand, note that

$$\begin{aligned} \max\{\|\Psi_t^{(2),x}\|_\infty, \|\phi_t^y\|_\infty\} &\lesssim t^{-n}, \\ \max\{\|\nabla\Psi_t^{(2),x}\|_\infty, \|\nabla\phi_t^y\|_\infty\} &\lesssim t^{-n-1}. \end{aligned}$$

Therefore, we get

$$|L_t(x,y)| \lesssim t^{-n} \lesssim \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}}.$$

We now turn to the second case when $|x-y| \geq 10t$. By representation of L_t, we can write

$$L_t(x,y) = \int_{\mathbb{R}^n\times\mathbb{R}^n} K(u,v)\phi_t(v-y)\Psi_t^{(2)}(u-x)dudv.$$

The singular points occur when $u=v$, but we will see that in the second case, this can be excluded for free. From the supports of ϕ_t and $\Psi_t^{(2)}$, we can confine u,v to $|u-x|\leq t$ and $|v-y|\leq t$, and from which and the triangle inequality, we get

$$\begin{aligned}|u-v| &= |(u-x)+(x-y)+(y-v)| \\ &\geq |x-y|-|u-x|-|v-y| \geq |x-y|-2t \\ &\geq 10t-2t = 8t \geq 8|u-x|.\end{aligned}$$

Hence we can constraint to this part, and K is bounded in this case since it is a C-Z kernel. Henceforth, we can apply Fubini's Theorem to get

$$L_t(x,y) = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} K(u,v)\Psi_t^{(2)}(u-x)du\right)\phi_t(v-y)dv. \tag{5}$$

Observe that

$$\int_{\mathbb{R}^n}\Psi_t^{(2)}(x)dx = 0, \tag{6}$$

since $x\phi(x)$ is "odd". From which we know $\Psi_t^{(2)} \in \mathcal{S}_0(\mathbb{R}^n)$. So we can write (5) into

$$|L_t(x,y)| = \left|\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}[K(u,v)-K(x,v)]\Psi_t^{(2)}(u-x)du\right)\phi_t(v-y)dv\right|.$$

By property of C-Z kernel,

$$|K(u,v)-K(x,v)| \lesssim \frac{|u-x|^\sigma}{|u-v|^{n+\sigma}}.$$

So we obtain

$$\begin{aligned}|L_t(x,y)| &\lesssim \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|u-x|^\sigma}{|u-v|^{n+\sigma}}|\Psi_t^{(2)}(u-x)|du\right)|\phi_t(v-y)|dv \\ &\lesssim \int\int \frac{|u-x|^\sigma}{|u-v|^{n+\sigma}}\cdot\frac{1}{t^n}\cdot\frac{|u-x|}{t}\cdot|\phi(t^{-1}(u-x))|\cdot\frac{1}{t^n}\cdot|\phi(t^{-1}(v-y))|dudv.\end{aligned}$$

We use $B_R(z)$ to denote a ball centered at z of radius R. Since ϕ is a Schwartz function supported in $B_1(0)$, we can further dominate the integral in the last expression by

$$\begin{aligned}\cdots &\leq C_N\int_{B_t(y)}\int_{B_t(x)}\frac{|u-x|^\sigma}{|u-v|^{n+\sigma}}\cdot\frac{1}{t^n}\cdot\frac{|u-x|}{t}\cdot\frac{1}{(1+\frac{|u-x|}{t})^N}\cdot\frac{1}{t^n}\cdot\frac{1}{(1+\frac{|u-y|}{t})^N}dudv \\ &= \frac{C_N}{t^{2n+1}}\int_{B_t(y)}\int_{B_t(x)}\frac{|u-x|^{1+\sigma}}{|u-v|^{n+\sigma}}\cdot\frac{1}{(1+\frac{|u-x|}{t})^N}\cdot\frac{1}{(1+\frac{|u-y|}{t})^N}dudv \\ &= \frac{C_N}{t^{2n+1}}\cdot\frac{1}{t^{n+\sigma}}\int_{B_t(y)}\int_{B_t(x)}\frac{|u-x|^{1+\sigma}}{(\frac{|u-v|}{t})^{n+\sigma}}\cdot\frac{1}{(1+\frac{|u-x|}{t})^N}\cdot\frac{1}{(1+\frac{|u-y|}{t})^N}dudv \\ &\lesssim \frac{C_N}{t^{3n+1+\sigma}}\int_{B_t(y)}\int_{B_t(x)}\frac{|u-x|^{1+\sigma}}{(1+\frac{|u-v|}{t})^{n+\sigma}}\cdot\frac{1}{(1+\frac{|u-x|}{t})^N}\cdot\frac{1}{(1+\frac{|u-y|}{t})^N}dudv.\end{aligned}$$

Here in the last step, we use $|u-v|\geq 8t\geq 8|u-x|$, which allows us to add 1 to $|u-v|/t$. We take $N\geq n+\sigma$ and employ elementary inequality

$$\frac{1}{(1+|a|)}\cdot\frac{1}{(1+|b|)}\leq\frac{1}{1+|a-b|}$$

to get

$$\begin{aligned}|L_t(x,y)| &\lesssim \frac{1}{t^{3n+1+\sigma}\left(1+\frac{|x-y|}{t}\right)^{n+\sigma}}\int_{B_t(y)}\int_{B_t(x)}|u-x|^{1+\sigma}dudv\\ &\lesssim \frac{1}{t^{3n+1+\sigma}\left(1+\frac{|x-y|}{t}\right)^{n+\sigma}}\cdot t^{1+\sigma}|B_t(x)|\cdot|B_t(y)|\\ &\lesssim \frac{1}{t^{n}\left(1+\frac{|x-y|}{t}\right)^{n+\sigma}},\end{aligned}$$

hence we're done with the proof. □

Corollary 12. The integral

$$\int_{\mathbb{R}^n}L_t(x,y)dy$$

converges absolutely.

Let $f\in\mathcal{S}(\mathbb{R}^n)$, we can represent

$$\mathbf{L}_t f(x)=\int_{\mathbb{R}^n}L_t(x,y)f(y)dy,$$

and the Schwartz function f can be extended to bounded function, say $f=1$ identically. Indeed, $\mathbf{L}_t1$ can be defined as

$$\mathbf{L}_t1(x)=\int_{\mathbb{R}^n}L_t(x,y)dy.$$

For any $\varphi\in\mathcal{S}(\mathbb{R}^n)$,

$$\langle\mathbf{L}_t1,\psi\rangle=\left\langle\mathbf{Q}_t^{(2)}T\mathbf{P}_t1,\psi\right\rangle=\left\langle T1,\mathbf{Q}_t^{(2)}\psi\right\rangle,$$

where we use the fact that $\mathbf{P}_t1=1$ since $\int\phi=1$. Hence by (6), we have

$$\int_{\mathbb{R}^n}\mathbf{Q}_t^{(2)}\psi(x)dx=\int\Psi_t^{(2)}*\psi(x)dx=\left(\int\psi dx\right)\int\Psi_t^{(2)}(x)dx=0,$$

which yields that each component of the vector-valued function $\mathbf{Q}_t^{(2)}\psi(x)$ lies in $\mathcal{S}_0(\mathbb{R}^n)$. As assumed in Theorem 43, $T1=0$, which gives $\left\langle T1,\mathbf{Q}_t^{(2)}\psi\right\rangle=0$, leading to

$$\langle\mathbf{L}_t1,\varphi\rangle=0$$

for any $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Therefore, for a.e. x, we have

$$\mathbf{L}_t1(x)=\int_{\mathbb{R}^n}L_t(x,y)dy=0. \tag{7}$$

We're now ready to prove Lemma 23.

Proof of Lemma 23. Suffice to show for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\int_\epsilon^{1/\epsilon} |\mathbf{L}_t(\mathbf{P}_t\varphi)|^2 \frac{dt}{t} \le C\|\varphi\|_2,$$

where C is a constant independent of choice of ϵ. From (7), we see

$$\text{LHS} = \int_{\mathbb{R}^n}\int_\epsilon^{1/\epsilon} \left|\int L_t(x,y)\left(\mathbf{P}_t\varphi(y) - \mathbf{P}_t\varphi(x)\right)dy\right|^2 \frac{dt}{t}dx. \tag{8}$$

By Lemma 25, it follows that RHS is controlled by

$$\int_{\mathbb{R}^n}\int_\epsilon^{1/\epsilon}\left(\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}}\left|\mathbf{P}_t\varphi(y) - \mathbf{P}_t\varphi(x)\right|dy\right)^2\frac{dt}{t}dx,$$

which can be dominated by using Cauchy-Schwartz inequality,

$$\int_{\mathbb{R}^n}\int_0^\infty\left(\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}}dy\right)\left(\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}}\left|\mathbf{P}_t\varphi(y) - \mathbf{P}_t\varphi(x)\right|^2dy\right)\frac{dt}{t}dx. \tag{9}$$

Observe that by changing of variables, we have

$$\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}}dy = \int_{\mathbb{R}^n}\frac{1}{(1+|y|)^{n+\sigma}}dy = C_{n,\sigma},$$

which is a constant independent of ϵ and t. Hence (9) becomes

$$C_{n,\sigma}\int_{\mathbb{R}^n}\int_0^\infty\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}}\left|\mathbf{P}_t\varphi(y) - \mathbf{P}_t\varphi(x)\right|^2dy\frac{dt}{t}dx.$$

Changing variables $x \to u+y$, we have

$$C_{n,\sigma}\int_{\mathbb{R}^n}\int_0^\infty\int \frac{t^\sigma}{(t+|u|)^{n+\sigma}}\left|\mathbf{P}_t\varphi(y) - \mathbf{P}_t\varphi(u+y)\right|^2du\frac{dt}{t}dy.$$

Use Fubini's Theorem, it becomes

$$C_{n,\sigma}\int_{\mathbb{R}^n}\int_0^\infty \frac{t^\sigma}{(t+|u|)^{n+\sigma}}\left(\int\left|\mathbf{P}_t\varphi(y) - \mathbf{P}_t\varphi(u+y)\right|^2dy\right)\frac{dt}{t}du.$$

By Plancherel Theorem, one can represent the inner integral as

$$\int\left|\mathbf{P}_t\varphi(y) - \mathbf{P}_t\varphi(u+y)\right|^2dy = \int |e^{2\pi iu\xi}-1|^2\cdot|\hat{\phi}(t\xi)|^2\cdot|\hat{\varphi}(\xi)|^2d\xi.$$

So we see (8) is dominated by

$$C_{n,\sigma}\int|\hat{\varphi}(\xi)|^2\left(\int_{\mathbb{R}^n}\int_0^\infty\frac{t^\sigma}{(t+|u|)^{n+\sigma}}\cdot|e^{2\pi iu\xi}-1|^2\cdot|\hat{\phi}(t\xi)|^2\frac{dt}{t}du\right)d\xi. \tag{10}$$

To finish the proof, it suffices to prove the integral in the parenthesis in (10) is bounded by a constant independent of ξ. This is true by the following lemma, and we're done! □

All we need is to prove the following lemma:

Lemma 26. There is a constant C independent of ξ s.t.

$$\int_{\mathbb{R}^n}\int_0^\infty\frac{t^\sigma}{(t+|u|)^{n+\sigma}}\cdot|e^{2\pi iu\xi}-1|^2\cdot|\hat{\phi}(t\xi)|^2\frac{dt}{t}du \le C.$$

Proof. Assume $\xi \neq 0$. Otherwise it is trivial because $e^{2\pi i u 0} - 1 = 0$. Let $0 < \delta < \sigma$, for instance, δ can be chosen as $\sigma/2$ because $\sigma \in (0, 1]$. Recall the elementary inequality

$$|e^{i\theta} - 1| \leq 2|\theta|^{\epsilon}$$

for any $\theta \in \mathbb{R}^n$ and any $\epsilon \in [0, 1]$. Using the inequality with $\epsilon = \delta/2$, we bounded LHS by

$$8\pi \int_{\mathbb{R}^n} \int_0^{\infty} \frac{t^{\sigma}}{(t + |u|)^{n+\sigma}} \cdot |u \cdot \xi|^{\delta} \cdot |\hat{\phi}(t\xi)|^2 \frac{dt}{t} du. \tag{11}$$

Since $\hat{\phi}$ is radial, $\hat{\phi}(t\xi) = \hat{\phi}(t|\xi|)$, (11) can be estimated by

$$\begin{aligned}
&8\pi \int_{\mathbb{R}^n} \int_0^{\infty} \frac{t^{\sigma}}{(t + |u|)^{n+\sigma}} \cdot |u|^{\delta} \cdot |\xi|^{\delta} \cdot |\hat{\phi}(t|\xi|)|^2 \frac{dt}{t} du \\
&= 8\pi \int_{\mathbb{R}^n} \int_0^{\infty} \frac{t^{\sigma}}{(t + t|u|)^{n+\sigma}} \cdot t^{\delta}|u|^{\delta} \cdot |\xi|^{\delta} \cdot |\hat{\phi}(t|\xi|)|^2 t^n \frac{dt}{t} du && \text{(change variables } u \to tu) \\
&= \left(\int_{\mathbb{R}^n} \frac{8\pi u^{\delta}}{(1 + |u|)^{n+\delta}} du \right) \left(\int_0^{\infty} (t|\xi|)^{\delta} |\hat{\phi}(t|\xi|)|^2 \frac{dt}{t} \right) && \text{(Fubini)} \\
&= \left(\int_{\mathbb{R}^n} \frac{8\pi u^{\delta}}{(1 + |u|)^{n+\delta}} du \right) \left(\int_0^{\infty} t^{\delta} |\hat{\phi}(t)|^2 \frac{dt}{t} \right) && \text{(change variables } t \to t/|\xi|) \\
&= C_{\delta,n}.
\end{aligned}$$

This finishes the proof. □

So far, the proof of $T1$ Theorem in simple version (Theorem 43), is completed.

17 BMO (Bounded Mean Oscillation) and Sharp Function

Let $f \in L^1_{loc}(\mathbb{R}^n)$ and Q be a cube in $\mathbb{R}^n$. Denote

$$f_Q = \frac{1}{|Q|}\int_Q f(x)dx.$$

Definition 30. The **BMO (Bounded Mean Oscillation)** space is the collection of local integrable functions of bounded mean oscillation, that is,

$$BMO(R^n) = \{f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_* < \infty\},$$

where the **BMO-norm** $\|f\|_*$ (or $\|f\|_{BMO}$) is defined by

$$\|f\|_* = \sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx.$$

Note 6. It is clear that $L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. If f is constant, then $\|f\|_* = 0$. One can view BMO space as the quotient of the BMO space by the space of constant functions. Moreover, using triangle inequality, it is not difficult to show

$$\|f\|_* \sim \sup_Q \inf_{c\in\mathbb{C}} \frac{1}{|Q|}\int_Q |f(x) - c|dx.$$

Theorem 44 (**John-Nirenberg**)**.** There exists positive constants C_1 and C_2 depending only on the dimension n, such that for any $f \in BMO(\mathbb{R}^n)$, any cube Q in $\mathbb{R}^n$, and any $\lambda > 0$,

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \leq C_1 e^{-C_2\lambda/\|f\|_*}|Q|.$$

In order to prove the theorem, we need a few lemmas.

Lemma 27. Let Q be a cube and $\lambda > 0$. Suppose that $f \in L^1(Q)$ and

$$\frac{1}{|Q|}\int_Q |f(x)|dx < \lambda.$$

Then there exists a sequence $\{Q_j\}$ of pairwise disjoint subcubes of Q such that

1. $|f(x)| \leq \lambda$ a.e. on $Q\backslash(\cup_j Q_j)$,
2. $\lambda \leq \frac{1}{|Q_j|}\int_{Q_j} |f(x)|dx < 2^n\lambda$ for any subcubes Q_j.

This lemma can be proved in a similar way as we did in the proof of Calderón-Zygmund decomposition (Lemma 11). From this lemma, we can derive a handy result for BMO functions.

Lemma 28. Suppose that $f \in BMO(\mathbb{R}^n)$ with $\|f\|_* = 1$. Let Q be any cube in $\mathbb{R}^n$. There exists a sequence $\{Q_j\}$ of pairwise disjoint subcubes of Q such that

$$|f(x) - f_Q| \leq \frac{3}{2} \tag{1}$$

a.e. on $Q\backslash(\cup_j Q_j)$, and

$$\sum_j |Q_j| \leq \frac{2}{3}|Q|, \tag{2}$$

and also

$$\frac{1}{|Q_j|}\int_{Q_j} |f(x) - f_Q|dx < 3 \cdot 2^{n-1}. \tag{3}$$

Proof. Note that
$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \le \|f\|_* = 1 \le \frac{3}{2},$$
so we can apply Lemma 27 for $\lambda = 3/2$ to function $f(x)-f_Q$. Then we get a disjoint subcubes Q_j satisfying (1), and for any Q_j,
$$\frac{3}{2} \le \frac{1}{|Q_j|}\int_{Q_j} |f(x)-f_Q|dx < \frac{3}{2}\cdot 2^n,$$
which leads to (3). Plus,
$$|Q_j| \le \frac{2}{3}\int_Q |f(x)-f_Q|dx.$$
Summing up all j, we end up with
$$\sum |Q_j| \le \frac{2}{3}\int_{Q_j} |f(x)-f_Q|dx \le \frac{2}{3}|Q|\cdot\|f\|_* = \frac{2}{3}|Q|,$$
which yields (2). $\square$

Proof of Theorem 44. By rescaling λ in Theorem 44, we can assume $\|f\|_* = 1$. It suffices to prove for any $\lambda > 0$,
$$|\{x \in Q : |f(x)-f_Q| > \lambda\}| \le C_1 e^{-C_2\lambda}|Q|.$$
Applying Lemma 28 for the given Q, we get a sequence $\{Q_j^{(1)}\}$ of subcubes of Q s.t. $|f(x)-f_Q| \le \frac{3}{2}$ for a.e. $x \in Q\backslash(\cup_j Q_j^{(1)})$, $\sum_j |Q_j^{(1)}| \le \frac{2}{3}|Q|$ and $\frac{1}{|Q_j^{(1)}|}\int_{Q_j^{(1)}} |f(x)-f_Q|dx < 3\cdot 2^{n-1}$. Let $\mathcal{Q}^{(1)}$ be the set of all cubes in the sequence $\{Q_j^{(1)}\}$.

Now, we apply Lemma 28 to each cube $Q_j^{(1)}$ in $\mathcal{Q}^{(1)}$. Then again we obtain a sequence $\{Q_j^{(2)}\}$ of subcubes $Q^{(1)}$ s.t. $|f(x)-f_{Q^{(1)}}| \le \frac{3}{2}$ for a.e. $x \in Q^{(1)}\backslash(\cup_{j\in\mathcal{J}(Q^{(1)})}Q_j^{(2)})$, where $\mathcal{J}(Q^{(1)}) = \{j : Q_j^{(2)} \subset Q^{(1)}\}$, and $\sum_{j\in\mathcal{J}(Q^{(1)})} |Q_j^{(2)}| \le \frac{2}{3}|Q^{(1)}|$ and $\frac{1}{|Q_j^{(2)}|}\int_{Q_j^{(2)}} |f(x)-f_{Q^{(1)}}|dx < 3\cdot 2^{n-1}$.

We consider all cubes generated in the second stage. Set
$$\bigcup_j Q_j^{(2)} = \bigcup_{Q^{(1)}}\bigcup_{j\in\mathcal{J}(Q^{(1)})} Q_j^{(2)},$$
and
$$\sum_j |Q_j^{(2)}| = \sum_{Q^{(1)}}\sum_{j\in\mathcal{J}(Q^{(1)})} |Q_j^{(2)}|.$$
For $x \in j \in \mathcal{J}(Q^{(1)})$, we see that, from setting if $x \notin Q^{(1)}$, then
$$|f(x)-f_Q| \le \frac{3}{2}.$$
If $x \in j \in \mathcal{J}(Q^{(1)})$, and x belongs to some $Q^{(1)}$ from the first stage,
$$|f(x)-f_Q| \le |f(x)-f_{Q^{(1)}}| + |f_{Q^{(1)}} - f_Q| \le |f(x)-f_{Q^{(1)}}| + \frac{1}{|Q^{(1)}|}\int_{Q^{(1)}} |f-f_Q|,$$
which leads to
$$|f(x)-f_Q| \le \frac{3}{2} + 3\cdot 2^{n-1}$$

a.e. on $Q\backslash(\cup_j Q_j^{(2)})$. On the other hand, we have

$$\sum_j |Q_j^{(2)}| \leq \frac{2}{3}\sum_{Q^{(1)}} |Q^{(1)}| \leq \left(\frac{2}{3}\right)^2 |Q|.$$

Iterating the process described above, at N-th stage, we get a collection $\{Q_j^{(N)}\}$, each of which is a subcube of Q, such that

$$|f(x) - f_Q| \leq \frac{3}{2} + 3(N-1)\cdot 2^{n-1} \leq 3N\cdot 2^{n-1}$$

for a.e. $x \in Q\backslash(\cup_j Q_j^{(N)})$, and also

$$\sum_j |Q_j^{(N)}| \leq \left(\frac{2}{3}\right)^N |Q|.$$

For any $\lambda \geq 3\cdot 2^{n-1}$, there exists $N \in \mathbb{N}$ such that $3N\cdot 2^{n-1} \leq \lambda \leq 3(N+1)\cdot 2^{n-1}$. Thus we see that for $\lambda \geq 3\cdot 2^{n-1}$,

$$\begin{aligned}|\{x \in Q : |f(x) - f_Q| > \lambda\}| &= |\{x \in \cup_j Q_j^{(N)} : |f(x) - f_Q| > \lambda\}| \\ &\leq \sum_j |Q_j^{(N)}| \leq \left(\frac{2}{3}\right)^N |Q| \leq e^{-C_2\lambda}|Q|,\end{aligned}$$

where $C_2 = \frac{\log\frac{3}{2}}{3\cdot 2^{n-1}}$. For $\lambda < 3\cdot 2^{n-1}$, we have

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \leq |Q| = e^{C_2\lambda}e^{-C_2\lambda}|Q| \leq e^{3C_2\cdot 2^{n-1}}e^{-C_2\lambda}|Q|.$$

Let $C_1 = 3/2$, we complete the proof of the theorem. □

Corollary 13. For any $1 < p < \infty$, let

$$\|f\|_{*,p} = \sup_Q \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx\right)^{1/p}.$$

Then the norm $\|\cdot\|_{*,p}$ on BMO is a norm equivalent to $\|\cdot\|_*$.

Proof. By Hölder's inequality, it suffices to check $\|f\|_{*,p} \leq C_p\|f\|_*$ for $f \in BMO(\mathbb{R}^n)$. By John-Nirenberg Theorem (Theorem 44), we get

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx &= \frac{p}{|Q|}\int_0^\infty \lambda^{p-1}|\{x \in Q : |f(x) - f_Q| > \lambda\}|d\lambda \\ &\leq pC_1\int_0^\infty \lambda^{p-1}e^{-C_2\lambda/\|f\|_*}d\lambda \\ &= \frac{pC_1}{C_2^p}\|f\|_*^p\int_0^\infty \lambda^{p-1}e^{-\lambda}d\lambda \qquad \text{(change variables } \lambda \to \|f\|_*\lambda/C_2\text{)} \\ &= \frac{pC_1\Gamma(p)}{C_2^p}\|f\|_*^p.\end{aligned}$$

□

We now turn to the sharp function.

Definition 31. For any $f \in L^1_{loc}(\mathbb{R}^n)$, we define the **sharp function** of f by

$$f^{\#}(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where the sup is taken over all cubes containing x.

By definition, it is clear that sharp function is closely related to the BMO space. In fact, it is easy to verify $\|f\|_* = \|f^{\#}\|_\infty$. Also it is clear that for $1 < p \leq \infty$, $\|f^{\#}\|_p \leq C_p \|f\|_p$, simply following from $f^{\#}(x) \lesssim Mf(x)$, and Hardy-Littlewood maximal function is bounded on L^p. We will also see that if $f \in L^p$, $\|f\|_p \leq C_p \|f^{\#}\|_p$. To see that, we need a good-λ inequality of the sharp function.

Definition 32. Let

$$\mathcal{D}_k = \left\{ \prod_{j=1}^{n} [2^{-k} n_j, 2^{-k}(n_j + 1)) : \text{each } n_j \in \mathbb{Z} \right\},$$

which is a family of cubes, open on the right, whose vertices are adjacent points of the lattice $(2^{-k}\mathbb{Z})^n$. A cube in $\cup_{k \in \mathbb{Z}} \mathcal{D}_k := \mathcal{D}$ is called a **dyadic cube**. The family of all dyadic cubes, $\mathcal{D}$, satisfies so-called **grid structure**, that is, any two dyadic cubes are either disjoint or one is contained in the other.

Define the **dyadic maximal function** $M_d f$ of f by

$$M_d f(x) = \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the sup is taken over all dyadic cubes containing x. We shall note that $M_d f(x) \leq C f^{\#}(x)$ does not hold pointwise.

Lemma 29 (**Good-λ Inequality**)**.** For $f \in L^1_{loc}(\mathbb{R}^n)$, and for any $\gamma > 0$ and any $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, f^{\#}(x) \leq \gamma\lambda\}| \leq 2^n \gamma |\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|.$$

Proof. When $\gamma \geq 2^{-n}$, the result is trivial. Suppose γ is small. By Calderón–Zygmund decomposition of f at level λ, the set $\{x : M_d f(x) > \lambda\}$ can be represented as a union of disjoint maximal dyadic cubes. Thus it suffices to show

$$|\{x \in Q : M_d f(x) > 2\lambda, f^{\#}(x) \leq \gamma\lambda\}| \leq 2^n \gamma \cdot |Q| \tag{4}$$

for any maximal dyadic cube Q in $\{x : M_d f(x) > \lambda\}$. For such a cube Q, if $x \in Q$ and $M_d f(x) > 2\lambda$, then

$$M_d(f\chi_Q)(x) > 2\lambda.$$

Exercise 16. Check this inequality.

Use Q^* to denote the unique dyadic cube containing Q, whose side length is twice as much as that of Q. The cube Q^* is called the parent of Q. By maximality of Q, we have

$$\frac{1}{|Q^*|} \int_{Q^*} |f(x)| dx \leq \lambda.$$

Henceforth, we see

$$M_d(f_Q \cdot \chi_Q)(x) \leq \frac{1}{|Q^*|} \int_{Q^*} |f(x)| dx \leq \lambda,$$

since $M_d(\chi_Q) \leq 1$ and Exercise 16. By triangle inequality, we get

$$M_d((f - \chi_{Q^*})\chi_Q) \geq M_d(f\chi_Q) - M_d(f_{Q^*} \cdot \chi_Q) \geq M_d(f\chi_Q) - \lambda > \lambda.$$

From the discussion above,

$$\{x \in Q : M_d f(x) > 2\lambda, f^\#(x) \leq \gamma\lambda\} \subset \{x \in Q : M_d(f_Q \cdot \chi_Q)(x) \geq \lambda\}. \tag{5}$$

It is not difficult to see that for any $f \in L^1$ and $\lambda > 0$, from C-Z decomposition,

$$|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| \leq \frac{\|f\|_1}{\lambda}.$$

Using weak (1,1) estimate, we have

$$\begin{aligned}
|\{x \in Q : M_d((f - \chi_{Q^*})\chi_Q)(x) > \lambda\}| &\leq \frac{\int_Q |f - f_{Q^*}|}{\lambda} \\
&\leq \frac{2^n|Q|}{\lambda} \cdot \frac{1}{|Q^*|} \int_{Q^*} |f - f_{Q^*}| \\
&\leq \frac{2^n|Q|}{\lambda} \inf_{x \in Q^*} f^\#(x) \leq \frac{2^n|Q|}{\lambda} \inf_{x \in Q} f^\#(x).
\end{aligned}$$

Assume that $\{x \in Q : f^\#(x) \leq \gamma\lambda\} \neq \varnothing$, otherwise the result is trivial. Under the assumption, we have

$$|\{x \in Q : M_d((f - \chi_{Q^*})\chi_Q)(x) > \lambda\}| \leq 2^n\gamma \cdot |Q|. \tag{6}$$

Now (4) follows from (5) and (6) immediately, hence the lemma follows. $\square$

Remark 12. This lemma is a special case of Cotlar-Stein Lemma, which we will present in Lecture 20.

Theorem 45. Let $1 \leq p < \infty$. Suppose that $f \in L^{p_0}$ for some $p_0 \in [1, p]$. Then there exists a constant $C_{p,n}$ independent of f such that

$$\|M_d f\|_p \leq C_{p,n} \|f^\#\|_p.$$

Proof. For any $N > 0$, let

$$I_N = \int_0^N p\lambda^{p-1} |\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| d\lambda.$$

For $f \in L^{p_0}$, we have

$$I_N \leq \frac{p}{p_0} N^{p-p_0} \int_0^N p_0 \lambda^{p_0-1} |\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| d\lambda \leq \frac{p}{p_0} N^{p-p_0} \|f\|_{p_0}^{p_0} < \infty.$$

Thus I_N is a real number. Furthermore,

$$\begin{aligned}
I_N &= 2^p \int_0^{N/2} p\lambda^{p-1} |\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda\}| d\lambda \qquad \text{(Change variables } \lambda \to 2\lambda) \\
&\le 2^p \int_0^{N/2} p\lambda^{p-1} |\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, f^{\#}(x) \le \gamma\lambda\}| d\lambda \\
&\quad + 2^p \int_0^{N/2} p\lambda^{p-1} |\{x \in \mathbb{R}^n : f^{\#}(x) > \gamma\lambda\}| d\lambda \\
&\le 2^{p+n}\gamma \int_0^{N/2} p\lambda^{p-1} |\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| d\lambda + \frac{2^p}{\gamma^p} \int_0^{\infty} p\lambda^{p-1} |\{x \in \mathbb{R}^n : f^{\#}(x) > \lambda\}| d\lambda \\
&\le 2^{p+n}\gamma I_N + \frac{2^p}{\gamma^p} \|f^{\#}\|_p^p,
\end{aligned}$$

which implies

$$(1 - 2^{p+n}\gamma) I_N \le \frac{2^p}{\gamma^p} \|f^{\#}\|_p^p.$$

We can take γ satisfying $1 - 2^{p+n}\gamma = 1/2$. Then we obtain

$$I_N \le 2^{p^2+(n+2)p+1} \|f^{\#}\|_p^p.$$

Letting $N \to \infty$ we obtained the desired result. $\square$

One can obtain an interpolation result from Theorem 45, from which we see that BMO space is a good substitute for L^∞ space.

Theorem 46. Let T be a linear operator which is bounded on L^{p_0} for some $p_0 \in (1, \infty)$. Suppose that T is also bounded from L^∞ to BMO. Then for any $p \in (p_0, \infty)$, T is bounded on L^p.

Proof. Define $T^{\#}$ by

$$T^{\#} f(x) = (Tf)^{\#}(x).$$

Then $T^{\#}$ is a sublinear operator. For any $f \in L^{p_0}$, we have, by $\|f^{\#}\|_p \le C_p \|f\|_p$,

$$\|T^{\#} f\|_{p_0} = \|(Tf)^{\#}\|_{p_0} \le C_{p_0} \|Tf\|_{p_0} \lesssim \|f\|_{p_0},$$

which shows that $T^{\#}$ is bounded on L^{p_0}. On the other hand, since T is bounded from L^∞ to BMO, we get

$$\|T^{\#} f\|_\infty = \|(Tf)^{\#}\|_\infty = \|Tf\|_* \lesssim \|f\|_\infty,$$

which yields the L^∞-boundedness of $T^{\#}$. By Marcinkiewicz Interpolation Theorem (Theorem 2), we see that $T^{\#}$ is bounded on L^p for any $p \in (p_0, \infty)$. Henceforth, for any $f \in L^p$ with $p \in (p_0, \infty)$, as a consequence of Theorem 45, it follows that

$$\|Tf\|_p \le \|(Tf)^{\#}\|_p \lesssim \|f\|_p.$$

Therefore the proof is finished. $\square$

18 Carlesen Measures

Let $\mathbb{R}^{n+1}_+$ be the upper half plane $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\}$. Let μ be a non-negative Borel measure on $\mathbb{R}^{n+1}_+$, and for any cube Q, we define the Carlesen box of Q by

$$\hat{Q} = \{(x,t) \in \mathbb{R}^{n+1}_+ : x \in Q, 0 \le t \le l(Q)\},$$

where $l(Q)$ is the side length of Q.

Definition 33. If the measure μ satisfying that for any cube $Q \subset \mathbb{R}^n$,

$$\mu(\hat{Q}) \le C|Q|,$$

where C is an absolute constant independent of Q, then μ is called a **Carlesen measure**. For any such measure, we denote its norm by

$$\|\mu\| = \sup_Q \frac{\mu(\hat{Q})}{|Q|}.$$

Definition 34. Denote $\Gamma(x) = \{(y,t) \in \mathbb{R}^n_+ : |y - x| < t\}$ be a **cone** in $\mathbb{R}^n_+$. We define that for any measurable function f on $\mathbb{R}^n_+$, the **non-tangential maximal function** by

$$\mathcal{N}^* f(x) = \sup_{(y,t)\in\Gamma(x)} |f(y,t)|,$$

where $x \in \mathbb{R}^n$.

We begin with a well-known theorem for Carlesen measure:

Theorem 47. Let f be a continuous function on $\mathbb{R}^n_+$, and μ be a Carlesen measure. Then for any $0 < p < \infty$, we have

$$\int_{\mathbb{R}^n_+} |f(x,t)|^p d\mu \le C\|\mu\| \int_{\mathbb{R}^n} |\mathcal{N}^* f(x)|^p dx. \tag{1}$$

This can also be represented as

$$\|f\|_{L^p(\mathbb{R}^n_+, d\mu)} \lesssim \|\mu\|^{1/p} \|\mathcal{N}^* f\|_{L^p(\mathbb{R}^n)}.$$

To prove this theorem, we need the following theorem:

Theorem 48 (Whitney decomposition). Let Ω be an open set in $\mathbb{R}^n$. Suppose that the complement Ω^c is not empty. Then there is a non-overlapping collection of cubes $\{Q_j\}$ such that

$$\Omega = \bigcup_j Q_j,$$

and

$$C_1 l(Q_j) \le \operatorname{dist}(Q_j, \Omega^c) \le C_2 l(Q_j),$$

where C_1 and C_2 are constants independent of Q_j.

Proof. Recall that for any $k \in \mathbb{Z}$,

$$\mathcal{D}_k = \left\{ \prod_{j=1}^{n} [2^{-k} n_j, 2^{-k}(n_j + 1)) : \text{each } n_j \in \mathbb{Z} \right\}$$

gives a collection of all dyadic cubes of side length 2^{-k}. For given $k \in \mathbb{Z}$, we set

$$\Omega_k = \{ x \in \Omega : 3\sqrt{n} \cdot 2^{-k} < \operatorname{dist}(x, \Omega^c) \leq 3\sqrt{n} \cdot 2^{-k+1} \}.$$

Let $\mathbf{Q}_k$ be defined by

$$\mathbf{Q}_k = \{ Q \in \mathcal{D}_k : Q \cap \Omega_k \neq \varnothing \},$$

and

$$\mathbf{Q} = \bigcup_k \mathbf{Q}_k.$$

We shall see that Ω can be represented as a union of those cubes in $\mathbf{Q}$, that is,

$$\Omega = \bigcup_{Q \in \mathbf{Q}} Q.$$

Indeed, from the definition of $\mathbf{Q}_k$, it follows that $Q \in \mathcal{D}_k$ touching Ω_k does not contain any point of Ω^c, and thus such a dyadic cube Q is contained in Ω. Henceforth, every $Q \in \mathbf{Q}$ is contained in Ω, i.e.

$$\bigcup_{Q \in \mathbf{Q}} Q \subset \Omega.$$

On the other hand, $\Omega = \cup_k \Omega_k$ and Ω_k is covered by those Q's in $\mathbf{Q}_k$, that is,

$$\Omega = \bigcup_k \Omega_k \subset \bigcup_k \bigcup_{Q \in \mathbf{Q}_k} Q = \bigcup_{Q \in \mathbf{Q}} Q.$$

Then the claim follows. Now prove the other part of theorem. For any $Q \in \mathbf{Q}$ of side length 2^{-k}, from the definition of $\mathbf{Q}_k$, we get that there is a point $x \in Q$ satisfying

$$3\sqrt{n} \cdot 2^{-k} < \operatorname{dist}(x, \Omega^c) \leq 3\sqrt{n} \cdot 2^{-k+1},$$

which implies

$$2\sqrt{n} \cdot l(Q) < \operatorname{dist}(Q, \Omega^c) \leq 6\sqrt{n} \cdot l(Q). \tag{2}$$

Here we use the triangle inequality $\operatorname{dist}(x, \Omega^c) \leq \operatorname{diam}(Q) + \operatorname{dist}(Q, \Omega^c)$. Notice that those cubes in $\mathbf{Q}$ may not be mutually disjoint. We let $\mathbf{Q}^*$ denote the collection of maximal dyadic cubes in $\mathbf{Q}$. Then write

$$\Omega = \bigcup_{Q \in \mathbf{Q}^*} Q,$$

which leads to a Whitney decomposition because $\mathbf{Q}^*$ is a family of disjoint dyadic cubes and hence the desired inequality follows from (2). $\square$

Return to Theorem 47. Let

$$E_\lambda = \{(x,t) \in \mathbb{R}^{n+1}_+ : |f(x,t)| > \lambda\},$$

and

$$E^*_\lambda = \{x \in \mathbb{R}^n : \mathcal{N}^* f(x) > \lambda\}.$$

Observe that Theorem 47 follows from

$$\mu(E_\lambda) \lesssim \|\mu\| \cdot |E^*_\lambda|. \tag{3}$$

In fact, notice that

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} |f(x,t)|^p d\mu &= p\int_0^\infty \lambda^{p-1}\mu(E_\lambda)d\lambda \\ &\lesssim \|\mu\| \cdot p\int_0^\infty \lambda^{p-1}|E^*_\lambda|d\lambda \\ &\lesssim \|\mu\| \int_{\mathbb{R}^n} |\mathcal{N}^* f(x)|^p dx,\end{aligned}$$

which is exactly the inequality (1). So it suffices to prove (3), when μ is a Carlesen measure. To do that, we can assume that the open set E^*_λ has a finite Lebesgue measure so that its complement is not empty. Then we can apply Whitney decomposition to represent

$$E^*_\lambda = \cup_j Q_j,$$

where Q_j's are mutually disjoint dyadic cubes in $\mathbb{R}^n$, satisfying

$$C_1 l(Q_j) \le \operatorname{dist}(Q_j, (E^*_\lambda)^c) \le C_2 l(Q_j). \tag{4}$$

We need a technical lemma to finish the proof:

Lemma 30. There is an absolute constant α such that

$$E_\lambda \subset \bigcup_j \widehat{\alpha Q_j},$$

where αQ_j stands for a dilation of Q_j by the constant α.

Proof. For any ball (or cube) B in $\mathbb{R}^n$, a **tent** based on B is given by $T(B) = \{(y,t) \in \mathbb{R}^{n+1}_+ : B(y,t) \subset B\}$, where $B(y,t)$ stands for a ball in $\mathbb{R}^n$ centered at $y \in \mathbb{R}^n$ and of radius t. Let $(y,t) \in E_\lambda$. We claim that $B(y,t) \subset E^*_\lambda$. To see why this claim is true, we observe first that for any $x \in B(y,t)$, $(y,t) \in \Gamma(x)$. This is because by definition of cone, $|y-x| < t$. Hence

$$\mathcal{N}^* f(x) = \sup_{(y',t)\in\Gamma(x)} |f(y',t)| \ge |f(y,t)| > \lambda.$$

This means that any point in the ball $B(y,t)$ belongs to E^*_λ, which leads to the claim.

Let $\alpha = 100C_2$, where C_2 is the constant in (4). We prove that

$$E_\lambda \subset \bigcup_j T(\alpha Q_j), \tag{5}$$

from which the Lemma follows. From the fact $B(y,t) \subset E_\lambda^*$ and Whitney decomposition for E_λ^*, we see that for any $(y,t) \in \mathbb{E}_\lambda$,

$$B(y,t) \subset \bigcup_j Q_j, \tag{6}$$

where Q_j satisfies (4). Hence there exists a dyadic cube Q_j such that $Q_j \cap B(y,t) \neq \varnothing$ and Q_j satisfies (4). We shall consider the magnitude of those Q_j's, which touch $B(y,t)$, comparing the size of the ball $B(y,t)$. Then we run into only two cases: every Q_j touching $B(y,t)$ is of side length smaller than $4t/\alpha$ or at least one of Q_j's touching $B(y,t)$ has its side length $\geq 4t/\alpha$. We will see that the first case cannot occur since α is much larger than C_2. The last claim can be proved by contradiction. Assume that $l(Q_j) < 4t/\alpha$ for every Q_j obeying $Q_j \cap B(y,t) \neq \varnothing$. Then (6) and the assumption yields that y, the center of the ball $B(y,t)$, is contained in a dyadic cube $Q_j \subset \mathbb{R}^n$ whose side length $l(Q_j) < 4t/\alpha$. Thus we see that

$$8C_2 Q_j \subset B(y,t). \tag{7}$$

On the other hand, (4) tells that

$$\operatorname{dist}(Q_j, (E_\lambda^*)^c) \leq C_2 l(Q_j),$$

which, combined with (7), yields

$$B(y,t) \cap E_\lambda^*)^c \neq \varnothing,$$

which contradicts to the fact $B(y,t) \subset E_\lambda^*$ since $E_\lambda^*)^c \cap E_\lambda^*) = \varnothing$. Hence, the second case must occur, i.e. there exists a dyadic cube Q_j such that $Q_j \cap B(y,t) \neq \varnothing$ and $l(Q_j) \geq 4t/\alpha$. Thus we have

$$B(y,t) \subset \alpha Q_j,$$

which implies that for any $(y,t) \in E_\lambda$,

$$(y,t) \in T(B(y,t)) \subset T(\alpha Q_j).$$

Therefore, we obtain (5), and the lemma follows. □

Proof of Theorem 47. Given Lemma 30, we get

$$\mu(E_\lambda) \leq \mu(\cup_j \widehat{\alpha Q_j}) \leq \sum_j \mu(\widehat{\alpha Q_j}) \lesssim \|\mu\| \sum_j |Q_j| \lesssim \|\mu\| \cdot |E_\lambda^*|,$$

hence the proof is done. □

The other part of the lecture is to construct a Carlesen measure. We can use a BMO function to generate one. Let $b \in BMO(\mathbb{R}^n)$ and

$$Q_t b(x) = \psi_t * b(x),$$

where $\psi_t(x) = t^{-n}\psi(x/t)$ and ψ is a radial function obeying $\int \psi = 0$ with

$$|\psi(x)| + |\nabla\psi(x)| \leq \frac{C}{(1+|x|)^{n+\epsilon}} \tag{8}$$

for some $\epsilon > 0$. For any Lebesgue measurable set $E \subset \mathbb{R}_+^{n+1}$, we define a measure μ by

$$\mu(E) = \int_E |\psi_t * b(x)|^2 \frac{dxdt}{t}.$$

Theorem 49. Let μ be the measure given as above for any given $b \in BMO$. Then μ is a Carlesen measure whose norm satisfies

$$\|\mu\| \lesssim \|b\|_*^2.$$

This can be rephrased by

$$d\mu = |\psi_t * b|^2 \frac{dxdt}{t}.$$

Proof. For aby cube $Q \subset \mathbb{R}^n$, we aim to show

$$\mu(\hat{Q}) \lesssim \|b\|_*^2 \cdot |Q|.$$

We write

$$b = (b - b_{2Q})\chi_{2Q} + (b - b_{2Q})\chi_{(2Q)^c} + b_{2Q} := b_1 + b_2 + b_3.$$

Notice that

$$\psi_t * b_3(x) = b_{2Q} \int \psi_t(x)dx = b_{2Q} \int \psi(x)dx = 0.$$

Thus by triangle inequality, we can dominate

$$\mu(\hat{Q}) \lesssim \int_{\hat{Q}} |\psi_t * b_1|^2 \frac{dxdt}{t} + \int_{\hat{Q}} |\psi_t * b_2|^2 \frac{dxdt}{t} := I_1 + I_2.$$

The first term I_1 can be estimated by using Littlewood-Paley Theorem.

Exercise 17. Prove that for radial function ψ with $\int_{\mathbb{R}^n} \psi = 0$ and (8), we have

$$\int_{\mathbb{R}^{n+1}_+} |\psi_t * f(x)|^2 \frac{dxdt}{t} \lesssim \|f\|_2^2,$$

for any $f \in L^2(\mathbb{R}^n)$.

Now use the Exercise above, we obtain

$$\begin{aligned} I_1 &\le \int_{\mathbb{R}^{n+1}_+} |\psi_t * b_1|^2 \frac{dxdt}{t} \\ &\lesssim \int_{\mathbb{R}^n} |b_1(x)|^2 dx \lesssim \int_{2Q} |b - b_{2Q}|^2 dx \lesssim \|b\|_*^2 \cdot |Q|. \end{aligned}$$

It remains to control the second term I_2. Since b_2 is supported outside $2Q$, we majorize

$$\begin{aligned} |\psi_t * b_2(x)| \le& \frac{1}{t^n} \int |\psi(\frac{x-y}{t})| \cdot |b_2(y)| dy \\ \le& \frac{C}{t^n} \int \frac{|b(y) - b_{2Q}|}{(1 + t^{-1}|x-y|)^{n+\epsilon}} dy \\ =& C \int_{(2Q)^c} \frac{t^\epsilon |b(y) - b_{2Q}|}{(t + |x-y|)^{n+\epsilon}} dy. \end{aligned}$$

When $(x,t) \in \hat{Q}$ and $y \notin 2Q$, we have

$$|x-y| \ge |y - c(Q)| - |x - c(Q)| \ge \frac{1}{2}|y - c(Q)|,$$

since $x \in Q$ and $y \notin 2Q$, where $c(Q)$ stands for the center of the cube Q. Using this observation, we can further control $|\psi_t * b_2(x)|$ by

$$Ct^{\epsilon}\int_{(2Q)^c}\frac{|b(y)-b_{2Q}|}{(t+|x-y|)^{n+\epsilon}} \lesssim \frac{t^{\epsilon}}{l(Q)^{\epsilon}}\|b\|_*,$$

if $(x,t)\in\hat{Q}$. (Exercise) Now we dominate the second term I_2 by

$$\int_{\hat{Q}}|\psi_t * b_2|^2\frac{dxdt}{t} \lesssim \|b\|_*^2\int_Q\int_0^{l(Q)}\frac{t^{2\epsilon-1}}{l(Q)^{2\epsilon}}dtdx \lesssim \|b\|_*^2\cdot|Q|$$

as desired. Hence we obtain $\mu(\hat{Q})\lesssim \|b\|_*^2\cdot|Q|$, and the theorem follows. $\square$

19 $T1$ Theorem in Full Version

We now present the full version of $T1$ theorem.

Theorem 50 ($T1$ **Theorem, David and Journé).** Suppose that T is a singular integral operator associated to a Calderón–Zygmund kernel. Then T extends to a bounded operator on L^2 iff T satisfies the WBP, $T1 \in BMO$ and $T^*1 \in BMO$.

Remark 13. We know in Lecture 16, $T1$ is defined as a linear functional on $\mathcal{S}_0(\mathbb{R}^n)$. So $T1 \in BMO$ means that there is a function $b \in BMO$ such that for any $\psi \in \mathcal{S}_0(\mathbb{R}^n)$,

$$\langle T1, \psi \rangle = \langle b, \psi \rangle ,$$

where the right side is the usual inner product in $L^2(\mathbb{R}^n)$, i.e., $\int b(x)\overline{\psi(x)}dx$. Of course, $T^*1 \in BMO$ has a similar meaning.

First we deal with the necessity part. We have seen in Lecture 16 that WBP is a necessary condition. Let T be an L^2-extendable SIO associated with to a C-Z kernel. It now remains to show $T1 \in BMO$ and $T^*1 \in BMO$ can be treated in a similar manner.

Lemma 31. Let T be an L^2-extendable SIO associated with to a C-Z kernel. Suppose that f is a bounded function of compact support. Then $Tf \in BMO$ and

$$\|Tf\|_* \lesssim \|f\|_\infty. \tag{1}$$

The implicit constant C in $\lesssim$ is independent of f.

Proof. When f is a bounded function supported in a compact set, it belongs to L^2. Thus Tf makes sense and it belongs to L^2 because T can be extended to L^2. Of course in the Tf, the operator T means the extension operator of the SIO T. We still use T to denote the extension. Hence we aim to prove the function Tf obeys (1). For any cube $Q \subset \mathbb{R}^n$, let

$$a_Q = \int_{\mathbb{R}^n} K(c(Q), y) f(y) \chi_{(5Q)^c}(y) dy = T(f\chi_{(5Q)^c})(c(Q)),$$

where $c(Q)$ stands for the center of Q. We estimate via the triangle inequality,

$$\frac{1}{|Q|}\int_Q |Tf(x) - a_Q| dx \le \frac{1}{|Q|}\int_Q |T(f\chi_{(5Q)^c})(x)| dx + \frac{1}{|Q|}\int_Q |T(f\chi_{(5Q)^c})(x) - a_Q| dx.$$

The first term in RHS can be controlled by

$$\left(\frac{1}{|Q|}\int_Q |T(f\chi_{(5Q)^c})(x)| dx\right)^{1/2} \lesssim \left(\frac{1}{|Q|}\int_{5Q} |f(x)|^2 dx\right)^{1/2} \lesssim \|f\|_\infty.$$

Here we used Cauchy-Schwartz first, then the L^2-boundedness of T. The second term in the RHS is majored by

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \int_{(5Q)^c} |K(x,y) - K(c(Q), y)| \cdot |f(x)| dy dx \\
\lesssim{}& \|f\|_\infty \cdot \frac{1}{|Q|}\int_Q \int_{(5Q)^c} \frac{|x - c(Q)|^\epsilon}{|x-y|^{n+\epsilon}} dy dx \qquad \text{(By smoothness condition of } K\text{)} \\
\lesssim{}& \|f\|_\infty.
\end{aligned}$$

Combining very last inequality with previous estimate, we obtain

$$\sup_Q \frac{1}{|Q|}\int_Q |Tf(x) - a_Q| dx \lesssim \|f\|_\infty.$$

By knowledge in BMO space (see Lecture 17), we know $\|Tf\|_* \lesssim \|f\|_\infty$, which prove the desired result. $\square$

Lemma 32. Let T be an L^2-extendable SIO associated to a C-Z kernel. Then T extends to a bounded operator from L^∞ to BMO.

Proof. For any $j \in \mathbb{Z}$, let $B_j = B(0, 2^j)$, a ball centered at the origin, of radius 2^j. When T is L^2-extendable, we can define Tf for any $f \in L^\infty$ by for any B_j with $j \geq 0$ and $x \in B_j$,

$$Tf(x) = T(f\chi_{5B_j})(x) + \int_{\mathbb{R}^n} [K(x,y) - K(0,y)]\, f(y)\chi_{(5B_j)^c}(y)dy.$$

Let $x \in B_j \subsetneq B_{j'}$. Then

$$\begin{aligned} b_{B_j}(x) - b_{B_{j'}}(x) =& T(f\chi_{5B_j} - f\chi_{5B_{j'}})(x) + \int_{(5B_j)^c} [K(x,y) - K(0,y)]\, f(y)dy \\ & - \int_{(5B_{j'})^c} [K(x,y) - K(0,y)]\, f(y)dy \\ =& -\int_{5B_{j'}\setminus 5B_j} K(0,y)f(y)dy = C_{B_j, B_{j'}}, \end{aligned}$$

where the constant $C_{B_j,B_{j'}}$ is independent of x. Recall that two functions that differ by a constant are treated as the same function in BMO. Thus in the BMO space, $Tf(x)$ is well-defined and it is independent of the choices of B_j's. Using Lemma 31, we have

$$\|T(f\chi_{B_j})\|_* \leq C\|f\|_\infty,$$

where C is a constant independent of f and B_j. On the other hand, for $x \in B_j$,

$$\left| \int_{\mathbb{R}^n} [K(x,y) - K(0,y)]\, f(y)\chi_{(5B_j)^c}(y)dy \right| \leq \|f\|_\infty c \int_{(5B_j)^c} |K(x,y) - K(0,y)| dy \leq C\|f\|_\infty,$$

since the C-Z kernel satisfies the Hörmander's condition. Combining the results above, we get

$$\|Tf\|_* \lesssim \|f\|_\infty$$

for any $f \in L^\infty$, as desired. $\square$

We now return to prove $T1 \in BMO$. We aim to find a BMO function b such that $\langle T1, \psi\rangle = \langle b, \psi\rangle$ for all $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ holds. We let b be defined by

$$b(x) = T(\chi_{5B_j})(x) + \int_{\mathbb{R}^n} [K(x,y) - K(0,y)]\, \chi_{(5B)^c}(y)dy.$$

for any $x \in B_j$. By Lemma 32, $b \in BMO(\mathbb{R}^n)$. Then for any $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ supported in B_J for some positive integer J, we see that

$$\langle T1, \psi\rangle = \langle T(\chi_{5B_J}), \psi\rangle + \left\langle \chi^c_{5B_J}, T^*\psi \right\rangle.$$

Since $\int \psi = 0$, we get

$$\left\langle \chi^c_{5B_J}, T^*\psi \right\rangle = \int \chi^c_{5B_J}(x) \left[\int [K(y,x) - K(0,x)] \overline{\psi(y)} dy \right] dx,$$

which, by Fubini's Theorem, equals to $\langle g, \psi \rangle$, where

$$g(x) = \int [K(y,x) - K(0,x)] \chi^c_{5B}(y) dy.$$

Henceforth, we end up

$$\langle T1, \psi \rangle = \langle T(\chi_{5B_J}), \psi \rangle + \langle g, \psi \rangle = \langle b, \psi \rangle .$$

Therefore, we obtain $T1 \in BMO$ and similarly $T^*1 \in BMO$. We finish the proof of "only if" part of Theorem 50.

We now turn to the proof of "if" part. Let ϕ, ψ be radial Schwartz functions on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \phi(x) dx = 1,$$
$$\int_{\mathbb{R}^n} \psi(x) dx = 0,$$
$$\int_0^\infty |\hat{\phi}(t)|^2 \frac{dt}{t} = 1.$$

In addition, we assume that ψ is $\mathbb{R}$-valued. Recall that

$$\mathbf{P}_t f(x) = \phi_t * f(x),$$
$$Q_t f(x) = \psi_t * f(x),$$

where $\phi_t(x) = t^{-n}\phi(x/t)$ and $\psi_t(x) = t^{-n}\psi(x/t)$. All those convolution operators are well-defined for $f \in L^p$ with $p \in [1,\infty]$ and $t > 0$. For any $b \in BMO$ and any $\epsilon > 0$, we define a paraproduct by

$$\Pi_{b,\epsilon} f(x) = \int_\epsilon^{\frac{1}{\epsilon}} Q_t(Q_t b \mathbf{P}_t f)(x) \frac{dt}{t}.$$

Define a SIO Π_b by

$$\langle \Pi_b \varphi, \psi \rangle = \lim_{\epsilon \to 0} \langle \Pi_{b,\epsilon} \varphi, \psi \rangle ,$$

for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. For any $b \in BMO(\mathbb{R}^n)$, $\Pi_b 1$ is a linear functional on $\mathcal{S}_0(\mathbb{R}^n)$, defined by

$$\langle \Pi_b 1, \varphi \rangle = \lim_{\epsilon \to 0} \langle \Pi_{b,\epsilon} 1, \varphi \rangle ,$$

for any $\varphi \in \mathcal{S}_0(\mathbb{R}^n)$.

Lemma 33. As a linear functional on $\mathcal{S}_0(\mathbb{R}^n)$, $\Pi_b 1 = b$.

Proof. First we have

$$\Pi_{b,\epsilon} 1(x) = \int_\epsilon^{\frac{1}{\epsilon}} Q_t(Q_t b \mathbf{P}_t 1)(x) \frac{dt}{t} = \int_\epsilon^{\frac{1}{\epsilon}} Q_t(Q_t b)(x) \frac{dt}{t},$$

since $\mathbf{P}_t 1(x) = 1$. Moreover, it follows from Fubini's Theorem that

$$\langle \Pi_{b,\epsilon} 1, \varphi \rangle = \int_\epsilon^{\frac{1}{\epsilon}} \langle b, Q_t^2 \varphi \rangle \frac{dt}{t} = \left\langle b, \int_\epsilon^{\frac{1}{\epsilon}} Q_t^2 \varphi \frac{dt}{t} \right\rangle.$$

By Calderón reproducing formula, wee see that

$$\lim_{\epsilon \to 0} \langle \Pi_{b,\epsilon} 1, \varphi \rangle = \langle b, \varphi \rangle,$$

because

$$\lim_{\epsilon \to 0} | \langle \Pi_{b,\epsilon} 1 - b, \varphi \rangle | = \|b\|_* \cdot \lim_{\epsilon \to 0} \left\| \int_\epsilon^{\frac{1}{\epsilon}} Q_t^2 \varphi \frac{dt}{t} - \varphi \right\|_{H^1} = 0.$$

Here H^1 is the dual space of BMO and $\| \cdot \|_{H^1}$ is the norm of H^1. We used a generalized Calderón reproducing formula (see below) in H^1 space or equivalently,

$$b(x) = \lim_{\epsilon \to 0} \int_\epsilon^{\frac{1}{\epsilon}} Q_t^2 b(x) \frac{dt}{t}$$

as a linear functional on $\mathcal{S}_0(\mathbb{R}^n)$. The proof to this is left to readers. □

Remark 14. Formally, one can derive

$$b(x) = \int_0^\infty Q_t^2 b(x) \frac{dt}{t},$$

in a non-rigorous way, say, by taking Fourier transform for both sides to see that

$$\hat{b}(\xi) = \int_0^\infty |\hat{\psi}(|\xi| t)|^2 \hat{b}(\xi) \frac{dt}{t} = \hat{b}(\xi) \int_0^\infty |\hat{\psi}(t)|^2 \frac{dt}{t} = \hat{b}(\xi).$$

Exercise 18. In this exercise, we aim to prove the generalized Calderón reproducing formula. Let $p \in (1, \infty)$, $f \in L^p$. Prove that

$$\lim_{\substack{\epsilon \to 0 \\ R \to \infty}} \left\| \int_\epsilon^R Q_t^2 f \frac{dt}{t} - f \right\|_p = 0.$$

Here Q_t is defined by $Q_t f = \psi_t * f$ with $\mathbb{R}$-valued radial Schwartz function ψ satisfying $\hat{\psi}(0) = 0$ and $\int_0^\infty |\hat{\psi}(t)|^2 \frac{dt}{t} = 1$.

Hint: First prove that there exists a function $\eta \in \mathcal{S}(\mathbb{R}^n)$ with $\hat{\eta}(0) = 1$ and

$$-t\partial_t(\eta_t * f) = \psi_t * \psi_t * f = Q_t^2 f. \tag{2}$$

The function η can be defined simply by setting

$$\hat{\eta}(\xi) = 1 - \int_0^1 \hat{\psi}(t\xi)^2 \frac{dt}{t}.$$

Such a function is a Schwartz function because its Fourier transform belongs to $\mathcal{S}$. The key observation is that $\hat{\eta}(t\xi) = \int_t^\infty \hat{\psi}(s\xi)^2 \frac{ds}{s}$ since ψ is radial, which implies $\partial_t(\hat{\eta}(t\xi)) = -t^{-1}(\hat{\psi}_t)^2(\xi)$. The last equality gives the identity (2), from which it follows that

$$\int_\epsilon^R Q_t^2 f(x) \frac{dt}{t} = -\int_\epsilon^R \partial_t(\eta_t * f)(x) dt = \eta_\epsilon * f(x) - \eta_R * f(x).$$

Note that by DCT (why?),

$$\lim_{R\to\infty} \|\eta_R * f\|_p = \| \lim_{R\to\infty} \eta_R * f\|_p = 0.$$

Verify this in details. Thus combining what have been proved, one can reach

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left\| \int_\epsilon^R Q_t^2 f \frac{dt}{t} \right\|_p = \lim_{\epsilon\to 0} \|\eta_\epsilon * f\|_p = \|f\|_p,$$

since $\{\eta_\epsilon\}$ is an approximation to the identity. One can also show the pointwise convergence (prove this).

From the definition of $\Pi_{b,\epsilon}$, the adjoint operator of $\Pi_{b,\epsilon}$ is

$$\Pi_{b,\epsilon}^* f(x) = \int_\epsilon^{\frac{1}{\epsilon}} \mathbf{P}_t^*(Q_t b Q_t^* f)(x) \frac{dt}{t},$$

where $\mathbf{P}_t^*$ and Q_t^* are adjoint to $\mathbf{P}_t$ and Q_t respectively. Then the adjoint operator Π_b^* of the SIO Π_b is given by

$$\langle \Pi_b^* \varphi, \psi\rangle = \lim_{\epsilon\to 0} \left\langle \Pi_{b,\epsilon}^* \varphi, \psi \right\rangle$$

for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. We then have

$$\Pi_b^* 1 = 0,$$

because

$$\begin{aligned}
\Pi_{b,\epsilon}^* 1 &= \int_\epsilon^{\frac{1}{\epsilon}} \mathbf{P}_t^*(Q_t b Q_t^* 1)(x) \frac{dt}{t} \\
&= \int_\epsilon^{\frac{1}{\epsilon}} \mathbf{P}_t^* 0(x) \frac{dt}{t} && (\text{by } Q_t^* 1 = 0) \\
&= 0. && (\text{by } \mathbf{P}_t^* 0 = 0)
\end{aligned}$$

We will see that, when $b \in BMO$, Π_b extends to an operator bounded on $L^p(\mathbb{R}^n)$. First we verify the L^2-boundedness of Π_b.

Lemma 34. Let $b \in BMO$. Then Π_b extends to an operator bounded on $L^2(\mathbb{R}^n)$.

Proof. It suffices to show that for any $f \in L^2$,

$$\|\Pi_{b,\epsilon} f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_2.$$

Here the hidden constant in $\lesssim$ is independent of f and ϵ. For any $f, g \in L^2$, we control

$$\begin{aligned}
|\langle \Pi_{b,\epsilon} f, g\rangle| &= \left| \int_\epsilon^{\frac{1}{\epsilon}} \int_{\mathbb{R}^n} Q_t(Q_t b \mathbf{P}_t f)(x) g(x) \frac{dxdt}{t} \right| \\
&= \left| \int_\epsilon^{\frac{1}{\epsilon}} \int_{\mathbb{R}^n} Q_t b(x) \mathbf{P}_t f(x) Q_t^* g(x) \frac{dxdt}{t} \right| \\
&\le \left(\int \int_{\mathbb{R}^n} |\mathbf{P}_t f(x) Q_t b(x)|^2 \frac{dxdt}{t} \right)^{\frac{1}{2}} \left(\int \int_{\mathbb{R}^n} |Q_t^* g(x)|^2 \frac{dxdt}{t} \right)^{\frac{1}{2}} \\
&\le \left(\int \int_{\mathbb{R}^n} |\mathbf{P}_t f(x)|^2 |Q_t b(x)|^2 \right)^{\frac{1}{2}} \|g\|_2.
\end{aligned}$$

Here we used Cauchy-Schwartz and the L^2-boundedness of $(\int |Q_t^* g(x)|^2 \frac{dt}{t})^{1/2}$, which is a consequence of Plancherel theorem as Theorem 38. By Theorem 49, we know that the measure $d\mu = |Q_t b(x)|^2 \frac{dxdt}{t}$ is a Carlesen measure since $b \in BMO$. We can employ Carlesen inequality in Theorem 47 to estimate the double integral in the last expression, so that we obtain

$$|\langle \Pi_{b,\epsilon} f, g\rangle| \lesssim \left(\int_{\mathbb{R}^n} \sup_{(y,t)\in\Gamma(x)} |\mathbf{P}_t f(y)|^2 dx \right)^{\frac{1}{2}} \|g\|_2,$$

which can be further bounded by $\lesssim \|Mf\|_2 \|g\|_2 \lesssim \|f\|_2 \|g\|_2$, from which the L^2-boundedness of Π_b follows. $\square$

To conclude that Π_b extends to an L^2-bounded operator, by Calderón–Zygmund theory, we only need to prove that Π_b is a SIO associated to a C-Z kernel.

Lemma 35. Let $b \in BMO$. Then Π_b is a SIO associated to a C-Z kernel.

Proof. We represent

$$Q_t(Q_t b \mathbf{P}_t f)(x) = \int_{\mathbb{R}^n} K_t(x,y) f(y) dy,$$

where K_t is given by

$$K_t(x,y) = \frac{1}{t^{2n}} \int_{\mathbb{R}^n} \psi(\frac{x-z}{t}) \phi(\frac{z-y}{t}) Q_t b(z) dz.$$

Then

$$\Pi_{b,\epsilon} f(x) = \int_{R^n} \left[\int_\epsilon^{\frac{1}{\epsilon}} K_t(x,y) \frac{dt}{t} \right] f(y) dy.$$

Thus we see that the SIO Π_b is associated to the kernel, in the sense of distribution,

$$\lim_{\epsilon\to 0} \int_\epsilon^{\frac{1}{\epsilon}} K_t(x,y) \frac{dt}{t} := K(x,y).$$

To finish the proof, it remains to show the kernel K is a C-Z kernel. The proof relies on the following two inequalities on $Q_t b$,

$$\|Q_t b\|_\infty \leq C \|b\|_* \tag{3}$$

and

$$\|\nabla_x Q_t b\|_\infty \leq C t^{-1} \|b\|_*, \tag{4}$$

where the constant C is independent of b and t. To see why (3) is true, we write

$$Q_t b(x) = \int_{\mathbb{R}^n} \psi_t(x-y) [b(y) - b_{Q(x,t)}] dy,$$

where $Q(x,t)$ is a cube centered at x and of side length t, and $b_{Q(x,t)}$ is the average of b over $Q(x,t)$ as usual. Inserting the absolute value into the integrand in the last integral, we further control

$$\begin{aligned}|Q_t b(x)| &\lesssim \int_{\mathbb{R}^n} \frac{1}{t^n(1+\frac{|x-y|}{t})^{n+1}}|b(y)-b_{Q(x,t)}|dy \\ &\lesssim \int_{2Q(x,t)} \frac{t}{(t+|x-y|)^{n+1}}|b(y)-b_{Q(x,t)}|dy \\ &\quad + \int_{(2Q(x,t))^c} \frac{t}{(t+|x-y|)^{n+1}}|b(y)-b_{Q(x,t)}|dy \\ &\lesssim \frac{1}{|Q(x,t)|}\int_{2Q(x,t)} |b(y)-b_{Q(x,t)}|dy \\ &\quad + \int_{(2Q(x,t))^c} \frac{t|b(y)-b_{Q(x,t)}|}{|y-x|^{n+1}}dy \\ &\lesssim \|b\|_* + \frac{t}{t}\|b\|_* \le C\|b\|_*,\end{aligned}$$

which yields (3). To verify (4), we write $\nabla_x Q_t b(x)$, in terms of convolution, as

$$\nabla_x Q_t b(x) = \int_{\mathbb{R}^n} \nabla_x(\psi_t(x-y))b(y)dy = \int_{\mathbb{R}^n} \frac{1}{t^{n+1}}\nabla\psi(\frac{x-y}{t})b(y)dy = \frac{1}{t}(\nabla\psi)_t * b(x).$$

Notice that

$$\int (\nabla\psi)_t(x)dx = \int \nabla\psi(x)dx = 0,$$

by integration by parts and $\int\psi = 0$, and for any $N \in \mathbb{N}$,

$$|(\nabla\psi)_t(x)| \lesssim \frac{1}{t^n}\cdot\frac{1}{(1+\frac{|x|}{t})^N}.$$

We see that $(\nabla\psi)_t$ behaves like ψ_t. Repeat the method we used for the proof of (3) and then we are able to obtain (4).

Now we will see how the Lemma 35 follows from (3) and (4). Indeed, from (3), we get

$$\begin{aligned}|K_t(x,y)| &\lesssim \frac{1}{t^{2n}}\|Q_t b\|_\infty \cdot \int_{\mathbb{R}^n} \frac{1}{(1+\frac{|x-z|}{t})^N}\cdot\frac{1}{(1+\frac{|z-y|}{t})^N}dz \\ &\lesssim \|b\|_* \cdot \frac{1}{t^n}\cdot\frac{1}{(1+\frac{|x-y|}{t})^N}.\end{aligned}$$

Henceforth,

$$|K(x,y)| \lesssim \int_0^\infty |K_t(x,y)|\frac{dt}{t} \lesssim \|b\|_* \int_0^\infty \frac{1}{t^n}\cdot\frac{1}{(1+\frac{|x-y|}{t})^N}dt,$$

which implies

$$|K(x,y)| \lesssim \frac{\|b\|_*}{|x-y|^n}.$$

Similarly, (4) yields

$$|\nabla K(x,y)| \lesssim \frac{\|b\|_*}{|x-y|^{n+1}}.$$

Hence we've shown that K is a C-Z kernel. $\square$

Finally we turn the proof of sufficiency condition of $T1$ theorem. We need to show that T extends to an L^2-bounded operator if T satisfies the WBP, $T1 \in BMO$ and $T^*1 \in BMO$. In the proof, the paraproduct Π_b plays a role of translation, making a BMO function to a zero function. To see that, we set $T1 = b_1$ and $T2 = b_2$ in the sense of distribution, where $b_1, b_2 \in BMO$. Define a SIO by

$$T_0 = T - \Pi_{b_1} - \Pi_{b_2}^*.$$

By Lemma 33 and $\Pi_b^* 1 = 0$, we have

$$T_0 1 = T1 - \Pi_{b_1} 1 - \Pi_{b_2}^* 1 = b_1 - b_1 = 0,$$
$$T_0^* 1 = T^* 1 - \Pi_{b_1}^* 1 - \Pi_{b_2} 1 = b_2 - b_2 = 0.$$

By Theorem 43, the simple version of $T1$ theorem, T_0 can be extended to an bounded operator on L^2. Therefore from $T = T_0 + \Pi_{b_1} + \Pi_{b_2}^*$ is L^2-extendable and $T1$ theorem, the full version of $T1$ theorem, or Theorem 50 is established.

20 Cotlar-Stein Lemma

When analyzing an operator T, very often we decompose it into $T = \sum_{j\in\mathbb{Z}} T_j$, where T_j's usually are well-localized in some sense and they satisfy

$$\sup_j \|T_j f\| \le C\|f\|_2. \tag{1}$$

However, this is not enough to characterize the L^2-boundedness of T. But if we have some orthogonality conditions from T_j's, say

$$T_{j_1}T_{j_2}^* = T_{j_1}^* T_{j_2} = 0$$

for any $j_1 \neq j_2$, and T^* is the adjoint of T. then we are able to conclude T is bounded on L^2 by (1). This is a special case of Cotlar-Stein lemma, and the orthogonality conditions can also be replaced by weak orthogonality conditions, which we will discuss below.

Lemma 36. For any operator T and any $k \in \mathbb{N}$, we have

$$\|T\| = \|(TT^*)^k\|^{1/2k}.$$

Recall that the norm is given by

$$\|T\| = \sup_{\substack{f\in L^2\\ f\neq 0}} \frac{\|Tf\|_2}{\|f\|_2}.$$

Exercise 19. Check that $\|T\| = \|T^*\|$ and $\|T_1T_2\| \le \|T_1\|\cdot\|T_2\|$.

Proof. By the preceding exercise, we see that

$$\|(TT^*)^k\|^{1/2k} \le \left(\|T\|^k\|T^*\|^k\right)^{1/2k} = \|T\|.$$

Now to show the reverse. First note that $\|T\| \le \|TT^*\|^{1/2}$. This is true because by duality,

$$\begin{aligned}
\|TT^*\| &= \sup_{\substack{f\in L^2\\ f\neq 0}} \frac{\|TT^*\|_2}{\|f\|_2} = \sup_{\substack{f\in L^2\\ f\neq 0}} \sup_{\substack{g\in L^2\\ g\neq 0}} \frac{\langle TT^*f, g\rangle}{\|f\|_2\|g\|_2}\\
&= \sup_{\substack{f\in L^2\\ f\neq 0}} \sup_{\substack{g\in L^2\\ g\neq 0}} \frac{\langle T^*f, T^*g\rangle}{\|f\|_2\|g\|_2} \ge \sup_{\substack{f\in L^2\\ f\neq 0}} \frac{\langle T^*f, T^*g\rangle}{\|f\|_2^2}\\
&= \sup_{\substack{f\in L^2\\ f\neq 0}} \frac{\|T^*f\|_2^2}{\|f\|_2^2} = \|T^*\|^2 = \|T\|^2.
\end{aligned}$$

It suffices to show for any $k \in \mathbb{N}$,

$$\|(TT^*)^k\|^{1/2k} \le \|(TT^*)^{k+1}\|^{1/2(k+1)}.$$

Since TT^* is self-adjoint operator, this is a consequence of the following strong result:

$$\|U^k\|^{1/k} \le \|U^{k+1}\|^{1/(k+1)}, \tag{2}$$

where U is self-adjoint. We prove it by induction. The base step when $k = 1$ is a consequence of $\|T\| \le \|TT^*\|^{1/2}$ with T replaced by U. Let $k \ge 2$. Assume for any integer m with $1 \le m \le k-1$,

$$\|U^m\|^{1/m} \le \|U^{m+1}\|^{1/(m+1)}.$$

Observe that $\|U^k\|^2 \le \|U^{k-1}\| \cdot \|U^{k+1}\|$. This is true because by self-adjoint property of U, we notice that

$$\|U^k f\|_2^2 = \langle U^k f, U^k f\rangle = \langle U^{k-1} f, U^{k+1} f\rangle \le \|U^{k-1}\| \cdot \|U^{k+1}\| \cdot \|f\|_2^2.$$

Using the inductive hypothesis with $m = k-1$, we obtain

$$\|U^k\|^2 \le \|U^k\|^{\frac{k-1}{k}} \|U^{k+1}\|,$$

so we prove (2), and we're done. □

Lemma 37 (Cotlar-Stein Lemma). Let $\{T_j\}_{j\in\mathbb{Z}}$ be a sequence of operators satisfying (1). Suppose that

$$\|T_{j_1} T_{j_2}^*\| \le a(j_1 - j_2)$$
$$\|T_{j_1}^* T_{j_2}\| \le a(j_1 - j_2),$$

where a is a non-negative function on $\mathbb{R}$. Then

$$\left\|\sum_{j\in\mathbb{Z}} T_j\right\| \le \sum_{j\in\mathbb{Z}} a(j)^{1/2}.$$

Here $\sum_{j\in\mathbb{Z}} T_j = \lim_{N\to\infty} \sum_{j=-N}^{N} T_j$.

Proof. Let $S = \sum_{j=-N}^{N} T_j$. It is sufficient to show that

$$\|S\| \le \sum_{j\in\mathbb{Z}} a(j)^{1/2}.$$

By Lemma 36, we know for any $k \in \mathbb{N}$, $\|S\| = \|(SS^*)^k\|^{1/2k}$. Expand $(SS^*)^k$ to get

$$(SS^*)^k = \sum_{-N\le j_1, j_2, \cdots, j_{2k} \le N} T_{j_1} T_{j_2}^* \cdots T_{j_{2k-1}} T_{j_{2k}}^*.$$

There are two ways that we can estimate the norm of each single term in RHS. First we see

$$\|T_{j_1} T_{j_2}^* \cdots T_{j_{2k-1}} T_{j_{2k}}^*\| \le \|T_{j_1} T_{j_2}^*\| \cdots \|T_{j_{2k-1}} T_{j_{2k}}^*\| \le \prod_{i=1}^{k} a(j_{2i-1} - j_{2i}).$$

Second we see

$$\|T_{j_1} T_{j_2}^* \cdots T_{j_{2k-1}} T_{j_{2k}}^*\| \le \|T_{j_1}\| \cdot \|T_{j_2}^{ast} T_{j_3}\| \cdots \|T_{j_{2k-2}}^* T_{j_{2j-1}}\| \cdot \|T_{j_{2k}}^*\|,$$

which is bounded by

$$a(0)^{1/2} \left(\prod_{i=1}^{k-1} \|T_{j_{2i}}^* T_{j_{2j+1}}\|\right) a(0)^{1/2} = a(0) \prod_{i=1}^{k-1} a(j_{2i} - j_{2i+1}).$$

Taking geometric mean of both bounds, we end up with

$$\begin{aligned}\|(SS^*)^k\| &\le a(0)^{\frac{1}{2}} \sum_{-N\le j_1, \cdots, j_{2k}\le N} \prod_{i=1}^{2k-1} a(j_{2i} - j_{2i+1})^{1/2} \\ &\le a(0)^{\frac{1}{2}} (2N+1) \left[\sum_j a(j)^{1/2}\right]^{2k-1}.\end{aligned}$$

Combine with $\|S\| = \|(SS^*)^k\|^{1/2k}$, we see that for any $k \in \mathbb{N}$,

$$\|S\| \leq a(0)^{\frac{1}{4k}}(2N+1)^{\frac{1}{k}}\left[\sum_j a(j)^{\frac{1}{2}}\right]^{\frac{2k-1}{2k}}.$$

Let $k \to \infty$, we obtain the result. $\square$

Now we provide an application of Cotlar-Stein lemma to Hilbert transform. Hilbert transform H can be partitioned into

$$H = \sum_{j \in \mathbb{Z}} T_j,$$

where T_j is defined by

$$T_j f(x) = \int_{2^j \leq |t| < 2^{j+1}} f(x-t)\frac{dt}{t}.$$

It is easy to see that there is a constant independent of j of such that $T_j f(x) \leq CMf(x)$, from which we get the uniform L^2-estimates for T_j's, namely,

$$\sup_j \|T_j f\|_2 \leq C\|f\|_2.$$

Notice that $T_j^* = -T_j$. To obtain the weak orthogonality, we need to verify

$$\|T_{j_1}T_{j_2}\| \lesssim 2^{-|j_1-j_2|}, \tag{3}$$

for any j_1, j_2. By Cotlar-Stein lemma, we get the L^2-boundedness of Hilbert transform H. This leads to an alternative proof for Hilbert transform without using Fourier transform.

We now turn to the proof (3), the almost orthogonality of T_j's. WLOG, we can assume that

$$j_1 < j_2.$$

We aim to show

$$\|T_{j_1}T_{j_2}\| \lesssim 2^{-(j_2-j_1)}. \tag{4}$$

For any $j \in \mathbb{Z}$, let

$$K_j(x) = \frac{\chi_{\Delta_j}(x)}{x},$$

where $\Delta_j = \{x \in \mathbb{R} : 2^j \leq |x| < 2^{j+1}\}$. Then it is clear that

$$T_j f(x) = K_j * f(x).$$

Henceforth we get for any $f \in L^2$,

$$\|T_{j_1}T_{j_2}f\|_2 = \|K_{j_1} * K_{j_2} * f\|_2 \leq \|K_{j_1} * K_{j_2}\|_1 \|f\|_2.$$

We shall analyze $K_{j_1} * K_{j_2}$ more carefully. We write

$$K_{j_1} * K_{j_2}(x) = \int \frac{1}{t}\chi_{\Delta_{j_1}}(t) \cdot \frac{1}{x-t}\chi_{\Delta_{j_2}}(x-t)dt.$$

By the support conditions of $\chi_{\Delta_{j_1}}$ and $\chi_{\Delta_{j_2}}$, we can localize

$$2^{j_1} \le |t| < 2^{j_1+1},$$
$$2^{j_2} \le |x-t| < 2^{j_2+1}.$$

Using triangle inequality and the last constraints for t and $x-t$, we see the range of x as follows:

$$2^{j_2} - 2^{j_1+1} \le |x-t| - |t| \le |x| \le |x-t| + |t| < 2^{j_2+1} + 2^{j_1+1}.$$

We only need to focus on those x obeying $2^{j_2} - 2^{j_1+1} \le |x| < 2^{j_2+1} + 2^{j_1+1}$, since $K_{j_1} * K_{j_2}(x)$ vanishes otherwise. We further break the range of x into three parts:

$$2^{j_2} - 2^{j_1+1} \le |x| < 2^{j_2} + 100 \cdot 2^{j_1}, \tag{5}$$
$$2^{j_2} + 100 \cdot 2^{j_1} \le |x| < 2^{j_2+1} - 100 \cdot 2^{j_1+1}, \tag{6}$$
$$2^{j_2+1} - 100 \cdot 2^{j_1+1} \le |x| \le |x| < 2^{j_2+1} + 2^{j_1+1}. \tag{7}$$

When $|x|$ obeying (5) or (7), we see that such x's only occupy a set E whose measure is at most $500 \cdot 2^{j_1}$. We then see that

$$\int_E |K_{j_1} * K_{j_2}(x)| dx \le \|K_{j_1} * K_{j_2}\|_\infty \cdot |E| \lesssim 2^{-(j_2 - j_1)},$$

because $\|K_{j_1} * K_{j_2}\|_\infty \lesssim 2^{-j_2}$. When x satisfies (6), notice that for $t \in \Delta_{j_1}$,

$$|x-t| \ge |x| - |t| \ge 2^{j_2} + 100 \cdot 2^{j_1} - 2^{j_1+1} > 2^{j_2},$$

and

$$|x-t| \le |x| + |t| \le 2^{j_2+1} - 100 \cdot 2^{j_1+1} + 2^{j_1+1} < 2^{j_2+1}.$$

Thus when x lies in the case (6), $x - t \in \Delta_{j_2}$ provided that $t \in \Delta_{j_1}$. Thus we get, for x obeying (6),

$$\begin{aligned}
|K_{j_1} * K_{j_2}(x)| &= \left| \int \frac{1}{t} \chi_{\Delta_{j_1}}(t) \cdot \frac{1}{x-t} dt \right| \\
&= \left| \int \frac{1}{t} \chi_{\Delta_{j_1}}(t) \cdot \left(\frac{1}{x-t} - \frac{1}{x} \right) dt \right| \\
&= \left| \int \chi_{\Delta_{j_1}}(t) \cdot \frac{1}{(x-t)x} dt \right| \lesssim 2^{j_1 - 2 \cdot j_2}.
\end{aligned}$$

Integrating in all such x, we have

$$\int_{2^{j_2} + 100 \cdot 2^{j_1} \le |x| < 2^{j_2+1} - 100 \cdot 2^{j_1+1}} |K_{j_1} * K_{j_2}(x)| dx \lesssim 2^{-(j_2 - j_1)}.$$

Combine previous results, we see that

$$\|K_{j_1} * K_{j_2}\|_1 \lesssim 2^{-(j_2 - j_1)}.$$

Hence we show (4), and we're done!

21 The Besicovitch Set

A needle is moved continuously in a plane to its opposite direction. What is the least area required to make such a movement? This is so called Kakeya needle problem, posted in 1927. It sounds plausible that the least area could be related to π. However, astonishingly the least area can be as small as possible. The solution to Kakeya needle problem relies on a fundamental construction of Besicovitch which yields a set of measure zero that contains line segments in all possible directions. Such a set plays a significant role in modern analysis. For example, it shows that Lebesgue differentiation theorem can not be extended to higher dimensional spaces arbitrarily. More precisely, let us consider

$$\lim_{\substack{\operatorname{diam}(R)\to 0\\ R\in\mathcal{R}}} \frac{1}{|R|}\int_R f(x-y)dy, \tag{1}$$

where $\mathcal{R}$ is a family of rectangles. It is natural to ask whether the limit in (1) converges to $f(x)$ a.e. if $f \in L^p$ with $1 < p < \infty$. Of course, we have learnt the convergence holds when $\mathcal{R}$ is a family of cubes or balls, by Lebesgue differentiation theorem. However, the convergence property relies on how many directions pointed by rectangles and the boundedness of eccentricity of rectangles. Closely related to the pointwise convergence is the problem of L^p-boundedness of the corresponding maximal operator $M_{\mathcal{R}}$, defined by

$$M_{\mathcal{R}}f(x) = \sup_{R\in\mathcal{R}} \frac{1}{|R|}\int_R |f(x-y)|dy.$$

There are different ways to produce the Besicovitch set, for instance, Kahane's construction by Cantor sets. In this section, we present the Besicovitch set in terms of a union of a large number of congruent thin rectangles in the plane with a high degree of overlap. Let N be a sufficiently large number. We use $\mathcal{R}_N$ to denote a family of rectangles of side lengths 1 and 2^{-N}. For any $R \in \mathcal{R}_N$, $\tilde{R}$ denotes the rectangle obtained by translating R two units in the positive direction (see Figure 1 below).

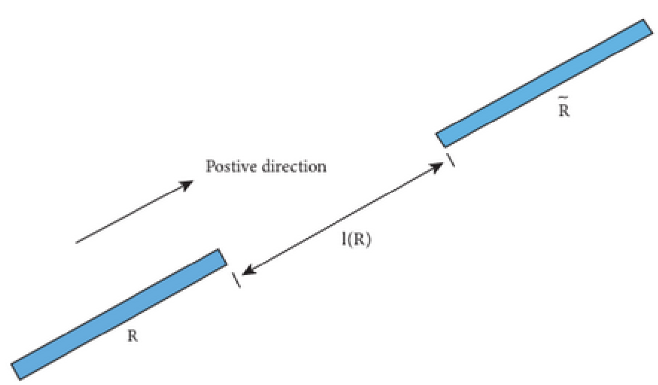

Figure 1: $\tilde{R}$, a translation of R

Theorem 51. Given any $\epsilon > 0$, there exists an integer N and 2^N many rectangles $R_1, \cdots, R_{2^N} \in \mathcal{R}_N$ such that

$$\left| \bigcup_{j=1}^{2^N} R_j \right| < \epsilon,$$

and the $\tilde{R}_j$'s are pairwise disjoint for $j = 1, \cdots, 2^N$, and so $|\cup_j \tilde{R}_j| = 1$. Here $\tilde{R}_j$ denotes the two-unit translation of R_j, as defined above.

A family of $R_1, \cdots, R_{2^N}$ in Theorem 51 can be made by cutting an initial triangle into a large number of subtriangles, obtained by equally by dividing the base of the original triangle, and then shift those subtriangles to make them overlap significantly so that their union has small measure. We will describe these in details now.

Start with a triangle T. Suppose T is the triangle $\triangle ABC$, with the base AB. The middle point M of the base yields two subtriangles, the "left" triangle $\triangle AMC$ and the "right" triangle $\triangle MBC$. Let $\alpha \in (1/2, 1)$ be a constant of proportionality. We shift $\triangle MBC$ leftward such that $\frac{|PB'|}{|B'C'|} = \frac{|PA|}{|AC|} = \alpha$. We end up with a overlapping figure, call $\Phi(T)$. See Figure 2 below:

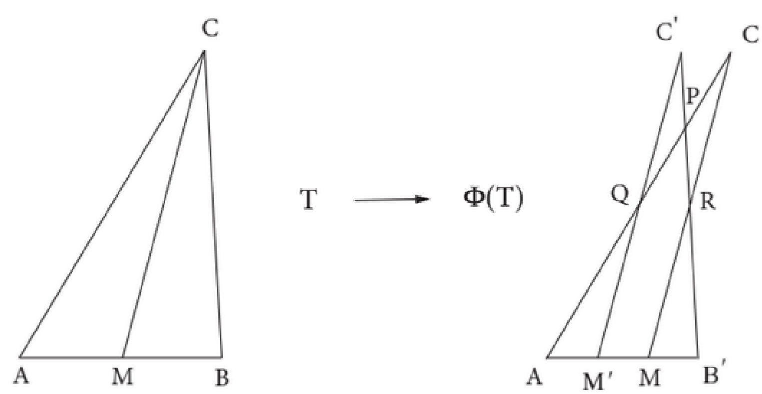

Figure 2: Bisecting T and shifting subtriangles

$\triangle AB'P$ is similar to the triangle $\triangle ABC$ with ratio α. We call $\triangle AB'P$ the "heart" of $\Phi(T)$, denoted by $\Phi_h(T)$. The remaining part of $\Phi(T)$ is called the "arms" of $\Phi(T)$, denoted by $\Phi_a(T)$, consisting of two small triangles $\triangle QPC'$ and $\triangle PRC$ in Figure 2. See Figure 3 below.

Because the ratio between two triangles $\Phi_h(T)$ and T is α, we see that

$$|\Phi_h(T)| = \alpha^2 |T|. \tag{2}$$

Now to evaluate the area of the arm $\Phi_a(T)$, we draw a line segment EF, parallel to the base AB', and passing through the intersection point P.

We use $\sim$ to mean the similar triangles, and $\cong$ to mean the congruent triangles from now on. It is clear $\triangle EPC' \sim \triangle M'B'C'$ with ratio $1 - \alpha$. By reflection, it is easy to see that $\triangle EPC' \cong \triangle PFR$.

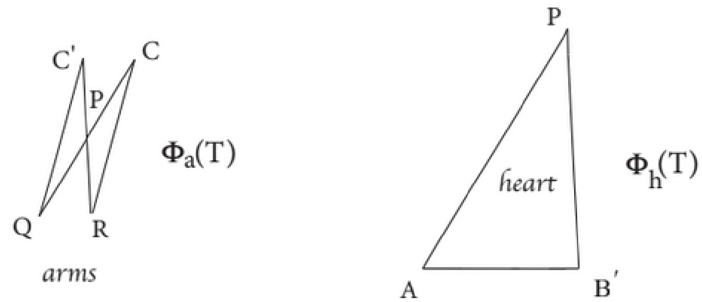

Figure 3: The arms and the heart of $\Phi(T)$

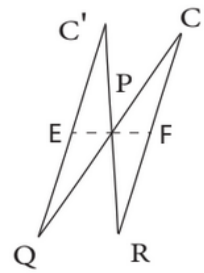

Figure 4: The arms $\Phi_a(T)$

Similiarly, $\triangle PFC \sim \triangle AMC$, with ratio $1-\alpha$, and it is congruent to the $\triangle PEQ$. Henceforth, we get

$$|\Phi_a(T)| = 2(1-\alpha)^2|T|. \tag{3}$$

Combine (2) and (3), we obtain

$$|\Phi(T)| = (\alpha^2 + 2(1-\alpha)^2)|T|. \tag{4}$$

We will iterate the above basic process sufficiently many times to obtain Theorem 51. Let's start with a large integer n and a triangle, say $\triangle ABC$. We subdivide the base AB into 2^n equal subintervals, with division points $A = A_0, A_1, \cdots, A_{2^n} = B$. In this way, we divide the original triangle $\triangle ABC$ into many smaller triangles. We are in particular interested in those 2^{n-1} many smaller triangles $A_{2j}A_{2j+2}C$, where $0 \le j < 2^{n-1}$. The base of such a triangle has midpoint A_{2j+1}.

Now for fixed $\alpha \in (1/2, 1)$, we preform the basic process, described in Figure 6 below, for each triangle $A_{2j}A_{2j+2}C$'s to get a figure $\Phi(A_{2j}A_{2j+2}C)$, for $j \in [0, 2^{n-1})$. In this way, we then obtain 2^{n-1} "hearts" and also 2^{n-1} pairs of "arms". By this construction, the right side of the heart $\Phi_h(A_{2j}A_{2j+2}C)$ is parallel to the side CA_{2j+2}, which is parallel to the left side of heart $\Phi_h(A_{2j+2}A_{2j+4}C)$. Here $0 \le j < 2^{n-1} - 1$.

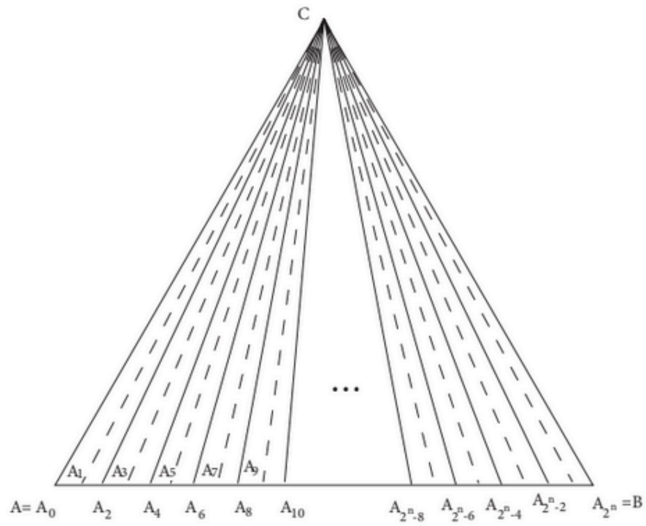

Figure 5: Divide $\triangle ABC$ into subtriangles $A_{2j}A_{2j+2}C$'s

Hence, we can translate $\Phi(A_{2j+2}A_{2j+4}C)$ leftwards so that the left side of $\Phi(A_{2j+2}A_{2j+4}C)$ coincides with the right side of $\Phi_h(A_{2j}A_{2j+2}C)$ (see Figure 7 below, in the right part of which, the point A'_{2j+2} coincides with A_{2j+2}.)

We shall carry out such a translation for all triangles $\triangle A_{2j}A_{2j+2}C$, $0 \le j < 2^{n-1}$. Then we can incorporate each of these 2^{n-1} hearts $\Phi_h(A_{2j}A_{2j+2}C)$ into one heart, which is similar to $\triangle ABC$. So far, we have shifted the 2^n subtriangles of $\triangle ABC$, forming a figure that we call $\Psi_1(ABC)$. This figure contains a heart, namely the disjoint union of the translates of the hearts $\Phi_h(A_{2j}A_{2j+2}C)$, $0 \le j < 2^{n-1}$. It is easy to see that

$$|\Phi_h(\Psi_1(ABC))| = \alpha^2|\triangle ABC|. \tag{5}$$

because $\Phi_h(\Psi_1(ABC)) \sim \triangle ABC$ with ratio α. The rest of $\Psi_1(ABC)$ consists of the union of the translated arms $\Phi_a(A_{2j}A_{2j+2}C)$, called the arms of $\Psi_1(ABC)$, or $\Phi_a(\Psi_1(ABC))$. It is clear that

$$|\Phi_a(\Psi_1(ABC))| \le \sum_{j=0}^{2^{n-1}-1} |\Phi_a(A_{2j}A_{2j+2}C))| = 2(1-\alpha)^2|\triangle ABC|. \tag{6}$$

Here we used (3) and $|\triangle A_{2j}A_{2j+2}C| = 2^{-n+1}|\triangle ABC|$. There can be considerable overlap among these translated arms, although we did not take advantage of this in the estimate (6). Putting (5) and (6) together, we obtain

$$|\Psi_1(ABC)| \le (\alpha^2 + 2(1-\alpha)^2)|\triangle ABC|. \tag{7}$$

We have seen that the heart of $\Psi(ABC)$ is a union of 2^{n-1} triangles (for translated hearts), which we will not further break into smaller triangles in order to maintain the original 2^n triangles $A_jA_{j+1}C$, $0 \le j < 2^n - 1$. The final figure we aim to create will be made of a union of translated $A_jA_{j+1}C$'s. In addition, we shall choose the proportionality constant α near 1 so that $1-\alpha$ is very tiny. In such

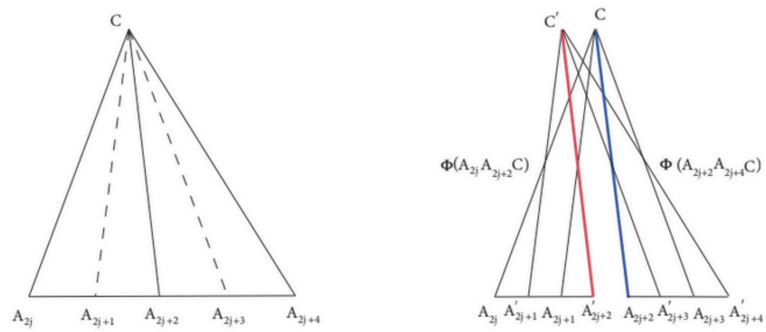

Figure 6: Making $\Phi(A_{2j}A_{2j+2}C)$ and $\Phi(A_{2j+2}A_{2j+4}C)$

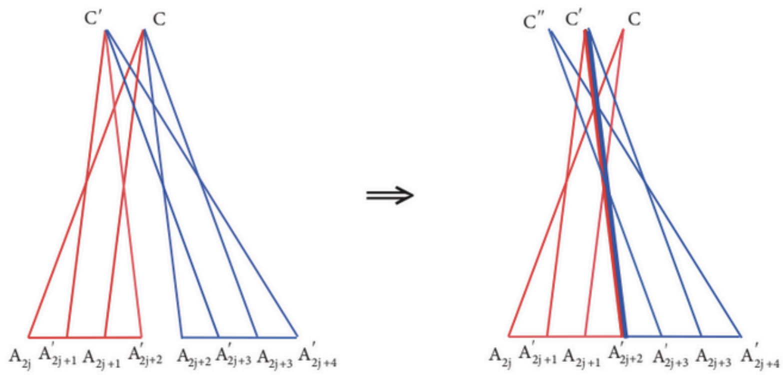

Figure 7: Shifting $\Phi(A_{2j+2}A_{2j+4}C)$ leftwards to $\Phi(A_{2j}A_{2j+2}C)$

a way, we can see that the contribution, in terms of the area, from those "arms" is insignificant since it is bounded by a continuous function of $1-\alpha$. Thus we focus on the main contribution, the heart $\Psi_1(ABC)$. The above algorithm for $\triangle ABC$ with 2^n+1 many division points in its base, can be carries out on the heart of $\Psi_1(ABC)$, a triangle with $2^{n-1}+1$ many division points in the base. Here n in the first stage is replaced by $n-1$ in the second stage. When shifting those 2^{n-1} translated "hearts" in the heart of $\Psi_1(ABC)$, we move the corresponding attached arms in the same way. When the process is completed, we end up a figure, called $\Psi_2(ABC)$ and containing a heart and some arms as $\Psi_1(ABC)$. Keep in mind that the translated triangles $A_jA_{j+1}C$ can NOT be broken into pieces. The picture made by those shifted arms may become messy due to the high degree of overlap. But we will see the contribution from those arms, even treated as they are disjoint mutually, is insignificant because we choose α near 1. To see this, notice that the arms of the figure $\Psi_2(ABC)$ consists of two parts, the arms in $\Psi_1(ABC)$ and the additional arms made by the algorithm acted on the heart of $\Psi_1(ABC)$. As we did in the first stage, we see that the area if the additional arms contributes at most

$$2(1-\alpha)^2|\Phi_h(\Psi_1(ABC))| = 2(1-\alpha)^2\alpha^2|\triangle ABC|.$$

Henceforth, the area of arms of $\Psi_2(ABC)$ is controlled by

$$|\Phi_a(\Psi_2(ABC))| + 2(1-\alpha)^2\alpha^2|\triangle ABC| = (2(1-\alpha)^2\alpha^2 + 2(1-\alpha)^2)|\triangle ABC|, \tag{8}$$

which is small when α is near 1. Meanwhile, the heart of $\Psi_2(ABC)$ has its area

$$\alpha^2|\Phi_h(\Psi_1(ABC))| = \alpha^2\alpha^2|\triangle ABC|,$$

which gets smaller since $\alpha \in (1/2, 1)$, compared to the contribution of the heart of $\Psi_2(ABC)$. Combining this with (8) yields

$$|\Psi_2(ABC)| \leq (\alpha^2\alpha^2 + 2(1-\alpha)^2 + 2(1-\alpha)^2\alpha^2)|\triangle ABC|. \tag{9}$$

The process can be iterated and finally we obtain $\Psi_n(ABC)$, where the algorithm terminates. It follows from (9) and induction that

$$|\Psi_n(ABC)| \leq (\alpha^{2n} + 2(1-\alpha)^2 + 2(1-\alpha)^2\alpha^2 + \cdots + 2(1-\alpha)^2\alpha^{2n-2})|\triangle ABC|. \tag{10}$$

The arms of $\Psi_n(ABC)$ contributes at most

$$2(1-\alpha)^2 + 2(1-\alpha)^2\alpha^2 + \cdots + 2(1-\alpha)^2\alpha^{2n-2} \leq 2(1-\alpha)^2\sum_{j=0}^{\infty}\alpha^{2j} \leq 2(1-\alpha).$$

Therefore we have

$$|\Psi_n(ABC)| \leq (\alpha^{2n} + 2(1-\alpha))|\triangle ABC|. \tag{11}$$

The set $\Psi_n(ABC)$ is essentially the **Besicovitch set** we are looking for, because its area can be made as small as we wish when α is near 1 and n is sufficiently large.

To finish the proof of Theorem 51, we make a crucial geometrical observation now. We had already seen that $\Psi_n(ABC)$ is a union of translated triangles $A_jA_{j+1}C$'s. Let us denote the triangle $A_jA_{j+1}C$ by T_j for $j = 0, 1, \cdots, 2^n - 1$. Those T_j's share a common vertex C. We use T_j' to denote the shifted T_j that comprises $\Psi_n(ABC)$. Let C_j denote the vertex of corresponding to the common vertex C. T_j^* is used to denote the triangle obtained by reflecting the T_j' through C_j. While the triangles T_j''s overlap to a very high degree, the reflected triangle T_j^*'s are mutually disjoint.

In fact, if T_{j_2} was originally to the right T_{j_1}, then by the algorithm T_{j_2} was moved leftwards to T_{j_1}, so the vertex C_{j_2} is to the left of C_{j_1}. The relative positions of the reflected triangles $T_{j_1}^*$ and $T_{j_2}^*$ are then described as in Figure 8, from which the disjointness of the reflected triangles is clear.

Finally we pass from the triangles above to rectangles. We choose the original triangle ABC to be an equilateral triangle whose height is 2. For any triangles T_j' that makes up $\Psi_n(ABC)$, we draw a line from its vertex C_j to the midpoint M_j of its base, marking off the points P_j and Q_j on it at distance 1/2 and 3/2 from the vertex C_j. We let R_j denote the rectangle whose major axis is P_jQ_j, whose side lengths are 1 and 2^{-N}. Here $N = n + L$, where L is a fixed large integer (see Figure 9 below). Since the angle of T_j' at the vertex C_j is larger than $c_1 \cdot 2^{-n}$, for some small positive constant c_1, we can always choose L large enough so that $R_j \subset T_j'$.

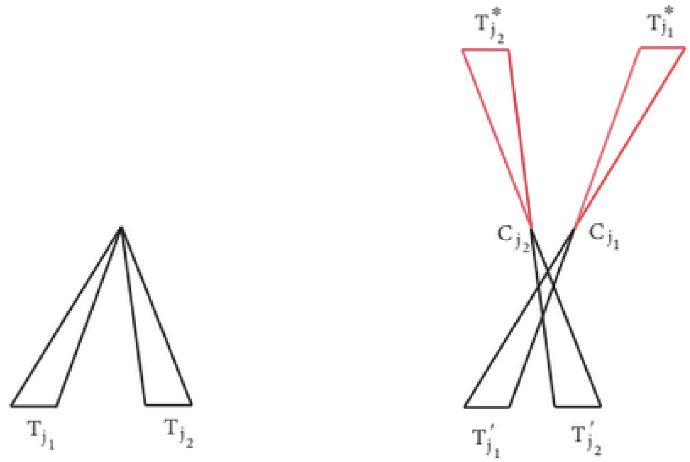

Figure 8: Reflected subtriangles are disjoint

We now have 2^n many rectangles R_j's of dimension 1×2^{-N}. To get $2^N = 2^L 2^n$ many such rectangles, notice that both $\Psi_n(ABC) = \cup_j T'_j$ and its reflection, given by $\cup_j T^*_j$, are covered by a 5×5 cube. By taking 2^L disjoint copies of $\Psi_n(ABC)$ and its reflection, we obtain 2^N rectangles with side lengths 1 and 2^{-N}. Those rectangles are contained in a set of measure at most

$$2^L(\alpha^{2n} + 2(1-\alpha))|\triangle ABC|,$$

which can be made smaller than arbitrary given $\epsilon > 0$ if we take n large enough and α sufficiently close to 1. Henceforth, there is an integer N depending on ϵ, such that

$$\left|\bigcup_{j=1}^{N} R_j\right| \le \epsilon.$$

Finally we verify the mutual disjointness of the translations $\tilde{R}_j$'s. According to the way how we select R_j in Figure 9, we see that $\tilde{R}_j$ is the reflection of R_j through C_j. The disjointness of $\tilde{R}_j$ follows from the crucial geometrical observation in Figure 8, as shown in Figure 10 below. Therefore we complete the proof of Theorem 51.

Remark 15. The existence of the Besicovitch set is really a striking phenomenon in analysis, going beyond common sense and usual imaginations. It indicates the significant difference between 1-dimensional analysis and higher dimensional analysis. Very often the main obstacle arises from the Besicovitch set in many analysis problems, for instance, the well-known Bochner-Riesz conjecture, restriction conjecture, and Kakeya conjecture, etc. To close the section, let us state another famous problem in analysis, a conjecture of Zygmund.

Let $v : \mathbb{R}^2 \to \mathbb{S}^1$ be a vector field in $\mathbb{R}^2$, consisting of unit vectors. Zygmund posed a question asking if

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x - tv(x))dt = f(x)$$

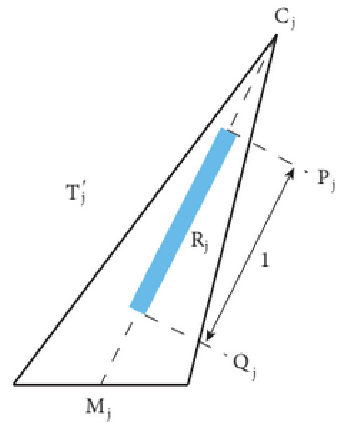

Figure 9: The rectangle R_j

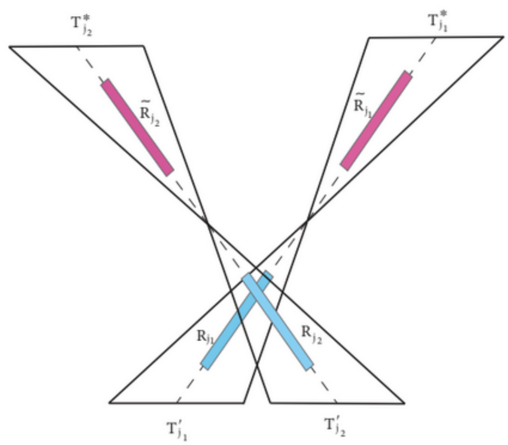

Figure 10: The disjointness of $\tilde{R}_j$'s

for a.e. $x \in \mathbb{R}^2$, where v is a Lipschitz vector field and $f \in L^2$. In other words, he asked whether any L^2 function is differentiable along Lipschitz directions. This is a longstanding problem and it turns out to be extremely challenging. It is even unknown for the C^∞ vector fields. The real enemy again is caused by the Besicovitch set. Some known positive results on the real analytic vector fields were proved by Bourgain, who was able to show that the Besicovitch set can not occur in the real analytic vector field case.

22 L^p ($p \neq 2$) Unboundedness of Disc Multipliers

We are interested in the **disc multiplier operator**, given by

$$\widehat{T_B f}(\xi) = \chi_B(\xi)\hat{f}(\xi),$$

for $f \in L^2(\mathbb{R}^2)$. Here B is a ball in R^2. It is easy to see, by Plancherel theorem, that T_B is bounded on L^2. It was proved by C. Fefferman in his Ph.D. thesis that T_B can not be bounded on any L^p if $p \neq 2$. Let D denote the unit ball in $\mathbb{R}^2$, centered at the origin. By a standard translation and dilation argument, it is easy to see that the L^p-boundedness of T_B is equivalent to that of T_D. More precisely, suppose that

$$\|T_D f\|_p \leq C_p \|f\|_p \tag{1}$$

for all $f \in L^p$. Then

$$\|T_B f\|_p \leq C_p \|f\|_p \tag{2}$$

for all $f \in L^p$. Conversely, (2) implies (1). Hence, we see that the L^p-norm of T_B (if exists) is independent of the location and the magnitude of B.

Theorem 52. The disc multiplier T_B is unbounded on $L^p(\mathbb{R}^2)$ provided $p \neq 2$.

Remark 16. This theorem is still valid in the higher dimensional $\mathbb{R}^n$ case. The unboundedness result follows from a surprising application of the Besicovitch set, discussed in Lecture 21. It reinforces what we said in the last lecture, there is a significant difference between 1-dimensional and higher-dimensional one. In the 1-dimensional case, the interval multiplier is an L^p-multiplier for all $p \in (1, \infty)$ because it can be represented as linear combinations in terms of the Hilbert transform. However, the L^p-boundedness if the ball multiplier operator turns out to be false in the higher dimensional $\mathbb{R}^n$.

We need some background before we give a proof to the theorem.

Definition 35. For any unit vector $u \in \mathbb{R}^2$, we define

$$S^u f(x) = \int_{\xi \cdot u > 0} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,$$

which is the multiplier operator whose multiplier is the characteristic function of the half plane $\{\xi : \xi \cdot u > 0\}$.

We will see some relation between the disc multiplier T_B and the operator S^u.

Lemma 38. Given $p \in [1, \infty)$, let $u_1, \cdots, u_N$ be unit vectors in $\mathbb{R}^2$ and $f_1, \cdots, f_N \in L^2 \cap L^p$. Suppose that the disc multiplier T_D is bounded on L^p. Then

$$\left\| \left(\sum_{j=1}^N |S^{u_j} f_j|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left(\sum_{j=1}^N |f_j|^2 \right)^{1/2} \right\|_p .$$

Here C_p is an absolute constant depending on p but independent of N, u_j and f_j's.

Proof. Let us recall a result in Lecture 11, following from Khinchin's inequality (Lemma 13). Any L^p-bounded linear operator T satisfies the following vector-valued inequality,

$$\left\| \left(\sum_{j=1}^{N} |Tf_j|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left(\sum_{j=1}^{N} |f_j|^2 \right)^{1/2} \right\|_p , \tag{3}$$

where C_p is independent of N and f_j's. We will use this inequality in the proof.

For any unit vector $u \in \mathbb{R}^2$, let B_R^u denote a ball of radius R, centered at Ru (See Figure 11 below). As $R \to \infty$, the B_R^u tends to the half plane $\{\xi \in \mathbb{R}^2 : \xi \cdot u > 0\}$.

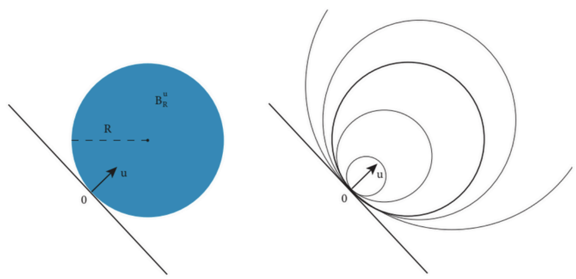

Figure 11: The ball B_R^u and its limit

From the definition of B_R^u and T_B, it is clear that

$$T_{B_R^u} f(x) = e^{2\pi Ru\cdot x} T_{B_R}(fe^{-2\pi i Ru\cdot(\cdot)})(x), \tag{4}$$

where B_R is a ball, of radius R, centered at the origin. Because of the L^p-boundedness of T_D, there exists a constant C_p such that

$$\|T_D f\|_p \leq C_p \|f\|_p$$

for all $f \in L^p$. By the equivalence of (1) and (2), we get for any $f \in L^p$,

$$\|T_{B_R} f\|_p \leq C_p \|f\|_p.$$

From (4), we also obtain

$$|T_{B_R^u} f_j| = |T_{B_R}(f_j e^{2\pi i Ru_j\cdot(\cdot)})|,$$

so that we can apply (3) to the operator T_{B_R} and the functions $f_j e^{2\pi i Ru_j\cdot(\cdot)}$'s, and then end up with

$$\left\| \left(\sum_{j=1}^{N} |T_{B_R^u} f_j|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left(\sum_{j=1}^{N} |f_j|^2 \right)^{1/2} \right\|_p . \tag{5}$$

Observe that, since the ball B_R^u increases to fill up the half plane $\{\xi \cdot u > 0\}$, DCT yields

$$\lim_{R\to\infty} \|T_{B_R^u} f - S^u f\|_2 = 0$$

whenever $f \in L^2$. We see that $T_{B_R^{u_j}} f_j$ converges to $S^{u_j} f_j$ in L^2, and consequently an appropriate subsequence converges to $S^{u_j} f_j$ a.e. Therefore the lemma follows from (5) by Fatou's Lemma. $\square$

Definition 36. For $f \in L^2(\mathbb{R}^n)$, $\epsilon > 0$, we define S^+ and S_ϵ by

$$\widehat{S^+ f}(\xi) = \chi_{(0,\infty)}(\xi)\hat{f}(\xi),$$
$$\widehat{S_\epsilon f}(\xi) = \chi_{(0,\infty)}(\xi)e^{-2\pi\epsilon\xi}\hat{f}(\xi).$$

By Fourier inversion theorem, S_ϵ can be written as

$$S_\epsilon f(x) = \int_0^\infty \hat{f}(\xi)e^{2\pi i(x+i\epsilon)\xi}d\xi.$$

Moreover, by Plancherel theorem, we see that for any $f \in L^2$,

$$S^+ f(x) = \lim_{\epsilon\to 0^+} S_\epsilon f(x) = \lim_{\epsilon\to 0^+}\int_0^\infty \hat{f}(\xi)e^{2\pi i(x+i\epsilon)\xi}d\xi, \tag{6}$$

where $x \in \mathbb{R}$ and the limits are taken in the L^2 sense.

Lemma 39. There is a positive constant C such that for any $|x| \geq 1$,

$$|S^+\chi_{(-1/2,1/2)}(x)| \geq \frac{C}{|x|}.$$

Proof. Notice that for any $f \in L^1 \cap L^2$ and any $\epsilon > 0$,

$$\begin{aligned}\int_0^\infty \hat{f}(\xi)e^{2\pi i(x+i\epsilon)\xi}d\xi &= \int_0^\infty\left(\int_R f(y)e^{-2\pi iy\xi}dy\right)e^{2\pi i(x+i\epsilon)\xi}d\xi \\ &= \int_{\mathbb{R}} f(y)\left(\int_0^\infty e^{-2\pi iy\xi}e^{2\pi i(x+i\epsilon)}d\xi\right)dy = \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{f(y)}{y-x-i\epsilon}dy.\end{aligned}$$

Here we used the Fourier transform of L^1 functions and Fubini's theorem, which is valid because of the integrability of the integrand. By (6), we can represent

$$S^+\chi_{(-1/2,1/2)}(x) = \lim_{k\to\infty}\int_0^\infty \hat{f}(\xi)e^{2\pi i(x+i\epsilon_k)}d\xi,$$

where $\{\epsilon_k\}$ is some positive sequence whose limit is 0. Employing this fact and the representation of $S_\epsilon f$ we just derived, we see that

$$|S^+\chi_{(-1/2,1/2)}(x)| \geq \lim_{k\to\infty}\frac{1}{2\pi}\left|\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{y-x-i\epsilon_k}dy\right| \geq \frac{C}{|x|},$$

since $|x| \geq 1$. □

Lemma 40. Let u be a unit vector and R denotes a thin rectangle of dimensions 1×2^{-N}, which is parallel to the vector u. $\tilde{R}$ represents the translation of R along u direction by two units. Then

$$|S^u\chi_R(x)| \geq C\chi_{\tilde{R}}(x).$$

Here C is an absolute constant independent of u, R and x.

Proof. Since the inequality in invariant under translation and rotation, we can set up an appropriate coordinate axes so that u is in the x_1-direction and

$$R = \left\{(x_1,x_2) : -\frac{1}{2} < x_1 < \frac{1}{2}, -2^{-N-1} < x_2 < 2^{-N-1}\right\},$$

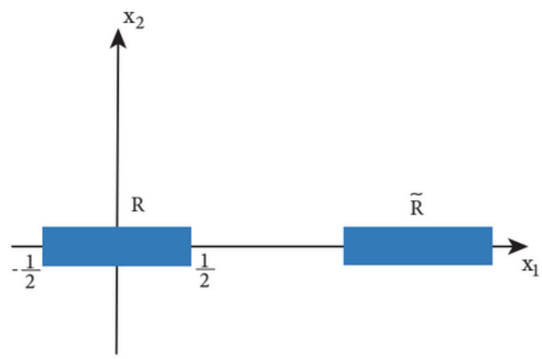

Figure 12: The rectangle R and $\tilde{R}$ in the new coordinate axes

and

$$S^u \chi_R(x_1, x_2) = \int_{\xi_1 > 0} \hat{\chi_R}(\xi_1, \xi_2) e^{2\pi i (x_1 \xi_1 + x_2 \xi_2)} d\xi_1 d\xi_2.$$

Since now we have $\chi_R(x_1, x_2) = \chi_{(-1/2,1/2)}(x_1)\chi_{(-2^{-N-1},2^{-N-1})}(x_2)$, the Fourier transform of χ_R can be represented as

$$\hat{\chi_R}(\xi_1, \xi_2) = \widehat{\chi_{(-1/2,1/2)}}(\xi_1)\hat{\chi}_{(-2^{-N-1},2^{-N-1})}(\xi_2),$$

from which we get

$$S^u \chi_R(x_1, x_2) = S^+ \chi_{(-1/2,1/2)}(x_1)\chi_{(-2^{-N-1},2^{-N-1})}(x_2).$$

When $x = (x_1, x_2) \in \tilde{R}$, $|x_1| > 1$ and $x_2 \in (-2^{-N-1}, 2^{-N-1})$. Consequently, for $x \in \tilde{R}$,

$$|S^u \chi_R(x)| \geq \frac{C}{|x_1|} \geq C,$$

following from Lemma 39 and the fact that $|x_1|$ is bounded above by $5/2$ when $(x_1, x_2) \in \tilde{R}$. The desired estimate then follows. $\square$

Proof of Theorem 52. We're ready to give the proof to our main result. By equivalence of (1) and (2), it suffices to show L^p-unboundedness of T_D, where D is the unit ball (disc) centered at the origin. We can also assume $1 < p < 2$ since $p > 2$ case follows by duality and the case $p = 1$ is a consequence of the complex interpolation. We prove the main result by contradiction. Assume that there is a number $1 < p < 2$ such that T_D is bounded on L^p. We aim to derive a contradiction under the assumption.

The main tool is Besicovitch set, discussed in Lecture 21. By Theorem 51, the Besicovitch construction, for any $\epsilon > 0$, we can take a collection of rectangles $R_1, \cdots, R_{2^N}$ such that each of those rectangles has side length 1×2^{-N},

$$\left| \bigcup_{j=1}^{2^N} R_j \right| < \epsilon,$$

and $\tilde{R}_j$'s are mutually disjoint so that $|\cup_j \tilde{R}_j| = 1$. Let u_j be the unit vector in the positive direction of the longest side of R_j. Then Lemma 40 yields

$$|S^{u_j} \chi_{R_j}(x)| \geq C\chi_{\tilde{R}_j}(x),$$

from which we see that

$$\left\| \left(\sum_{j=1}^{2^N} |S^{u_j} \chi_{R_j}|^2 \right)^{1/2} \right\|_p \geq C \left\| \left(\sum_{j=1}^{2^N} |\chi_{\tilde{R}_j}|^2 \right)^{1/2} \right\| \geq C \left| \bigcup_j \tilde{R}_j \right|^{1/p} = C.$$

On the other hand, by Lemma 38, we obtain for $1 < p < 2$,

$$\begin{aligned} \left\| \left(\sum_{j=1}^{2^N} |S^{u_j} \chi_{R_j}|^2 \right)^{1/2} \right\|_p &\leq C_p \left\| \left(\sum_{j=1}^{2^N} |\chi_{R_j}|^2 \right)^{1/2} \right\|_p \\ &\leq C_p \left\| \left(\sum_{j=1}^{2^N} |\chi_{R_j}|^2 \right)^{1/2} \right\|_2 \cdot \left| \bigcup_{j=1}^{2^N} R_j \right|^{\frac{1}{p}-\frac{1}{2}} && \text{(Hölder)} \\ &= C_p \left(\sum_j |R_j| \right)^{1/2} \cdot \left| \bigcup_{j=1}^{2^N} R_j \right|^{\frac{1}{p}-\frac{1}{2}} \\ &\leq C_p \epsilon^{\frac{1}{p}-\frac{1}{2}}. \end{aligned}$$

Putting this upper bound together with the lower bound above, we get

$$C \leq C_p \epsilon^{\frac{1}{p}-\frac{1}{2}}.$$

By letting $\epsilon \to 0$, it is clear that this is impossible since C is positive. Therefore, the L^p-unboundedness of the disc multiplier, or Theorem 52, is established. $\square$

References

[1] Folland, G. B. *Real analysis : Modern Techniques and Their Applications (*2^{nd} *edition)*. John Wiley, 1999. ISBN 978-0-471-31716-6

[2] David, G. & Journé, J-L. *A boundedness criterion for generalized Calderón-Zygmund operators*. Annals of Mathematics, Second Series, 120 (2): 371–397, 1984. doi:10.2307/2006946

[3] Stein, E. M. & Murphy, T. S. *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, 1993. ISBN 0-691-03216-5

[4] Bass, R. F. *Real Analysis for Graduate Students*. Createspace Ind Pub, 2013. ISBN 1-481-86914-0

[5] 周民强. 调和分析讲义（实变方法）. 北京大学出版社, 2003. 6. ISBN 978-7-301-04137-6

[6] Stein, E. M. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, 1971. doi:10.2307/j.ctt1bpm9w6

[7] Stein, E. M. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, 1971. ISBN 978-0-691-08079-6

[8] Duoandikoetxea, J. *Fourier analysis*. Graduate studies in math. Vol. 29., AMS, 2001. ISBN 978-0-821-82172-5

[9] Wolff, T. H. *Lectures on Harmonic Analysis*. Univ. Lecture Series, Vol. 29, AMS, 2003. ISBN 978-0-821-83449-7