Practical Higher Algebra

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Outline

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Collection of Homotopy Theory

Final Goal

Get to know the motivic homotopy theory, including examples, computations, applications, etc.

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It is a kind of homotopy theory, or the homotopy theoretic applications anyway! Natural questions to ask:

- What do homotopy theorists care?
- What are the techniques?
- How to apply these methods to the algebraic geometry?

Quick reminder of spectra

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Upshot

Spectra, or the category of spectra Sp, is one of the most important things that algebraic topologists care about.

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Similarly for the morphisms!

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Let $A = \mathbb{Z}[\alpha, \beta]/(f(\alpha, \beta))$ and $g : A \to \mathbb{Z}$. The knowledge of elements and operations of A, and the knowledge of the morphism g are compressed in their descriptions and can be completely retrieved from the definitions of A and g.

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Example

Let X be a CW complex. $\pi_n(X) = [S^n, X]$ encodes the information about attaching maps and building blocks (cells) of X up to weak equivalence. You can completely get the information of X from its homotopy groups, up to weak equivalence (Whitehead's theorem).

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One has $\eta\nu = \nu\eta = 0$, but $\langle \eta, \nu, \eta \rangle = \{\pm\sigma\}$. Here $\langle f, g, h \rangle$ is the Toda bracket for $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$ with $fg \simeq 0 \simeq gh$, which is the collection of homotopy classes of maps from $\Sigma X \to W$.

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To understand the higher associativity/commutativity, and actually retrieve the information of these properties from the new "zip file", we need knowledge of higher algebra. Higher algebra lets us both encode and decode these layers of structures.

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- Let $Q = \mathrm{colim}_n \Omega^n \Sigma^n$, then the following diagram commutes:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \Omega^\infty \Sigma^\infty (X) \arrow[d, "\alpha"] \\ & QX \end{tikzcd}$$

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To fulfill all requirements, one needs to work in the ∞-category.

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We need the higher algebra!

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Hochschild homology	topological Hochschild homology

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- **Compute the homotopy groups of spectra!**
- Atiyah-Hirzebruch SS, Adams SS, Adams-Novikov SS, slice SS, Bockstein SS, algebraic slice SS, etc. (in details in the later lectures)

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Slogan

Motivic homotopy theory $=$ encoding/decoding the information of schemes via sheaves of spaces or spectra (i.e. the functor-of-points viewpoint), within the framework of higher algebra.

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Simplicial Objects

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If $\mathcal{C} = \mathrm{Set}$, then a simplicial object X in $\mathcal{C}$ is called a **simplicial set**. The category of simplicial sets is denoted $s\mathrm{Set}$.

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Proposition

The face maps and degeneracy maps satisfy

$$d_i d_j = d_{j-1} d_i, \quad i < j; \tag{1}$$

$$s_j s_i = s_{i+1} s_j, \quad j \leq i; \tag{2}$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j; \\ \text{id} & i = j, j+1; \\ s_j d_{i-1} & i > j+1. \end{cases} \tag{3}$$

Face and Degeneracy

Face and Degeneracy

Theorem

For any $f \in \mathrm{Hom}_\Delta([n],[m])$*, f can be uniquely decomposed into* $f = d^{i_1} \cdots d^{i_r} s^{j_1} \cdots s^{j_s}$*, where* $m = n - s + r$*,* $i_1 < \cdots < i_r$*,* $j_1 < \cdots < j_s$*, up to linear order.*

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For example, if we write $f : [4] \to [2]$. Then $f = s^0 \circ s^2$ because s^0 doubles 0 and s^2 doubles 2.

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Similarly, for any simplicial set X, any morphism from $X_n \to X_m$ can be uniquely decomposed into the face maps d_i and the degeneracy maps s_j.

Example: Standard n-simplex

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Definition

By Yoneda embedding, any $[n] \in \Delta$ associates to $\mathrm{Hom}_\Delta(-,[n])$. Write $\Delta[n] = \mathrm{Hom}_\Delta(-,[n]) \in s\mathrm{Set}$, with $\Delta[n]_k = \mathrm{Hom}_\Delta([k],[n])$. This is called a **standard n-simplex**.

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So standard n-simplices recover the information of the simplicial sets.

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A **left Kan extension** of F along G is a functor $\mathrm{Lan}_G F : \mathcal{E} \to \mathcal{D}$ together with a natural transformation $\eta : F \Rightarrow \mathrm{Lan}_G F \circ G$ that is universal from F to $\mathrm{Lan}_G F \circ G$.

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$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\eta'"] \arrow[d, Rightarrow, "\eta"'] & S \circ G \\
\mathrm{Lan}_G F \circ G \arrow[ur, Rightarrow, "\varphi\circ"'] &
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Dually, one can define the right Kan extension.

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Intuitively, a left Kan extension is a map such that the diagram commutes at each object and morphism:

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The importance of the Kan extensions is revealed in the definition of geometric realization.

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Let $Y : \Delta \to s\mathrm{Set}$ be the Yoneda embedding sending $[n]$ to $\Delta[n]$, and $\Delta^* : \Delta \to \mathrm{Top}$ be the Δ-complex functor sending $[n]$ to Δ^n.

Geometric Realization

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Let $Y : \Delta \to s\mathrm{Set}$ be the Yoneda embedding sending $[n]$ to $\Delta[n]$, and $\Delta^* : \Delta \to \mathrm{Top}$ be the Δ-complex functor sending $[n]$ to Δ^n. The left Kan extension of Δ^* along Y is then called the **geometric realization**, denoted by $|-| := \mathrm{Lan}_Y \Delta^*$. One can visualize it as the following diagram:

$$\begin{tikzcd}
\Delta \arrow[r, "Y"] \arrow[d, "\Delta^*"'] & s\mathrm{Set} \arrow[dl, dashed, "\mathrm{Lan}_Y \Delta^* = |-|"] \\
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\end{tikzcd}$$

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For X_* a simplicial set,

$$|X_*| = \left(\coprod_{n \geq 0} X_n \times \Delta^n \right) / \sim,$$

where $(f_*(x), t) \sim (x, f^*(t))$ for any $x \in X_n$, $t \in \Delta^n$, and $f_* = X_*(f)$, $f^* = \Delta^*(f)$ are induced by $f : [m] \to [n]$ in Δ.

Outline

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Intuitively, a horn Λ_j^n is the union of all faces of $\Delta[n]$ except the j-th one.

Horns

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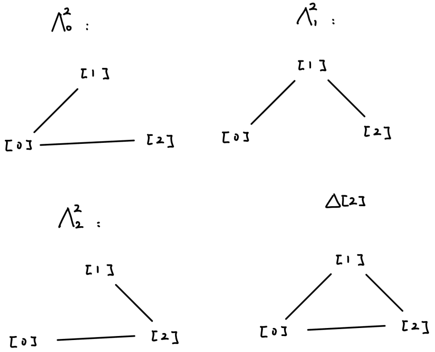
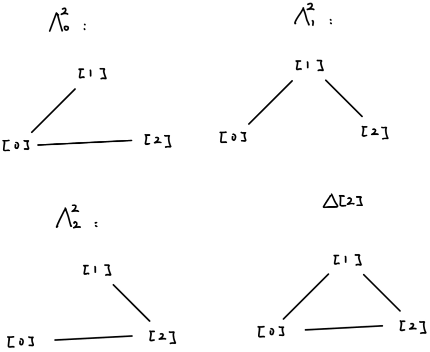
Horns Λ_j^2 for $j = 0, 1, 2$:



Horns

Horns Λ^2_j for $j = 0, 1, 2$:



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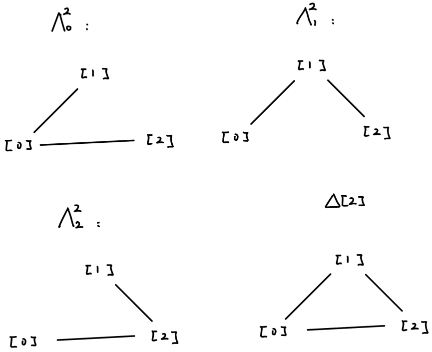

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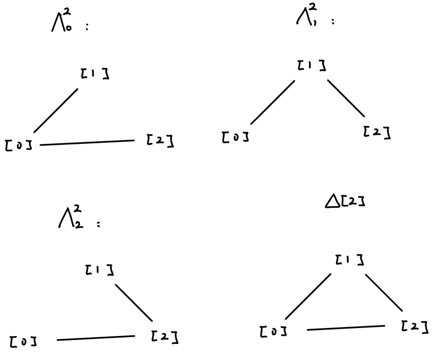

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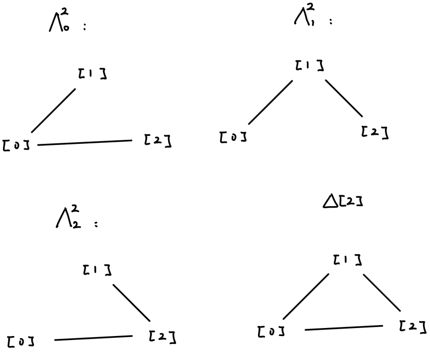

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- If $0 < j < n$, It is called an **inner horn**.
- If $j > 0$, it is called an **right horn**.
- If $j < n$, it is called an **left horn**.

Nerves

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In other words,

$$N\mathcal{C}_n = \operatorname{Hom}_{\mathsf{Cat}}([n], \mathcal{C}).$$

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$$d_i : [c_0 \to \cdots \to c_n] \mapsto [c_0 \to \cdots \to c_{i-1} \to \widehat{c_i} \to c_{i+1} \to \cdots \to c_n]$$

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Theorem

The nerve functor $N : \mathrm{Cat} \to s\mathrm{Set}$ is fully faithful, i.e.

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Hom}_{s\mathrm{Set}}(N\mathcal{C}, N\mathcal{D}).$$

∞-categories

Definition

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An ∞-category is a simplicial set, not a category in the usual sense!

∞-categories

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Definition

A natural transformation $\eta : F_0 \Rightarrow F_1$ between $F_0, F_1 : \mathcal{C} \to \mathcal{D}$ is the map

$$\eta : \mathcal{C} \times \Delta[1] \to \mathcal{D}$$

in $s\mathrm{Set}$ such that $\eta |_{\mathcal{C} \times [i]} = F_i$.

Sub-∞-categories

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Definition

Let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcomplex. It is a **sub-∞-category** if for $n \geq 2$, $0 < j < n$, every $f : \Delta[n] \to \mathcal{C}$ such that $f(\Lambda^n_j) \subseteq \mathcal{C}'$ satisfies $f(\Delta[n]) \subseteq \mathcal{C}'$.

A sub-∞-category $\mathcal{C}' \subseteq \mathcal{C}$ is **full**, if for all n and $x = (x_0 \to x_1 \to \cdots \to x_n) \in \mathcal{C}_n$, one has $x \in \mathcal{C}'_n$ iff $x_i \in \mathcal{C}'_0$ for all $0 \leq i \leq n$.

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K is a **Kan complex**, if for all $0 \le j \le n$, $n \ge 1$, the map

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An ∞-**groupoid** is an ∞-category such that every morphism is an equivalence.

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Theorem (Joyal)

Every Kan complex is an ∞-groupoid, and vice versa.

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Let Kan be the full subcategory of sSet spanned by the collection of Kan complexes. The ∞-**category of spaces** is then defined to be

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There are other ways to define a suitable notion of the ∞-category of spaces. However, we will end up with some new ∞-category which is equivalent to $\mathcal{S} = N(\mathrm{Kan})$.

Stable ∞-categories

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- fiber = kernel.
- cofiber = cokernel.

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Example

Let $\mathcal{C}$ be a pointed ∞-category with finite limits. Its **stabilization**, denoted $\mathrm{Sp}(\mathcal{C})$, is the homotopy inverse limit of the tower

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$\mathrm{Sp}(\mathcal{C})$ is guaranteed to be stable, no matter $\mathcal{C}$ is or not.

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References

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Thank you!