Lecture 1: Rapid Review of Algebraic Geometry

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Modern Techniques in Homotopy Theory Learning Seminar

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Note

The purpose of this lecture is to introduce a list of topics in algebraic geometry that the participants can refer back to if helpful. The main references here are:

1. Ravi Vakil's *The Rising Sea: Foundations of Algebraic Geometry*.
2. Marc Levine's *Background from Algebraic Geometry* in Motivic Homotopy Theory: Lectures at a Summer School in Nordfjordeid, Norway, August 2002
3. Robin Hartshorne's *Algebraic Geometry* GTM 52.

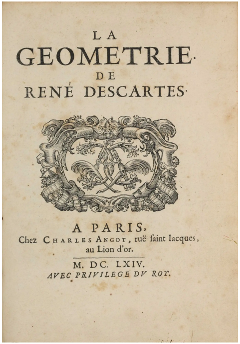
... among other sources.

Outline

Outline

At its inception, algebraic geometry (AG) studies the geometric properties of solutions to systems of polynomial equations.

AG Version 1 (Descartes 1630s and more):



[1] I learned this introduction from Eric Larson.

AG Version 1 (Descartes 1630s and more)

The idea behind AG Version 1 is to study the following:

Take $f_1, ..., f_r \in \mathbb{R}[x_1, ..., x_n]$ a system of polynomials. Write

$$V(\{f_1, ..., f_r\}) = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid f_i(x_1, ..., x_n) = 0 \quad \forall i\}.$$

This is the (affine) algebraic variety vanishing on $f_1, ..., f_r$.
There are two problems with this set-up:

1. How many points do two different lines meet in a plane? We are tempted to say that "every scuh lines meet in a point", but parallel lines do not actually meet at all.
2. In how many points does a line meet a circle? It could be two points, one point, or zero points on the reals.

AG Version 2 (Poncelet 1810s and more)

The focus of the second version is to make the changes:

1. Complexify: turning $\mathbb{R}$ to $\mathbb{C}$.
2. Compactify: adding points at infinity. Specifically, we replace $\mathbb{C}^n$ with $\mathbb{C}P^n$. This is called the projective compactification of $\mathbb{C}^n$.

In this case, two different lines do always meet on $\mathbb{C}P^2$! However, there are still some flaws:

1. Limited to algebraically closed fields.
2. Intersections still does not account multiplicity.
3. The objects (varieties) we study are not intrinsic.

Imagine A World Where ...

In your math classes,

1. A manifold is defined only as a subspace of $\mathbb{R}^n$ satisfying some properties.
2. A group is a subset of $n \times n$ matrices that are closed under multiplication and inverses.

We want an intrinsic object in AG to study that works well with intersections and over any ring[2]!

[2] By a ring, we almost always mean commutative with unity

AG Version 3 (Grothendieck 1960s)

Alexander Grothendieck invented the theory of schemes to address these questions!

Outline

Going Back to the Variety Perspective:

Let $f_1, \ldots, f_r$ be polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. Let $I = \langle f_1, \ldots, f_r \rangle$ be the ideal they generate, observe that

$$V(\{f_1, \ldots, f_r\}) = V(I) \subseteq \mathbb{C}^n.$$

Each point $(a_1, \ldots, a_n) \in V(I)$ corresponds to a maximal ideal of the form $(x_1 - a_1, \ldots, x_n - a_n)$.

The first idea is to enrich the information of a variety by considering prime ideals too. Instead of polynomial rings, we can do this over any ring.

Definition

Let A be a ring, the prime spectrum of A, as a set, is the prime ideals of A.

We would like to endow $\mathrm{Spec}(A)$ with more structure.

Let S be a subset of A, we define

$$V(S) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid S \subseteq \mathfrak{p}\}.$$

Observe that the collection τ of all $V(S)$'s satisfies:

1. $\emptyset, \mathrm{Spec}(A) \in \tau$.
2. τ is closed under arbitrary intersections.
3. τ is closed under finite unions.

In other words, τ defines a topology of closed sets on $\mathrm{Spec}(A)$ known as the Zariski topology.

Topology is Not Enough

$\mathrm{Spec}(A)$, as it stands, is still undesirable:

1. Let k_1, k_2 be two distinct fields, then $\mathrm{Spec}(k_1)$ and $\mathrm{Spec}(k_2)$ are both topological spaces with 1 point and are hence homeomorphic.
2. Let k be your favorite field, then $\mathrm{Spec}(k)$ and $\mathrm{Spec}(k[x]/(x^2))$ are both homeomorphic. This is not accounting for multiplicity.

The next idea is to add some geometry onto $\mathrm{Spec}(A)$, which comes in the form of a sheaf.

Sheaf

Let X be a topological space and $\mathrm{Open}(X)$ be the poset category of open sets ordered by inclusion.

A presheaf of values in a category $\mathcal{C}$ is a contravariant functor $F : \mathrm{Open}(X)^{op} \to \mathcal{C}$ (ie. for $U \subset V$, there is a map $\mathrm{res}_{U,V} : F(V) \to F(U)$).

A presheaf F is a sheaf if it satisfies the following descent condition[3]: For any open cover $\{U_\alpha\}_{\alpha \in I}$ of X,

$$0 \to F(U) \xrightarrow{\prod_a \mathrm{res}_{U_a,U}} \prod_a F(U_a) \rightrightarrows \prod_{a,b} F(U_a \cap U_b)$$

is an equalizer. A morphism of sheaves is a natural transformation as functors.

[3] In more concrete terms, it means F satisfies a suitable gluability condition and identity condition.

Examples and Non-Examples of Sheaves

Here are some examples and non-example of sheaves:

1. Let M be a smooth manifold. The functor $F : \mathrm{Open}(M)^{op} \to \mathrm{Rings}$ with $F(U) = \mathrm{C}^\infty(\mathrm{U}, \mathbb{R})$ and $\mathrm{res}_{U,V}$ being actual restriction maps is a sheaf.
2. Let $f : Y \to X$ be a continuous map. The presheaf of sets F with

$$F(U) = \{\text{continuous maps } s : U \to Y \text{ such that } f \circ s = id|_U\}$$

   is a sheaf.
3. Let X be any space and B be a non-zero abelian group. The constant functor $\mathcal{B}(U) := B$ (and sends morphisms to identity) is not a sheaf[4].
4. For any space X, the locally constant presheaf $\underline{B}(U) := \mathrm{Hom}_{ct}(U, B^{\mathrm{discrete}})$ is a sheaf!

[4] Look at the empty set

Sheafification

Let F be a presheaf of sets on X, there exists a sheaf $\mathcal{F}$ and morphism $i : F \to \mathcal{F}$ such that any morphism $F \to \mathcal{G}$ ($\mathcal{G}$ a sheaf) factors:

$$\begin{array}{ccc} F & \xrightarrow{\ i\ } & \mathcal{F} \\ & \searrow & \Big\downarrow {\scriptstyle \exists ! \phi} \\ & & \mathcal{G} \end{array}$$

$\mathcal{F}$ is called the sheafification of F.

Ex: $\underline{B}$ is the sheafification of $\mathcal{B}$ from last slide.

More Homotopical Perspective: There is in fact a model structure that can be put on set-valued presheaves such that sheaves = fibrant objects. The sheafification is exactly the fibrant replacement procedure.

Affine Schemes

Given a ring A and $X = \operatorname{Spec} A$, we define a sheaf of rings $\mathcal{O}_X$ on X as follows:

1. If $D(f) = X - V(f)$ for a single element $f \in A$, then
$$\mathcal{O}(D(f)) := A_f := S^{-1}A,$$
where $S \subset A$ is the collection of $g \in A$ such that $V(g) \subset V(f)$. For $D(f) \subset D(f')$, there is an induced map by the universal property $\operatorname{res}_{f,f'} : A_{f'} \to A_f$.
2. For any open set U, write $U = \bigcup_{f \in I(U)^5} D(f)$. We define
$$\mathcal{O}_X(U) := \ker\Big(\prod_{f \in I(U)} \mathcal{O}_X(D(f))) \xrightarrow{\operatorname{res}_{fg,f} - \operatorname{res}_{fg,g}} \prod_{f,g \in I(U)} \mathcal{O}_X(D(fg))$$
with natural restriction maps given by universal properies.

[5] For any subset $S \subset \operatorname{Spec}(A)$, define $I(S) = \bigcap_{[\mathfrak{p}] \in S} \mathfrak{p} \subset A$.

(Locally) Ringed Spaces

The pair $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is called an affine scheme.

An affine scheme is more generally an example of a (locally) ringed space (which also includes manifolds):

Definition:

A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings.

For $p \in X$, the stalk of $\mathcal{O}_X$[6] at p is the categorical direct limit $\mathcal{O}_{X,p} := \lim_{U \ni p} \mathcal{O}_X(U)$.

A ringed space $(X, \mathcal{O}_X)$ is locally ringed if $\mathcal{O}_{X,p}$ is a local ring for all $p \in X$.

One can check $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}}$ is the localization of A at $\mathfrak{p}$.

[6] More generally, for any presheaves.

The Definition of Schemes

"A scheme is to a ring what a manifold is to an open chart."

Definition:

A scheme $(X, \mathcal{O}_X)$ is a ringed space such that for all $p \in X$, there is an open subset $U \ni x$ with $(U, \mathcal{O}_U)$ is isomorphic to an affine scheme.

Note that schemes are clearly locally ringed.

Definition:

A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (f, ϕ) where $f : X \to Y$ is continuous and $\phi : \mathcal{O}_Y \to f_*\mathcal{O}_X$[7] is a morphism of sheaves.
A morphism of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a ringed space morphism such that the induced map $f^* : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ gives $f^*(\mathfrak{m}_{Y,f(x)}) \subseteq \mathfrak{m}_{X,x}$[8].

[7] $f_*(\mathcal{O}_X)(U) := \mathcal{O}_X(f^{-1}(U))$

[8] In a precise sense, this is equivalent to saying the morphism is locally a morphism of affine schemes

Affine and Non-affine Schemes

The category of affine schemes Aff is the full subcategory of the category of schemes Sch.

Theorem:

$\mathrm{Hom}_{\mathrm{Rings}}(A, B) \cong \mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} B, \operatorname{Spec} A)$. Furthermore, there is an equivalence of categories

$$\operatorname{Spec} : \mathrm{Rings}^{op} \to \mathrm{Aff}\,.$$

"A scheme glues together piecewise commutative algebra." We write $\mathbb{A}^n_A := \operatorname{Spec}(A[x_1, ..., x_n])$ to denote the affine n-space.

Affine Line with 2 Origins Let $X = \mathbb{A}^1_k := \operatorname{Spec} k[t]$. Then consider $Y = \mathbb{A}^1_k := \operatorname{Spec} k[u]$. Note that X contains $U = \operatorname{Spec} k[t, t^{-1}]$ and Y contains $V = \operatorname{Spec} k[u, u^{-1}]$. U and V are isomorphic by sending t to u. The quotient Z of X and Y by identifying U and V is not affine.

Fibered Product in Schemes

Theorem

Fibered products (ie. pullbacks) exist in the category of schemes Sch.

We are interested in fibered products for many reasons, not limited to:

1. When restricted to affine schemes, fibered products correspond exactly to tensor product of rings.
2. Just as how intersections of open sets are pullbacks, we can look at analogs of intersections using pullbacks in algebraic geometry.
3. Fibered products give rise to one definition of fibers, which we will not elaborate more on this lecture.

Sheaf of $\mathcal{O}_X$-Modules

Just as how rings have modules, we also want our sheaf of rings to have an associated sheaf of modules.

Definition

Let $(X, \mathcal{O}_X)$ be a ringed space, an $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ of abelian groups with a morphism of sheaves

$$\mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$$

satisfying conditions on associativity and unitaliy.

More concretely, each $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module and the diagram commutes for $U \subset V$:

$$\begin{array}{ccc} \mathcal{O}_X(V) \times \mathcal{F}(V) & \xrightarrow{\cdot} & \mathcal{F}(V) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(U) \times \mathcal{F}(U) & \xrightarrow{\cdot} & \mathcal{F}(U) \end{array}$$

An Example from Manifolds

Consider a smooth manifold X and $\mathcal{O}_X$ the sheaf of smooth functions on X. Suppose $\pi : V \to X$ is a vector bundle over X, and define the sheaf of abelian groups $\mathcal{F}$ as

$$\mathcal{F}(U) = \{\text{smooth sections } \sigma : U \to V\}.$$

This is an $\mathcal{O}_X$-module. Consider $s_1, s_2 \in \mathcal{F}(U)$ as sections, we can consider $s_1 + s_2$ as a section. Given $f \in \mathcal{O}_X(U)$ and a section s, we can consider $f \cdot s$ by scaling.

Consider the affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. Let M be an A-module, we can define a sheaf of abelian group $\tilde{M}$ such that

$$\tilde{M}(D(f)) := M_f = A_f \otimes M,$$

and the restrictions are defined by universal property. This extends to general open sets in a similar way. This is an $\mathcal{O}_{\operatorname{Spec} A}$-module!

Definition

Let X be a scheme, an $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if for each $p \in X$, there exists a affine open neighborhood $(U, \mathcal{O}_U) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ such that the restriction of $\mathcal{F}$ to U is isomorphic to $\tilde{M}$ for some A-module M.

Quasicoherent Sheaves

Write $\mathrm{QCoh}(X)$ as the category of quasicoherent sheaves. Quasicoherent sheaves should be thought of as an enlargement of vector bundles.

- In topology, the category of vector bundles over the same space need not be abelian.
- In AG, $\mathrm{QCoh}(X)$ is abelian[9]. Note that $\mathcal{O}_X - \mathrm{Mod}$ is also abelian.

To locate the proper analog of vector bundles for AG, we should be thinking of locally free sheaves.

Definition

An $\mathcal{O}_X$-module $\mathcal{F}$ is free if its of the own $\mathcal{O}_X^{\oplus I}$ for some index set I. $\mathcal{F}$ has finite rank if I is finite. $\mathcal{F}$ is an algebraic vector bundle if it is locally free of finite rank.

[9] There is in fact a general definition of q.c. sheaves on any ringed space, which will also be abelian.

Ideal Sheaf

A special class of quasi-coherent sheaves we want to pay attention to are ideal sheaves - which are analogous of ideals for schemes.

Definition

A sheaf $\mathcal{I}$ of $\mathcal{O}_X$-modules is an ideal sheaf if for every point $p \in X$, there is a neighborhood $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ such that $\mathcal{I} \cong \tilde{I}$ for some ideal $I \subset A$.

Just like how ideals $I \subset A$ induce a closed subset $\operatorname{Spec} A/I \subset \operatorname{Spec} A$. An ideal sheaf $\mathcal{I}$ of $(X, \mathcal{O}_X)$ defines a closed subscheme $(Z, \mathcal{O}_X/\mathcal{I})$ where:

1. $\mathcal{O}_X/\mathcal{I}$ is the cokernel of the natural map $\mathcal{I} \to \mathcal{O}_X$.
2. Z is the closed subset of X of $p \in X$ such that the stalk $(\mathcal{O}_X/\mathcal{I})_p \neq 0$.

Closed and Open Immersion

A morphism of schemes $(f, \phi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a closed immersion if:

1. f sends $|Y|$ homeomorphically to a closed subset of $|X|$.
2. $\phi : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ is surjective and the kernel is an ideal sheaf.

Note that the inclusion of closed subscheme by an ideal sheaf is always a closed immersion.

A morphism of schemes $(f, \phi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is an open immersion if it induces an isomorphism $(Y, \mathcal{O}_Y) \cong (U, \mathcal{O}_U := \mathcal{O}_X|_U)$ for some open subset U.

Outline

Noetherian Schemes

For a scheme X, we use $|X|$ to denote its underlying topological space.

Definition

A scheme X is Noetherian if:

1. $|X|$ is Noetherian, that is, its open subsets satisfy the ascending chain conditions.
2. X has an affine cover from rings that are all Noetherian.

X is locally Noetherian if it only satisfies the second axiom.

- For any Noetherian ring A, $\operatorname{Spec} A$ is Noetherian as a scheme[10].
- Noetherian topological spaces are very limited in Hausdorff spaces. Every Noetherian Hausdorff space is a finite set with discrete topology.

[10] The converse need not hold

Quasi-Compact

A scheme X is quasicompact if

1. Every open cover of $|X|$ has a finite subcover
2. Equivalently, X admits a finite cover of open affine subset.

In particular, every affine scheme is quasicompact.

A morphism of schemes $f : X \to Y$ if quasicompact if for every open affine subset U of Y, $f^{-1}(U)$ is quasicompact.

Quasi-Separated

A scheme X is quasiseparated if

1. The finite intersections of quasicompact open subsets is quasicompact[11].
2. Equivalently, the intersection of two affine open subsets is a finite union of affine open subsets.

In particular, every affine scheme is quasiseparated.

A morphism of schemes $f : X \to Y$ if quasiseparated if for every open affine subset U of Y, $f^{-1}(U)$ is quasiseparated.

Note: Every Noetherian scheme is quasicompact and quasiseparated.

[11] I just mean compact in point-set topology.

Connected, Irreducible, Reduced, Integral, Normal, Factorial

A scheme X is:

1. connected if $|X|$ is connected.
2. irreducible if $|X|$ is irreducible as a topological space, meaning it is not the union of two proper closed subsets. Note that irreducible implies connected, but not vice versa.
3. reduced if $\mathcal{O}_X(U)$ is reduced for all U.
4. integral if $\mathcal{O}_X(U)$ is an integral domain for all U. Note that integral $\iff$ reduced + irreducible.
5. normal if $\mathcal{O}_{X,p}$ is a normal ring (meaning integral domain and integrally closed in its field of fraction) for all $p \in |X|$.
6. factorial if $\mathcal{O}_{X,p}$ is a UFD for all $p \in |X|$. UFDs are integrally closed, so factorial implies normal[12].

[12] When I first learned this, my instructor said this is basically taught in high school (ie. rational root test)

Universal Property of Reduced Schemes

Fact:[13] Let X be a scheme, there exists an unique closed subscheme $X_{\text{red}} \subset X$ such that:

1. $|X_{\text{red}}| = |X|$.
2. X_{red} is reduced.
3. For any morphism $Y \to X$ with Y reduced, the map factors as

$$\begin{array}{ccc} & & X \\ & \nearrow & \uparrow \\ Y & \longrightarrow & X_{\text{red}} \end{array}$$

The construction is to consider the ideal sheaf associated to $\mathcal{O}_X$ given by the nilradical!

[13] PK's favorite exercise in Hartshorne.

Generic Point

If the scheme X is irreducible, then there is a unique point $x \in X$ such that the closure of x is $|X|$. This point is called the generic point.

The construction of the point is as follows:

- Since we only care about topology, we can without loss assume X is reduced.
- Thus, X is integral. Take any non-empty open affine subset U of X, this must be dense by irreducibility.
- Since X is integral, U is integral (which clearly has an unique generic point given by the zero ideal).

Rational Functions on Schemes

Let X be irreducible and x its unique generic point, the ring of rational functions on X is

$$K(X) \coloneqq \mathcal{O}_{X,x}.$$

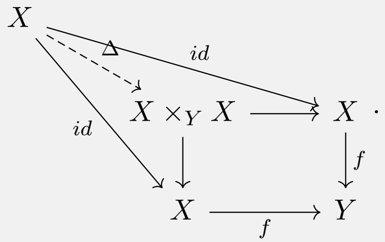
Fact: If X is integral, $K(X)$ is in fact a field.

Separated Schemes

It is not hard to check that the affine line with two origins is quasi-separated. But we actually want it to be consider as "non-Hausdorff". Therefore, we want an analog of Hausdorffness in algebraic geometry.

Definition

A morphism of schemes $f : X \to Y$ is separated if the diagonal map Δ is a closed immersion:



Equivalently, $\Delta(|X|)$ is a closed subset of $|X \times_Y X|$.

Separated Schemes

Note that $\operatorname{Spec} \mathbb{Z}$ is the terminal object in Sch. A scheme X is separated if the natural diagonal map $X \to X \times_{\operatorname{Spec} \mathbb{Z}} X$ is separated.

Properties of Separated:

1. Every affine scheme is separated.
2. Separated is stronger than quasi-separated - if X is separated and U, V are affine open subschemes of X, then $U \cap V$ is affine open (as opposed to a finite union of affines).
3. The affine line with two origins is not separated.

Finite Type

Recall a ring morphism $A \to B$ is of finite type if B, as an A-algebra, is isomorphic to a quotient of $A[x_1, ..., x_n]$.

Definition

A morphism of schemes $f : X \to Y$ is of finite type at $p \in X$ if there exists a neighborhood $\mathrm{Spec}(B)$ of p and $\mathrm{Spec}(A)$ of $f(p)$ such that $f(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$, and the induced ring map $A \to B$ is finite type.

f is locally fo finite type if it is finite type at every point $p \in X$.

f is of finite type if it is locally of finite type and quasi-compact. If $Y = \mathrm{Spec}\, A$, we say X is a finite type A-scheme.

Closed, Universally Closed, Quasi-Finite, Finite, Proper, Integral

A morphism of schemes $f : X \to Y$ is:

1. closed if f sends closed sets to closed sets topologically.
2. universally closed if for each map $Z \to Y$, the map in the pullback $Z \times_Y X \to Z$ is closed.
3. quasi-finite if $f^{-1}(y)$ is a finite set for each $y \in |Y|$.
4. finite if Y has an open cover of affine scheme $\operatorname{Spec} B_i$ such that $f^{-1}(\operatorname{Spec} B_i)$ is also open affine of the form $\operatorname{Spec} A_i$. Furthermore the induced maps $B_i \to A_i$ makes A_i a finitely generated module over B_i.
5. proper if it is separated, finite type, and universally closed.
6. integral if there is an open affine cover $\operatorname{Spec} B_i$ of Y such that $f^{-1}(\operatorname{Spec} B_i)$ is affine and the induced ring maps are integral.

Note that finite $\iff$ proper + quasi-finite.
Note that finite implies integral implies closed.

What is a Variety in Scheme Land?

Under the Vakil camp of introductions, a variety over a field k is generally agreed upon to be an integral separated scheme of finite type. We call such schemes "k-varieties".

Dimensions and Codimensions

Let X be a scheme, the dimension of X is maximum possible length of strict inclusion of closed irreducible subspaces.

Recall the Krull dimension of a ring A is the maximum possible length of strict subsets of prime ideals in A. It turns out there is a correspondence between prime ideals and irreducible subsets, and hence

$$\dim \operatorname{Spec}(A) = \text{Krull dimension of } A$$

Let $Y \subset X$ be an irreducible subspace, the codimension of Y is the maximum possible length of strict inclusions of closed irreducible subspaces, starting at the closure of Y (which is also irreducible).

More on Dimensions

We say a scheme X is equi-dimensional (or pure dimensional) if all of its irreducible components have the same Krull dimension.

Let X be an irreducible k-variety, its dimension can be computed in terms of transcendence degree of the field of rational functions. In other words,

$$\dim X = \operatorname{trdeg} K(X)/k.$$

Let X be a scheme and $p \in |X|$, the Zariski cotangent space $T^v_{X,p}$ at p is the quotient $\mathfrak{m}_{X,p}/\mathfrak{m}^2_{X,p}$, viewed as a vector space of the residue field. The Zariski tangent space $T_{X,p}$ is the dual of $T^v_{X,p}$.

Fact: Let $\mathfrak{m}$ be a maximal ideal of A and let $f \in \mathfrak{m}$ be any element:

$$\dim T_{\operatorname{Spec} A, \mathfrak{m}} = \dim T_{\operatorname{Spec} A/\langle f \rangle, \mathfrak{m}/\langle f \rangle}.$$

Example: The point $[(2, x)]$ in $\operatorname{Spec} \mathbb{Z}[x]/(x^2 + 4)$ has a Zariski tangent space of dimension 2:

1. Note that $x^2 + 4 \in (2, x)$, so the fact above shows we can just calculate $\dim T_{\mathbb{A}^1_{\mathbb{Z}}, (2,x)}$. The residue field is $\mathbb{Z}/2$.
2. $T_{\mathbb{A}^1_{\mathbb{Z}}, (2,x)}$ has only 4 elements.

Regularity

Recall a regular local ring A is a Noetherian local ring with unique maximal ideal $\mathfrak{m}$ such that $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

Let X be a locally Noetherian scheme, we say

1. X is regular at $p \in |X|$ if $\mathcal{O}_{X,p}$ is a regular local ring.
2. X is singular at $p \in |X|$ if $\mathcal{O}_{X,p}$ is not a regular local ring.
3. X is regular if it regular for all points.

Regularity Continued

Auslander-Buchsbaum Theorem

Every regular local ring is a UFD. As a consequence, every regular scheme is factorial.

Although we will not get in the details, it turns out for finite type $\overline{k}$-schemes, regularity of closed points can be checked by what is called the Jacobian criterion. This method is limited however:

1. The converse of Jacobian criterion may not hold.
2. This works mainly over algebrically closed fields[14].

[14] Technically, over a field k, it works for k-valued points.

Smooth Schemes

There is a refined notion of regularity known as smoothness.

A k-scheme X is smooth of dimension d over k if

1. X is equidimensional of dimension d.
2. X has a cover of affine open subschemes of the form $\operatorname{Spec} k[x_1, ..., x_n]/(f_1, ..., f_r)$ such that its associated Jacobian matrix (ie. $(\frac{\partial f_i}{\partial x_j})$) has corank d for all points on each open cover.

Note that the data of a smooth scheme naturally imposes a finite type condition.

Every smooth k-scheme is regular. If k is a perfect field and X is a finite type k-scheme, then X is also smooth.

Smooth Maps

A finite type morphism $\pi : X \to Y$ is smooth of relative dimension n if there are open covers $\{U_i\}, \{V_i\}$ of X, Y, with $\pi(U_i) \to V_i$, such that the following diagram commutes:

$$\begin{array}{ccc} U_i & \xrightarrow{\sim} & W \\ {\scriptstyle \pi|_{U_i}} \downarrow & & \downarrow {\scriptstyle \rho|_W} \\ V_i & \xrightarrow[\sim]{} & \operatorname{Spec} B \end{array}$$

Here $\rho : \operatorname{Spec} B[x_1, ..., x_{n+r}]/(f_1, ..., f_r) \to \operatorname{Spec} B$, and W is an open subscheme of the domain such that the following determinant is invertible:

$$\det\left(\frac{\partial f_j}{\partial x_i}\right)_{i,j \leq r}.$$

When we say f is smooth, we mean it is smooth of some relative dimension without specifying the dimension. Note that when $Y = \operatorname{Spec} k$, X is a smooth k-scheme.

Étale Maps

An étale map is smooth of relative dimension 0. Note that locally,

Definition

A ring homomorphism $\phi : A \to B$ is étale if:

1. (Formally étale): For every map of A-algebras $R' \to R$ such that the kernel squares to 0, the map
$$\mathrm{Hom}_A(B, R') \to \mathrm{Hom}_A(B, R)$$
is bijective.
2. (Essentially of Finite Presentation:) The map $A \to B$ factors as $A \to C \to B$ where $A \to C$ is of finite presentation and the map $C \to B$ is "C-isomorphic" to a localization map of the form $C \to S^{-1}C$.

An Example of Étale Morphism

For $d \geq 1$, the natural map

$$\phi : \mathbb{C}[t, t^{-1}] \to \mathbb{C}[x, x^{-1}, y]/(y^d - x)$$

is an étale map. Geometrically, we can interpret ϕ as follows:

1. Consider the map $\mathbb{C}P^1 \to \mathbb{C}P^1$ by $z \mapsto z^d$.
2. This is almost a covering space map except at $0, \infty \in \mathbb{P}^1_{\mathbb{C}}$
3. $\mathbb{C}[t, t^{-1}]$ removes $0, \infty$, so the map ϕ really does become a degree d covering space map.

Flatness

Recall for a ring A, an A-module M is flat if the tensor product $- \otimes_A M$ is an exact functor. A ring map $f : A \to B$ is flat is B is flat as an A-module.

Definition

A morphism of schemes $f : X \to Y$ is flat if for each $x \in X$, the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

Outline

Grothendieck's Functor of Points Perspective

So far, we have been thinking of schemes as spaces. There is an alternative perspective of them as functors that also admits a generalization known as stacks, which will be important later.

- An alternative perspective: Let X be a scheme, then there is a functor

$$h_X : \mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Set}, h_X(R) = \mathrm{Hom}_{Sch}(\mathrm{Spec}\, R, X).$$

- Moreover, it turns out there is a fully faithful embedding $\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Alg}_{\mathbb{Z}}, \mathrm{Set})$ given by $X \mapsto h_X$.
- Why not just consider schemes as special functors from $\mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Set}$?

Remark: Why Not The Other Way?

Remark: Can we consider functors of the form

$$h'_X : \mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Set}, h'_X(R) = \mathrm{Hom}_{\mathrm{Sch}}(X, \operatorname{Spec} R)?$$

Fact: The maps $X \to \operatorname{Spec} R$ are in natural bijection with ring morphisms $A \to \Gamma(X, \mathcal{O}_X)$. Here $\Gamma(X, \mathcal{O}_X)$ denotes the global sections on X.

Definitions/Examples for Functor of Points Approach

1. Affine schemes are exactly the representable functors in $\mathrm{Fun}(\mathrm{Alg}_{\mathbb{Z}}, \mathrm{Set})$.
2. If we want to work with R-schemes for a fixed ring R, then they arise as special functors from $\mathrm{Alg}_R \to \mathrm{Set}$.
3. A scheme $h_X : \mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Set}$ is a group scheme if the functor can be lifted to the category of groups Grp.
4. Consider the functor $F_a : \mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Grp}$ given by

   $$F_a(\mathrm{R}) = (R, +).$$

   This is representable by the scheme $\operatorname{Spec} \mathbb{Z}[t]$. This is known as the additive formal group.
5. Consider the functor $F_m : \mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Grp}$ given by

   $$F_a(\mathrm{R}) = R^{\times}.$$

   This is representable by the scheme $\operatorname{Spec} \mathbb{Z}[t^{\pm 1}]$. This is known as the multiplicative formal group.

For us it will be more convenient to define projective spaces and (more generally) Grassmanians as follows.

We define $\mathrm{Gr}(r, N)$, the Grassmanians of r-planes in N-dimensional space, as the functor

$$A \mapsto \{\text{projective } A\text{-modules P of rank r,}$$

$$\text{equipped with an epimorphism } A^{\oplus N} \twoheadrightarrow P\}$$

It turns out this functor does indeed come from a scheme. When $r = 1$, this yields the projective space $\mathbb{P}^{N-1}$.

Outline

Injective and Projective Objects

Let $\mathcal{A}$ be an abelian category:

1. An object $A \in \mathcal{A}$ is injective if every exact sequence of the following form splits:

$$0 \to A \to B \to C \to 0.$$

2. An object $C \in \mathcal{A}$ is projective if every exact sequence of the following form splits:

$$0 \to A \to B \to C \to 0.$$

Example: In RMod, injective and projective objects are exactly injective and projective modules.

Enough Injectives / Projectives

Let $\mathcal{A}$ be an abelian category:

1. $\mathcal{A}$ has enough injectives if for every object $A \in \mathcal{A}$, there is a monomorphism $A \to I$ where I is injective.
2. $\mathcal{A}$ has enough projectives if for every object $A \in \mathcal{A}$, there is a epimorphism $P \to A$ where P is projectives.

Note that if $\mathcal{A}$ has enough injectives, then any object $A \in \mathcal{A}$ admits an injective resolution:

$$0 \to A \xrightarrow{f} I^0 \xrightarrow{g} I^1 \to \dots$$

where each I^k is injective. Here, the map g is the composition of canonical maps $I^0 \to \operatorname{coker} f$ and $f' : \operatorname{coker} f \hookrightarrow I^1$, and so on.

There is a similar notion of projective resolutions.

Examples of Categories with Enough Injectives

Theorem:

1. For any $\mathcal{C}$ a small category, the presheaves (ie. contravariant functors) of abelian groups $\mathrm{PShv}_{\mathrm{Ab}}(X)$ has enough injectives.
2. It is a fun exercise to check that for X a Noetherian schemes, $\mathrm{QCoh}(X)$ has enough injectives.
3. It is a lot harder to show that $\mathrm{QCoh}(X)$ has enough injectives for any scheme X.
4. For a scheme X, the category of $\mathcal{O}_X$-modules has enough injectives.

Right Derived Functors

Let $f : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories, and suppose $\mathcal{A}$ has enough injectives.

We construct the right derived functors $R^i f : \mathcal{A} \to \mathcal{B}$ as follows:

1. Given $A \in \mathcal{A}$, take an injective resolution

$$0 \to A \to I^0 \to I^1 \to \ldots$$

From here we define

$$R^i f(A) := H^i(f(I^0) \to f(I^1) \to \ldots).$$

2. A morphism $A \to B$ lifts to a morphism of their respective resolutions that is unique up to chain homotopy. This gives a canonical map $R^i f(A) \to R^i f(B)$[15].

[15] Note this is covariant

Right Derived Functors

By an argument in the flavor of the snake lemma, given an exact sequence in $\mathcal{A}$ of the form

$$0 \to A \to B \to C \to 0,$$

we have a naturally induced long exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^0 f(A) & \longrightarrow & R^0 f(B) & \longrightarrow & R^0 f(C) \\ & & R^1 f(A) & \longrightarrow & R^1 f(B) & \longrightarrow & R^1 f(C) \\ & & R^2 f(A) & \longrightarrow & \dots & & \end{array}$$

Thus, the construction of $R^i f$ should be thought of as "cohomology theories" and $R^i f(A)$ is the "i-th cohomology in coefficient A".

Left Exactness in Right Derived Functors

Fact:

If f is also left exact, then $R^0 f(A) \cong f(A)$ for all $A \in \mathcal{A}$.

In the left exact case, the long exact sequence becomes

$$0 \longrightarrow f(A) \longrightarrow f(B) \longrightarrow f(C)$$

$$\longrightarrow R^1 f(A) \longrightarrow R^1 f(B) \longrightarrow R^1 f(C)$$

$$\longrightarrow R^2 f(A) \longrightarrow \ldots$$

1. Let $\mathcal{A} = \text{RMod}$ for a ring R, $M \in \text{RMod}$, and $f : \text{RMod} \to \text{RMod}$ be the functor $\text{Hom}_{\text{RMod}}(M, -)$. The functors $R^i f$ are exactly the Ext-functors $\text{Ext}^i_R(M, -)$.
2. Motivated by the previous item, for a general object $M \in \mathcal{A}$, the right derived functors of $\text{Hom}_{\mathcal{A}}(M, -)$ are defined as $\text{Ext}^i_{\mathcal{A}}(M, -)$.
3. Let k be a field and G be a group, the right derived functors of the fixed points functor $(-)^G : k[G] - \text{Mod} \to k[G] - \text{Mod}$ is exactly group cohomology. In fact, this is a special case of Ext functors.
4. Let A be a k-algebra and consider the functor

$$(A, A)\text{−Bimod} \to k\text{−Mod}, M \mapsto \{x \in M \mid ax = xa, \forall a \in A\}.$$

The right derived functors are Hochschild cohomology.

The Definition of Sheaf Cohomology

Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of $\mathcal{O}_X$-modules over X, the global sections functor given by $\Gamma : \mathcal{F} \to \mathcal{F}(X)$ is left exact, ie. the following is exact

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X).$$

Definition

The sheaf cohomology of $\mathcal{F}$ on X is

$$H^i(X, \mathcal{F}) := R^i\Gamma(\mathcal{F}).$$

Remark: A similar definition can be given for abelian sheaves!

Examples of Sheaf Cohomology

1. (Serre's Affine Vanishing Theorem): Let X be affine scheme and $\mathcal{F}$ be a quasicoherent sheaf, then

$$H^i(X, \mathcal{F}) = 0 \text{ for all } i > 0.$$

2. Let B be an abelian group, if X is locally contractible, then

$$H^i_{\text{sing}}(X, B) \cong H^i(X, \underline{B})$$

3. In many good cases the sheaf cohomology is more computable: Let X be a Noetherian and separated scheme and U be an affine open cover of X. For any quasicoherent sheaf $\mathcal{F}$ on X, there is an isomorphism between the Čech cohomology[16] and sheaf cohomology:

$$\check{H}^i(U, \mathcal{F}) \cong H^i(X, \mathcal{F}).$$

[16] The one from topology

What About Sheaf Homology?

- Similar to the injective case, if an abelian category $\mathcal{A}$ has enough projectives, then there is an analogous definition for left derived functors.
- Unfortunately, it is not true in general that the category sheaves of $\mathcal{O}_X$-modules has enough projectives.

Outline

The Promise of Intersection Theory

- The Chow groups are an analog of homology theories in algebraic geometry by replacing "k-simplicies" with "k-dimensional subvarieties".
- If there is some more regularity in the set-up (ie. smoothness), the Chow groups can in fact admit an intersection product.
- Thus, they have wide applications in intersection theory.

The Cycle Groups

Definition:

Let X be a scheme, the group of cycles of X is $Z(X)$: the free abelian group generated by Y, where Y ranges over irreducible subvarieties of X.

Note that $Z(X)$ admits a grading of the form

$$Z(X) := \bigoplus_k Z_k(X)$$

where $Z_k(X)$ are generated by the k-dimensional subvarieties. Elements of $Z_k(X)$ are called k-cycles.

If X is equidimensional, codimension-1 cycles are also known as (Weil) divisors sometimes.

Rational Equivalence

Let $\mathrm{Rat}(X)$ be generated by formal differences of the form:

$$V \cap \{t_0\} \times X - V \cap \{t_1\} \times X,$$

where $V \subseteq \mathbb{P}^1 \times X$ is a sub-variety not contained in any fiber $\{t\} \times X$.

Definiton:

We say two varieties V_1, V_2 are rationally equivalent if $V_1 - V_2 \in \mathrm{Rat}(X)$.

Chow Group

Definition:

The Chow group $\mathrm{CH}(X)$ of X is $Z(X)$ modulo rational equivalence. In other words,

$$\mathrm{CH}(X) := Z(X)/\operatorname{Rat}(X).$$

Note that there is a grading $\mathrm{CH}(X) = \bigoplus_k \mathrm{CH}_k(X)$ by dimension due to the following lemma.

Lemma:

Two non-empty rationally equivalent varieties have the same dimension.

Note that clearly $\mathrm{CH}(X) = \mathrm{CH}(X_{\mathrm{red}})$.

The Chow Ring

Now we restrict to smooth k-varieties. If X is equidimensional, we can further write:

$$\mathrm{CH}(X) := \bigoplus_i \mathrm{CH}^i(X) := \mathrm{CH}_{\dim X - i}(X).$$

Let A, B be irreducible sub-varieties that are generically transversally[17], then we define

$$[A] \cdot [B] := [A \cap B] \quad (\dagger)$$

As a consequence of the moving lemma, there is an unique product structure on $\mathrm{CH}(X)$ whose restriction to generaically transversal pairs is $(\dagger)$. This defines the Chow ring structure.

[17] Meaning each component has points they are transverse on. Transverse is defined with tangent spaces replaced with Zariski tangent spaces.

Examples of Chow Rings

$\mathrm{CH}(\bullet)$ admit their own versions of Meyer-Vietoris and Excision! Here are some examples of Chow rings:

1. $\mathrm{CH}(\mathbb{A}_k^n) = \mathbb{Z}\{[\mathbb{A}_k^n]\}$. This can be done by showing that every strict subvariety V of $\mathbb{A}^n$ is rationally equivalent to 0. The idea is to without loss change coordinates such that $0V$.
2. By a combination of excision and deduction, it can be shown that
$$\mathrm{CH}(\mathbb{P}_k^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1})$$
where ζ is the equivalence class of a hyperlane.

Bezout's Theorem

As a corollary of the Chow ring computation on $\mathbb{P}_k^n$, we have that:

Theorem

Let $X_1, ..., X_r$ be subvarieties of $\mathbb{P}_k^n$ of codimension $a_1, ..., a_r$ such that $a_1 + ... + a_r \leq n$, each intersecting generically transversely, then

$$\deg(X_1 \cap ... \cap X_k) = \prod \deg(X_i).$$

By a long chain of deductions, this is emblematic of the idea that two lines on $\mathbb{C}P^2$ should intersect in the history section.

Chern Classes

Just as there are Chern classes for vector bundles in topology valued in integral cohomology, there is an analog of Chern classes in algebraic geometry valued in Chow rings.

Definition

Let X be smooth k-variety. An (algebraic) vector bundle $\mathcal{E} \to X$ is globally generated if there exists sections $s_1, ..., s_r : X \to \mathcal{E}$ such that the span of $s_1(x), ..., s_r(x)$ is $\mathcal{E}_x$ for all $x \in X$.

Chern Classes

The Chern classes are defined axiomatically as follows:

Let $\mathcal{E} \to X$ be globally generated vector bundle of rank n. There is an unique element $c(\mathcal{E}) = \sum_{i \geq 0} c_i(\mathcal{E}) \in \mathrm{CH}(X)$ such that:

1. $c_0(\mathcal{E}) = 1$.
2. Naturality: $c(\bullet)$ is natural with respect to morphisms.
3. Whitney Sum Formula: If $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is a SES of globally generated vector bundles, then
$$c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G})$$
4. If $\mathcal{E}$ is a line bundle, $c_1(\mathcal{E})$ is the subvariety on X where the zero section and a generic section agree.
5. Let $s_0, ..., s_{n-p}$ be global sections of $\mathcal{E}$ and
$$Y(s_0, ..., s_{n-p}) = \{x \in X \mid s_0(x), ..., s_{n-p}(x) \text{ are linearly dependent}\}$$
Suppose Y has codimension p, $c_p(\mathcal{E}) = [Y] \in \mathrm{CH}^p(X)$.

Properties of Chern Classes

1. If X is a smooth k-variety, the first Chern class defines a map
$$c_1 : \mathrm{Pic}(X) \to \mathrm{CH}^1(X)$$
that is an isomorphism.
2. Chern classes behave how you would expect in the case of topology. For example:
   - There is an analog of the splitting principle.
   - If $\mathcal{E}$ has rank n, then $c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$.
3. The Chern class technology can be used to show that there are 27 lines on a smooth cubic over an algebraically closed field.

Connections to K-theory

The Grothendieck-Riemann-Roch theorem implies:

Theorem:

There is a rational equivalence of the form

$$K_0(X) \otimes \mathbb{Q} \cong \bigoplus_k \mathrm{CH}^k(X) \otimes \mathbb{Q}.$$

A result of Bloch shows this extends to higher K-theories too:

Theorem:

There is a rational equivalence of the form

$$K_n(X) \otimes \mathbb{Q} \cong \bigoplus_k \mathrm{CH}^k(X, n) \otimes \mathbb{Q}.$$

Here $\mathrm{CH}^k(X, n)$ is a variant defined as certain cycles in $X \times \mathbb{A}^n$.

There is also a way to connect Chow rings to sheaf cohomology.

The Bloch-Quillen Formula

Let X be a smooth k-variety[18]. Consider the presheaf on X given by sending $U \subset X$ to $K_q(U, \mathcal{O}_U)$[19], and let $\mathcal{K}_q(X, \mathcal{O}_X)$ denotes its associated sheaf.

$$\mathrm{CH}^q(X) \cong H^q(X, \mathcal{K}_q(X, \mathcal{O}_X))$$

Remark: Recall K_1 returns the units (in reasonable cases), and hence the case $q = 1$ recovers the isomorphism

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*).$$

[18] In fact, this works over any regular k-schemes of finite type

[19] The algebraic K-theories of the scheme

What is Next?

So far, our main interests in algebraic geometry have been sheaves valued in sets, abelian groups, or rings. In (unstable) motivic homotopy theory, we would be interested in sheaves valued in the ∞-category of spaces.

In next lecture, we will discuss

1. Simplicial Sets
2. ∞-categories
3. The ∞-category of spaces
4. And possibly more