Lecture 0

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Modern Techniques in Homotopy Theory Learning Seminar

May 28th, 2025

Meeting Plan

- The plan is to meet once a week for 1.5 hours from the week of **May 26th-30th** to the week of **August 18th-22nd**.
- Currently, it appears 10:00 am to 11:30 am EST on some of the weekdays is most preferable to everyone who filled out the when2meet.
- The idea is that the two organizers would give Lecture 0, 1, 2, and the other participants would volunteer to give the rest of the lectures.
- A tentative outline of the lectures would be released soon.

Outline

High-Level Introduction

Here we will give a fairly informal introduction to motivic homotopy theory. It is okay if you have not seen some of the concepts in the introduction, many of the concepts mentioned below will be covered as background materials in the lectures.

What is Motivic Homotopy Theory?[1]

Motivic homotopy theory is a general paradigm that seeks to apply techniques in homotopy theory to the study of algebraic geometry.

The founding motivation of the field is based on the following observation. In topology, a free homotopy between two maps $f, g : X \to Y$ is a continuous function $H : X \times [0,1] \to Y$ such that:

1. $H(x, 0) = f(x)$ for all $x \in X$.
2. $H(x, 1) = g(x)$ for all $x \in X$.

Problem

What even is $[0,1]$ in algebraic geometry?

[1] I learned part of this introduction from Jonathan Block, Thomas Brazelton, and Ben Spitz.

Instead of using $[0, 1]$, it appears that the affine line $\mathbb{A}^1_R$ is the best possible analog of a 1-parameter family in AG.

Therefore, motivic homotopy theory can be thought of as the study of the "homotopy theory" of schemes where $\mathbb{A}^1_R$ replaces the role of $[0, 1]$.

What Problems Can Motivic Homotopy Theory Solve?

There are certainly many more problems motivic homotopy theory can solve, but motivic perspective have proven to be quite effective in attacking the following program of questions:

1. Take statements about algebraic topology.
2. Try to make similar statements about algebraic geometry.
3. Get algebraic questions and solve them.

What Problems Can Motivic Homotopy Theory Solve?

Another class of problems the motivic perspective can help with come from questions that involve data that are "invariant under $\mathbb{A}^1$". There are many such properties, for example,

1. For R a reduced ring, $R^\times \cong (R[x])^\times$ (equivalently $\mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(R), \mathbb{G}_m) \cong \mathrm{Hom}_{\mathrm{Sch}}(\mathbb{A}^1_R, \mathbb{G}_m)$.
2. Let X be a scheme, then there is an isomorphism of Chow groups $\mathrm{CH}_*(X) \cong \mathrm{CH}_{*+1}(X \times \mathbb{A}^1)$.
3. Let X be a Noetherian regular scheme, then there is an isomorphism of algebraic K-theories of schemes $K(X) \cong K(X \times \mathbb{A}^1)$.

Here, all isomorphisms are induced by the natural projection map.

Corank Zero Splitting (Topology)

Let us look at a concrete example where this perspective is carried out in action.

Theorem (Folklore)

Let X be a CW complex of dimension $2n > 0$ and E be a rank n complex vector bundle over X, then $c_n(E) = 0 \in H^{2n}(X, \mathbb{Z})$ if and only if E splits off a trivial summand. (ie. There exists vector bundle $E' \to X$ such that E is the direct sum of E' and the trivial line bundle)

Translating to Algebraic Geometry

If we want an analogous statement in algebraic geometry, we should look at the proper analog of each topology terms in the theorem:

1. CW complex $X \to$ "nice" scheme X
2. vector bundle $\to$ locally free $\mathcal{O}_X$-module of finite rank (aka. algebraic vector bundles)
3. dimension of $X \to$ Krull dimension of X (and $2n$ should be changed to n)
4. $H^{2n}(X, \mathbb{Z}) \to$ the Chow group

Corank Zero Splitting (Algebraic Geometry)

Let X be an affine smooth k-variety of Krull dimension r over k an algebraically closed field.

Theorem (Murthy-Swan, Kuma-Murthy, Murthy)

An algebraic vector bundle $E \to X$ of rank r **splits** off a trivial summand if and only if $c_r(E) = 0$ in $\mathrm{CH}^r(X)$ (the Chow group).

- For $r = 1$ this is an exercise to the audience.
- $r = 2$ is resolved in Murthy, M. P. and Swan, R. G. (1976).

  Vector bundles over affine surfaces.
  Inventiones Mathematicae, 36(1):125–165
- $r = 3$ is resolved in Kuma, N. M. and Murthy, M. P. (1982).

  Algebraic cycles and vector bundles over affine three-folds.
  Annals of Mathematics, 116(3):579–591

Corank Zero Splitting (Algebraic Geometry)

- For $r > 3$, this was resolved in Murthy, M. P. (1994). Zero cycles and projective modules. *Annals of Mathematics*, 140(2):405–434

The original proofs for the theorem took a lot of effort and time, but there is a much more streamlined version of the proof using motivic homotopy theory.

This is because the motivic perspective seeks to build the proper set-up in algebraic geometry so that an analogous proof in the topology world can be carried over directly.

The Topological Proof

Theorem (Folklore)

Let X be a CW complex of dimension $2n > 0$ and E be a rank n complex vector bundle over X, then $c_n(E) = 0 \in H^{2n}(X, \mathbb{Z})$ if and only if E splits off a trivial summand.

Proof: Recall the bundle $E \to X$ is classified by a map $\phi : X \to \mathrm{BU}(n)$. E splits off a trivial summand if and only if the following lifts:

$$\begin{tikzcd} & \mathrm{BU}(n-1) \arrow[d] \\ X \arrow[ur] \arrow[r, "\phi"'] & \mathrm{BU}(n) \end{tikzcd}$$

Here the map $\mathrm{BU}(n-1) \to \mathrm{BU}(n)$ is induced by the natural inclusion $U(n-1) \hookrightarrow U(n)$.

The Topological Proof (Continued)

The map $\mathrm{BU}(n-1) \to \mathrm{BU}(n)$ admits fibers and cofibers[2]:

$$\begin{tikzcd} S^{2n-1} \arrow[r] & \mathrm{BU}(n-1) \arrow[r] & \mathrm{BU}(n) \arrow[r, "f"] & S^{2n} \\ & & X \arrow[ul, dashed] \arrow[u, "\phi"] \arrow[ur] & \end{tikzcd}$$

It can be shown that ϕ lifts if and only if $f \circ \phi : X \to S^{2n}$ is null-homotopic.

By general Postnikov tower arguments, the map $f \circ \phi$ is null-homotopic if and only if the following elements (called k-invariants) are all zero.

$$k_t \in H^{t+2}(X, \pi_{t+1}(S^{2n-1})).$$

[2] In fact, it is a principal fibration.

The Topological Proof (Continued)

$$k_t \in H^{t+2}(X, \pi_{t+1}(S^{2n-1})).$$

Observe that

- For $t \geq 2n-1$, $t+2 > 2n$. Since X is a CW complex of dimension $2n$, $k_t = 0$.
- For $t < 2n-2$, $t+1 < 2n-1$ and hence $\pi_{t+1}(S^{2n-1}) = 0$, so $k_t = 0$.
- For $t = 2n-2$, $k_{2n-2} \in H^{2n}(X, \mathbb{Z})$ can be identified as the n-th Chern class!

Theorem

An algebraic vector bundle $E \to X$ of rank r **splits** off a trivial summand if and only if $c_r(E) = 0$ in $\mathrm{CH}^r(X)$.

Motivic homotopy theory builds the theory such that:

1. There is an analog of $\mathrm{BU}(n)$ known as BGL_n (which is an example of what is called a motivic space).
2. There is an analog of spheres called motivic spheres.
3. The splitting of the algebraic vector bundle is equivalent to a lift

$$\begin{tikzcd} & \mathrm{BGL}_{r-1} \arrow[d, dashed] \\ X \arrow[ur] \arrow[r] & \mathrm{BGL}_r \end{tikzcd}$$

Translating from Topology to Algebraic Geometry

Theorem

An algebraic vector bundle $E \to X$ of rank r **splits** off a trivial summand if and only if $c_r(E) = 0$ in $\mathrm{CH}^r(X)$.

4. The motivic spheres give a fiber sequence of the form

$$S^{2r-1,r} \to \mathrm{BGL}_{r-1} \to \mathrm{BGL}_r$$

5. In this case, the map lifts if and only if the following k-invariants are all zero

$$k_t \in H^{t+2}(X, \pi_{t+1}(S^{2r-1,r})(\det E)).$$

6. Since X has Krull-dimension r, $k_t = 0$ for $t \geq r-1$.
7. Due to an analogous connectivity statement for motivic spheres, $k_t = 0$ for $t < r-2$.

Theorem

An algebraic vector bundle $E \to X$ of rank r **splits** off a trivial summand if and only if $c_r(E) = 0$ in $\mathrm{CH}^r(X)$.

8. $k_{r-2} \in H^r(X, \pi_{r-1}(S^{2r-1,r})(\det E))$ is the remaining invariant.
9. Murthy showed that $\pi_{r-1}(S^{2r-1,r})(\det E)$ is the same as this sheaf K_r^{MW} called Milnor-Witt K-theory.
10. There is also another sheaf K_r^M called Milnor K-theory whose cohomology gives Chow groups and fits in an exact sequence
$$0 \to I^{r+1} \to K_r^{MW} \to K_r^M \to 0$$
11. This induces a long exact sequence

$$\ldots \to H^r(X, I^{r+1}(\det E)) = 0 \to H^r(X, K_r^{MW}(\det E)) \xrightarrow{\psi} \mathrm{CH}^r(X) \to 0$$

and $\psi(k_{r-2}) = c_r(E)$.

The following is a recent application of motivic homotopy theory for a much more difficult scenario.

Theorem (Asok-Bachmann-Hopkins, 2023)

Let k be characteristic 0 and algebraically closed, X be a smooth affine k-variety of Krull dimension r. An algebraic vector bundle $E \to X$ of rank $r-1$ splits off a trivial summand if and only if $c_{r-1}(E) = 0$ in $\mathrm{CH}^{r-1}(X)$.

- Asok, A., Bachmann, T., and Hopkins, M. J. (2023). On P^1-stabilization in unstable motivic homotopy theory

So How Does (Unstable) Motivic Homotopy Theory Do It?

1. Let Sm_k denote the category of smooth k-schemes. To construct motivic spaces, the idea is to consider a Grothendieck topology on the category Sm_k known as the Nisnevich topology.

$$\text{Zariski} \leq \text{Nisnevich} \leq \text{étale}.$$

2. We would like to consider sheaves with respect to the Nisnevich topology. In AG, typically sheaves are valued in rings or abelian groups, but we would like to capture homotopical information, so instead we want to consider Nisnevich sheaves that are valued in (the ∞-category of) spaces Spc:

$$\mathcal{F} : \mathrm{Sm}_k^{op} \to \mathrm{Spc}$$

So How Does (Unstable) Motivic Homotopy Theory Do It?

3. A motivic space is a Nisnevich sheaf valued in Spc that is also $\mathbb{A}^1$-invariant as a presheaf.
4. Motivic spaces allow the definition for analogs of many familiar constructions in topology - including motivic spheres and motivic Eilenberg-Maclane spaces.

This is what our first few lectures will be about. After that, we move into the stable version of motivic homotopy theory. Here, instead of considering Spc, we instead consider the stable ∞-category of Spectra Sp.

References I

Asok, A., Bachmann, T., and Hopkins, M. J. (2023).
On P^1-stabilization in unstable motivic homotopy theory.

Kuma, N. M. and Murthy, M. P. (1982).
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