Grothendieck topologies and sheaves

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Outline

Motivation

Slogan

Motivic homotopy theory is doing homotopy theory on schemes.

In order to do that, we need to view a scheme X as a **functor**:

The functor of points perspective

For each scheme X, we can define a functor h_X as follows:

$$h_X \colon \mathrm{Sch}^{op} \to \mathrm{Set}, \quad Y \mapsto \mathrm{Hom}(Y, X)$$

The correspondence $h \colon \mathrm{Sch} \to \mathrm{Fun}(\mathrm{Sch}^{op}, \mathrm{Set})$ is fully faithful by Yoneda embedding, therefore we can identify X with the functor h_X.

Zariski topology

For a scheme X, the underlying space of X is endowed with a topology which we call the **Zariski topology**.

Definition

For an affine scheme $X = \mathrm{Spec}(R)$, the (Zariski) open sets of X are of the form $X \setminus V(I)$ where $I \subset R$ is an ideal of R and $V(I) = \{p \in \mathrm{Spec}(R),\ p \supset I\}$.

- For $X = \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[x])$, the underlying space of X is $\mathbb{C}$, and the nontrivial (Zariski) open sets are of the form $\mathbb{C} \setminus \{\mathrm{pts}\}$.
- If X is irreducible, the space of X is also irreducible, and in particular not Hausdorff (if X is not a single point).

Sheaf structure

h_X is actually a **sheaf** with respect to the Zariski topology.

Proposition (Sheaf property)

Suppose $\{U_i\}$ is a (Zariski) open cover of a scheme Y, and $f_i \in h_X(U_i) = \mathrm{Hom}(U_i, X)$. Suppose $f_i = f_j$ when both restricted to $U_i \cap U_j$ for all i, j, then there exists a unique $f \in h_X(Y) = \mathrm{Hom}(Y, X)$ such that f restricts to f_i for all i.

However, the Zariski topology is too coarse (has too few open sets) to work with.

Theorem (Grothendieck)

For an irreducible scheme X, $H^r_{\mathrm{Zar}}(X; \mathcal{F}) = 0$ for all constant sheaves $\mathcal{F}$ and $r > 0$.

Refined topologies

Our respected category of motivic spaces will actually consist of sheaves over smooth schemes with respect to a carefully chosen topology, the **Nisnevich topology**.

Nisnevich topology is closely related to **étale** topology, which plays an essential role in Grothendieck's solution of Weil conjectures.

Proposition (Comparison theorem)

For a smooth complex variety X and a finite abelian group Λ, there is an isomorphism

$$H^r(X(\mathbb{C}); \Lambda) \simeq H^r_{\text{ét}}(X; \Lambda).$$

Outline

Étale morphisms of varieties

Let k be an algebraically closed field.

Recall that for a variety X over k, the Zariski tangent space at a point $x \in X$ is defined as $T_x X := (\mathfrak{m}_x/\mathfrak{m}_x^2)^\wedge$ where $\mathfrak{m}_x$ is the maximal ideal corresponding to x, and $(-)^\wedge$ is the dual vector space. We say X is smooth if $\dim_k(T_x X) = \dim X$ for all x.

Definition

Let X, Y be smooth varieties over k. Then a morphism $f\colon X \to Y$ is called étale if for any $x \in X$, the induced map on tangent spaces $T_x f\colon T_x X \to T_{f(x)} Y$ is an isomorphism.

Étale morphisms of schemes

Definition (Flat morphisms)

A ring map $A \to B$ is called **flat** if B is a flat A-module. In other words, the functor $A\text{-mod} \to B\text{-mod}, \quad M \mapsto M \otimes_A B$ is exact, i.e. sends exact sequences to exact sequences.
A morphism $f\colon X \to Y$ between schemes X, Y is called **flat** if for each $x \in X$, the local homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

A flat morphism $f\colon X \to Y$ is the algebraic geometric analogue of a continuous family of manifolds $X_y = f^{-1}(y)$. In fact if f is flat, we have

$$\dim f^{-1}(y) = \dim X - \dim Y$$

provided that $f^{-1}(x)$ is nonempty.

Étale morphisms of schemes

Definition (Unramified morphisms)

A local homomorphism $f\colon A \to B$ of local rings is called **unramified** if $B/f(\mathfrak{m}_A)B$ is a finite seperable field extension of $A/\mathfrak{m}_A$.
A morphism $f\colon X \to Y$ between schemes is called **unramified** if for each $x \in X$, the local homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is unramified.

Unramified morphisms are generalizations of seperable extensions. There is another characterization of unramified morphisms:

Proposition

Let $f\colon X \to Y$ be a map between schemes of finite type. Then f is unramified if and only if the sheaf of differentials $\Omega_{Y/X} = 0$.

Étale morphisms of schemes

Definition (Étale morphisms)

We say a morphism $f\colon X \to Y$ between schemes is **étale** if it is flat and unramified.

An Étale morphisms is the algebraic geometric analogue of a local isomorphism for manifolds. For smooth varieties X, Y, if $f\colon X \to Y$ is étale then all the fibers $f^{-1}(x)$ are either empty or disjoint single points (of multiplicity 1).

Proposition (local description)

A finite type morphism $f\colon X \to Y$ of schemes is étale if and only if there are open covers $\{U_i\}, \{V_i\}$ of X, Y with $f\colon U_i \to V_i$, such that $f\colon U_i \to V_i$ is isomorphic to $\mathrm{Spec}(B[x]_h/(g)) \to \mathrm{Spec}(B)$ for some ring B and $g, h \in B[x]$, with g monic and g' invertible in $B[x]_h/(g)$.

Basic properties

- An open immersion is étale.
- The composition of two étale morphisms is étale.
- A base change of an étale morphism is étale.
- An étale map is open.
- A finite étale map is the analogue of a covering map in topology. We can define the étale fundamental group of a scheme from the category of finite étale maps, similar to the covering space theory in topology. For a complex varietie X, the étale fundamental group of X is the profinite completion of $\pi_1(X)$.

Nisnevich morphism

Definition (Nisnevich morphism)

A morphism $f\colon X \to Y$ is called **Nisnevich**, if it is étale, and for each $y \in Y$, there exists $x \in f^{-1}(y)$, such that the induced map on residue fields $k(y) \to k(x)$ is an isomorphism.

Étale morphisms are not necessarily Nisnevich. For fields $k \subset L$, the map $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is étale if and only if L is a finite separable extension of k, but it is Nisnevich if and only if $L = k$.

Local rings

Definition

Let X be a scheme and $x \in X$. An **étale neighborhood** of x is an étale map $(U, u) \to (X, x)$. The connected affine étale neighborhoods forms a directed set by setting $(U, u) \leq (U', u')$ if there exists a map $(U, u) \to (U', u')$. Then we define the **local ring at** x **for étale topology** as

$$\mathcal{O}^{ét}_{X,x} = \varinjlim \Gamma(U, \mathcal{O}_U)$$

Note that if we replace "étale" by "Zariski" in the above definition, we will get the usual local ring $\mathcal{O}_{X,x}$ of $x \in X$. We can also replace "étale" by "Nisnevich" and obtain the Nisnevich local ring $\mathcal{O}^{\mathrm{Nis}}_{X,x}$.

Since every open immersion is Nisnevich, and every Nisnevich map is étale, we have maps

$$\mathcal{O}_{X,x} \to \mathcal{O}^{\mathrm{Nis}}_{X,x} \to \mathcal{O}^{\mathrm{ét}}_{X,x}.$$

Local rings

Definition (Henselian rings)

A local ring $(R, \mathfrak{m})$ is called **Henselian** if Hensel's lemma holds. This means if $p \in R[x]$, then any factorization of its image in $R/\mathfrak{m}[x]$ into a product of coprime monic polynomials can be lifted to a factorization in $R[x]$. It is called **strict Henselian** if it is Henselian and the residue field $R/\mathfrak{m}$ is separably closed.

In particular, fields are Henselian, separably closed fields are strict Henselian.

Local rings

Proposition (Henselization)

Let $(R, \mathfrak{m})$ be a local ring. Then there exists a unique Hensel ring R^h together with a map $R \to R^h$, such that any local map $R \to B$ where B is Henselian can be uniquely extended to R^h. We call R^h the **Henselization** of the ring R.
Similarly, there exists a unique strict Hensel ring R^{sh} together with a map $R \to R^{sh}$, such that any local map $R \to B$ where B is strict Henselian can be extended to R^{sh}, which is unique up to an automorphism of $R^{sh}/\mathfrak{m}^{sh}$. We call R^{sh} the **strict Henselization** of the ring R.

Proposition

$\mathcal{O}_{X,x}^{\mathrm{Nis}}$ is the Henselization of the local ring $\mathcal{O}_{X,x}$, and $\mathcal{O}_{X,x}^{\text{ét}}$ is the strict Henselization of the local ring $\mathcal{O}_{X,x}$.

Outline

Definition

The étale/Nisnevich topology we will define is not a topology on a space, but rather a topology on **category**, which we call the Grothendieck topology.

Definition (Grothendieck topology)

Let C be a category with pullbacks. Then a **Grothendieck topology** τ on C is a collection of families of morphisms $\{U_i \to U\}_{i\in I}$ called *coverings*, which satisfy the following conditions:

(1) *Closed under pullback*. If $\{U_i \to X\}_{i\in I}$ is a covering and $Y \to X$ is a morphism, then $\{U_i \times_X Y \to Y\}_{i\in I}$ is also a covering.

(2) *Closed under refinement*. If $\{U_i \to X\}$ is a covering, and $\{V_{ij} \to U_i\}$ are coverings of U_i, then the composition $\{V_{ij} \to U_i \to X\}_{i,j}$ is also a covering.

(3) *Isomorphisms*. Any isomorphism $\{U \xrightarrow{\sim} X\}$ is a covering.

A category C equipped with a Grothendieck topology is called a **site**.

Examples

- Let X be a topological space. Then we can define a site X_{top} whose objects are open subsets of X and morphisms are inclusions. The coverings are $\{V_i \to V\}_{i \in I}$ where $\bigcup_i V_i = V$.
- Let G be a group. We can define a site T_G whose underlying category is the category of G-sets and G-maps, and coverings are $\{S_i \xrightarrow{f_i} S\}_{i \in I}$ such that $\bigcup_i f_i(S_i) = S$.
- Let C be the category of n-dimensional polytopes in $\mathbb{R}^n$, whose morphisms are inclusions. We say $\{P_i \to P\}$ is a covering if $\bigcup_i P_i = P$. Then all the coverings form a Grothendieck topology on C.

We say that a family of maps $\{U_i \to U\}_{i \in I}$ is *jointly surjective* if the disjoint union $\bigsqcup_i U_i \to U$ is surjective.

Examples in algebraic geometry

Now let X be a scheme.

- **The Zariski site**. The site X_{Zar} is the site associated to the (Zariski) topological space X.
- **The small étale site**. The site $X_{\text{ét}}$ has the underlying category $\text{Ét}/X$, whose objects are étale maps $U \to X$ and morphisms are X-morphisms. The coverings are étale morphisms $\{U_i \to U\}$ that are jointly surjective.
- **The big étale site**. The site $X_{\text{Ét}}$ has the underlying category Sch/X, whose objects are maps $Y \to X$ and morphisms are X-morphisms. The coverings are jointly surjective étale morphisms.

Examples in algebraic geometry

- **The small Nisnevich site**. The site X_{nis} has the underlying category $\acute{\mathrm{E}}\mathrm{t}/X$, whose coverings are $\{U_i \to U\}$ such that $\bigsqcup_i U_i \to U$ is Nisnevich and surjective.
- **The big Nisnevich site**. The site X_{Nis} has the underlying category Sch/X, whose coverings are $\{U_i \to U\}$ such that $\bigsqcup_i U_i \to U$ is Nisnevich and surjective.
- **The fppf site**. The site X_{fppf} has the underlying category Sch/X, whose covering are jointly surjective morphisms $\{U_i \xrightarrow{f_i} U\}$ such that each f_i is flat and locally of finite presentation.
- **The fpqc site**. The site X_{fpqc} has the underlying category Sch/X, whose covering are jointly surjective morphisms $\{U_i \xrightarrow{f_i} U\}$ such that $\bigsqcup_i f_i$ is faithfully flat and quasi compact.

Hierarchy

For a category C and two topologies τ and σ on C, we say that $\tau \leq \sigma$ if every τ-cover is a σ-cover. In this case, we say that σ is finer than τ (or τ is coarser than σ).

Topologies on Sch/X

$$\mathrm{Zar} \leq \mathrm{Nis} \leq \mathrm{\acute{E}t} \leq \mathrm{fppf} \leq \mathrm{fpqc}.$$

An extended picture: Belmans: topologies comparison.

Remark

The difference between small and big sites ($X_{ét}$ and $X_{Ét}$ with different underlying categories: $\mathrm{Ét}/X$ and Sch/X):

- Small sites are easier to describe.
- In terms of computing sheaf cohomology, they are the same.
- But if we want to view schemes as sheaves over corresponding sites, it is necessary to use big sites. For example, if $Z \to X$ is a closed immersion, then $\mathrm{Hom}_X(-, Z)$ is the empty sheaf on the small site $\mathrm{Ét}/X$.

Therefore, in our setting for motivic homotopy theory, we mainly use the big sites. (Or Sm/X, the category of smooth schemes over X)

Fields

Let k be a field, $X = \mathrm{Spec}(k)$.

- X_{Zar} consists of two objects $\emptyset, \mathrm{Spec}(k)$. ($X$ is a single point)
- X_{nis} consists of objects $\emptyset, \bigsqcup \mathrm{Spec}(k)$.
- $X_{\text{ét}}$ consists of $\emptyset$, and objects of the form $\bigsqcup_i \mathrm{Spec}(k_i)$ where k_i/k is a finite separable extension.

Outline

Definition (Presheaf)

Let C be a site. Then a **presheaf** on C is a functor $C^{op} \to \mathrm{Set}/\mathrm{Ab}$.

When $C = X_{\mathrm{top}}$ for a topological space X, this definition obviously coincide with our classical definition of a presheaf.

Sheaf condition

In the classical case, we say a presheaf $\mathcal{F}$ on a space X is a sheaf if for any open cover $\{U_i\}_{i\in I}$ of an open subset $U \subset X$ and $x_i \in \mathcal{F}(U_i)$ such that $x_i|_{U_i\cap U_j} = x_j|_{U_i\cap U_j}$ for all i, j, there exists a unique $x \in \mathcal{F}(U)$ such that $x|_{U_i} = x_i$. A categorical way to describe this condition is that for any covering $\{U_i\}_{i\in I}$ of $U \subset X$, the diagram

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is a equalizer.

Sheaves

Definition (Sheaf)

Let C be a site. Then a sheaf $\mathcal{F}$ on C is a presheaf which satisfies the following sheaf condition: for any covering $\{U_i \to U\}_{i \in I}$, the diagram

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is a equalizer.

Proposition (A criterion)

In order for a presheaf $\mathcal{F}$ to be a sheaf in the étale (Nisnevich) topology, it suffices to check the sheaf condition for Zariski open coverings and for étale (Nisnevich) coverings $V \to U$ where both V and U are affine.

Examples

Let X be a scheme.

- **Structure sheaf**. Similar to the structure sheaf $\mathcal{O}_X$ in the Zariski case, we can define structure sheaves on different topologies. For example, for étale topology, we can define the structure sheaf $\mathcal{O}_X$ by setting
$$\mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_U).$$
- **Corepresentable sheaf**. let S be a scheme, X be an S-scheme. Then $h_X := \mathrm{Hom}_S(-, X)$ is a sheaf on X_{Zar}, X_{Nis}, $X_{\mathrm{\acute{E}t}}$, etc.
- **Constant sheaf**. Let Λ be a ring. For each étale map $U \to X$, we define
$$\mathcal{F}_\Lambda(U) = \Lambda^{\pi_0(U)}$$
then $\mathcal{F}_\Lambda$ is a sheaf on $X_{\mathrm{\acute{e}t}}$ (X_{nis}), called the constant sheaf. It is the sheaf corepresented by $X \times \Lambda$.

Examples

- **Sheaf of units**. Let $\mathcal{O}_X^\times(U) = \Gamma(U, \mathcal{O}_U)^\times$, then $\mathcal{O}_X^\times$ is the sheaf of units on $X_{\text{ét}}$. It is corepresented by $\mathbb{A}^1 \setminus 0$.
- **Sheaf of $\mathcal{O}_X$-modules**. A sheaf $\mathcal{F}$ is called a sheaf of $\mathcal{O}_X$-modules if for each $U \in C$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and restriction maps are compatible with module structures. Similar to the Zariski case, if $X = \mathrm{Spec} R$ is affine, every R-module M gives rise to an étale sheaf of $\mathcal{O}_X$-modules by setting

$$\mathcal{F}_M(U) = \Gamma(U, f^*M)$$

for $f \colon U \to X$.

Examples

- **Sheaves over fields**. Let $X = \mathrm{Spec}(k)$ for a field k. Then an étale sheaf $\mathcal{F}$ over X is determined by the sets $\mathcal{F}(\mathrm{Spec}(k'))$ where k'/k is a finite separable extension, and satisfies that

$$\mathcal{F}(\mathrm{Spec}(k')) \cong \mathcal{F}(\mathrm{Spec}(K))^{\mathrm{Gal}(K/k')}$$

for any finite Galois extension K/k'.

Let $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. For each sheaf $\mathcal{F}$ over $\mathrm{Spec}(k)_{\text{ét}}$, define $M_{\mathcal{F}} = \varinjlim \mathcal{F}(k')$. Then $M_{\mathcal{F}}$ is a discrete G-module. We can show that this correspondence gives an equivalence between the category of étale sheaves on $\mathrm{Spec}(k)$ and the category of discrete G-modules.

Stalks

Definition

Let X be a scheme, $x \in X$, and $\mathcal{F}$ be a sheaf on $X_{\text{ét}}(X_{\text{nis}}, \text{etc.})$. We define the **stalk** at x of $\mathcal{F}$ as

$$\mathcal{F}_x := \lim_{x \in U} \mathcal{F}(U).$$

- For the structure sheaf $\mathcal{O}_X$, the stalks are the local rings we defined in the previous sections: for Nisnevich topology $\mathcal{O}^{\text{nis}}_{X,x} = \mathcal{O}^h_{X,x}$, and for étale topology $\mathcal{O}^{\text{ét}}_{X,x} = \mathcal{O}^{sh}_{X,x}$.
- The stalk of a corepentable sheaf h_X on Sch/S at $x \in S$ is $\mathrm{Hom}_S(\mathcal{O}_{X,x}, X)$.
- The stalk of an étale sheaf $\mathcal{F}$ on $\mathrm{Spec}(k)$ is the $\mathrm{Gal}(k^{\text{sep}}/k)$-module $M_{\mathcal{F}}$ we defined.

Sheaf cohomology

Let C be a site, we denote by $\mathrm{PSh}(C)$ the category of presheaves, and $\mathrm{Sh}(C)$ the category of sheaves.

Proposition

$\mathrm{Sh}(C)$ is an abelian category with enough injectives. The global section functor Γ is defined by $\Gamma(\mathcal{F}) := \mathrm{Hom}_{\mathrm{Sh(C)}}(\underline{\mathrm{pt}}, \mathcal{F})$, where $\underline{\mathrm{pt}}$ is the sheaf that assigns each object U the point set pt. For the sites $C = X_\tau$, it is the usual global section functor $\Gamma(\mathcal{F}) = \mathcal{F}(X)$. It is a left exact functor.

Definition (Sheaf cohomology)

For a sheaf $\mathcal{F}$ on C, define the **sheaf cohomology** of $\mathcal{F}$ by

$$H^i(C; \mathcal{F}) = R\Gamma^i(\mathcal{F}).$$

In particular, if X is a scheme and $\mathcal{F}$ is a sheaf on X for the topology τ (could be Zar, Nis, ét, etc.), we have the sheaf cohomology $H^i_\tau(X; \mathcal{F}) = H^i(X_\tau; \mathcal{F})$.

Sheaf cohomology over fields

Let k be a field, $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ be the absolute Galois group. We have seen that there is an equivalence between étale sheaves over $\mathrm{Spec}(k)$ and G-modules.

In fact, in this case the sheaf cohomology recovers Galois cohomology:

Proposition

For each i, there is an isomorphism

$$H^i_{\mathrm{\acute{e}t}}(\mathrm{Spec}(k); \mathcal{F}) \cong H^i(G; M_{\mathcal{F}})$$

On the other hand, the Zariski and Nisnevich cohomology of sheaves are trivial (vanishes above degree 0).

A tool for computation: Čech cohomology

Let $\mathcal{U} = \{U_i \to X\}$ be an étale covering of X, $\mathcal{F}$ be an étale sheaf on X. Define

$$C^r(\mathcal{U}, \mathcal{F}) = \prod_{i_0, i_1, \dots, i_r} \mathcal{F}(U_{i_0 \dots i_r}), \quad \text{where } U_{i_0 \dots i_r} = U_{i_0} \times_X \times \cdots \times_X U_{i_r}.$$

For $s = (s_{i_0 \dots i_r}) \in C^r(\mathcal{U}; \mathcal{F})$, we define $d^r s \in C^{r+1}(\mathcal{U}; \mathcal{F})$ by

$$(d^r s)_{i_0 \dots i_{r+1}} = \sum_{j=0}^{r+1} (-1)^j s_{i_0 \dots i_{j-1} i_{j+1} \dots i_{r+1}} |_{U_{i_0 \dots i_{r+1}}}.$$

Čech cohomology

We can verify that

$$C^0(\mathcal{U};\mathcal{F}) \to C^1(\mathcal{U};\mathcal{F}) \to \cdots$$

is a chain complex, which we call the Čech complex.

Definition (Čech cohomology)

We define the **Čech cohomology** with respect to the cover $\mathcal{U}$ as $\check{H}^i(\mathcal{U};\mathcal{F}) = H^i(C(\mathcal{U};\mathcal{F}))$, and the Čech cohomology of the scheme X as

$$\check{H}^i(X;\mathcal{F}) = \varprojlim \check{H}^i(\mathcal{U};\mathcal{F})$$

where the inverse limit is taken for all open coverings of X.

Comparison

Proposition

For $r = 0$ *or* 1, $\check{H}^r(X;\mathcal{F}) = H^r(X;\mathcal{F})$.

For étale topology, Čech cohomology and sheaf cohomology coincide in many cases.

Proposition

Let X be a scheme. Suppose that every finite subset of X is contained in an open affine and X is quasi-compact (for example, X is a quasi-projective variety). Then

$$\check{H}^r(\mathcal{U};\mathcal{F}) = H^r(X;\mathcal{F})$$

for all r and sheaf $\mathcal{F}$ for étale topology.

Comparison

Theorem (The comparison theorem)

Let X be a smooth complex variety, $X(\mathbb{C})$ be the corresponding compact analytic space, Λ be an finite abelian group. Then we have an isomorphism

$$H^r_{\text{ét}}(X;\Lambda) \cong H^r(X(\mathbb{C});\Lambda).$$

for $r \geq 0$.

Corollary

Let X be a smooth variety over $\mathbb{C}$, σ be a field automorphism of $\mathbb{C}$. Then $H^i(X(\mathbb{C});\mathbb{Q}) \cong H^i(\sigma X(\mathbb{C});\mathbb{Q})$ for $i \geq 0$.

Note that in general σ can be not continuous, and the topology of X and σX could be very different.

The first cohomology groups

For the sheaf of units $\mathcal{O}^\times$, the first cohomology groups recovers Picard groups:

Proposition

There are isomorphisms

$$\check{H}^1(X;\mathcal{O}_X^\times) \cong H^1_{\mathrm{Zar}}(X;\mathcal{O}_X^\times) \cong H^1_{\mathrm{Nis}}(X;\mathcal{O}_X^\times) \cong$$
$$H^1_{\text{ét}}(X;\mathcal{O}_X^\times) \cong H^1_{\mathrm{fppf}}(X;\mathcal{O}_X^\times) \cong H^1_{\mathrm{fpqc}}(X;\mathcal{O}_X^\times) \cong \mathrm{Pic}(X).$$

All of which represents the isomorphism classes of line bundle over X.

Why Nisnevich?

Nisnevich topology combines the advantages of both Zariski topology and étale topology:

- Nisnevich topology has the same cohomological dimension as Zariski topology, the Krull dimension. (**Zariski**)
- Fields have trivial shape in Nisnevich topology. (**Zariski**)
- Algebraic K-theory satisfies Nisnevich decent (not true for étale), which mean algebraic K theory is a sheaf in Nisnevich topology (then moreover a motivic space). (**Zariski**)
- Nisnevich cohomology can be computed using Čech cohomology. They coincide for all quasicompact separated schemes over k. (**étale**)
- The purity theorem. (**étale**)

Sheaves valued in an ∞-category

Let C be a site, A be an ∞-category (for example, Spc). We can define a notion of sheaves that take values in A, which is useful in the definition of motivic spaces.

Definition (∞-sheaves)

A presheaf valued in A on C is a functor $C^{op} \to A$. A sheaf $\mathcal{F}$ valued in A on C is a presheaf that satisfies the sheaf condition: for any open cover $\{U_i \to U\}_{i \in I}$, there is an equivalence

$$\mathcal{F}(U) \simeq \lim(\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_{i_0 i_1}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod \mathcal{F}(U_{i_0 i_1 i_2}) \cdots)$$

- If $A = \mathrm{Set}$ or $A = \mathrm{Ab}$ the discrete 1-categories, this definition coincide with the usual definition of **sheaves**.
- If $A = \mathrm{Gpd}$ the 2-category of all groupoids, this definition coincide the usual definition of **stacks**.

Next lecture: unstable motivic homotopy theory!