Lecture 11: Motivic Characteristic Classes and Grothendieck-Riemann-Roch Theorem

By Mattie Ji

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Overview

This talk is about characteristic classes in motivic homotopy theory, leading to a version of the Grothendieck Riemann-Roch theorem.

We will mainly follow:

1. The PCMI lectures by Frédéric Déglise in [Déglise, 2024].
2. And parts of the accompanied papers [Déglise, 2016, Déglise, 2018, Déglise et al., 2021].

Throughout this talk:

- **Smooth** means smooth and of finite type.
- The base **schemes** S are Noetherian unless otherwise mentioned.
- $[a]$ means $\wedge(S^1)^a$ and (b) means $\wedge(\mathbb{G}_m)^b$.

Outline

Projective Bundle Formula

Recall from **last time**, we defined the first Chern class c_1 with respect to any oriented ring spectra $(\mathbb{F}, c)$. Here, we outline a general construction of higher Chern classes.

Let us recall the projective bundle formula over oriented ring spectra $(\mathbb{E}, c)$.

Theorem

Let V/X be a vector bundle of rank n and $p : \mathbb{P}(V) \to X$ with canonical line bundle λ_P. There is an isomorphism of $\mathbb{E}^{*,*}(X)$-modules with:

$$\bigoplus_{i=0}^{n-1} \mathbb{E}^{*,*}(X) \to \mathbb{E}^{*,*}(\mathbb{P}(V))$$

$$\lambda_i \mapsto \sum_i p^*(\lambda_i) c_1(\lambda_p)^i.$$

Application: Vanishing of $\lim^1$-Term

For a motivic ring spectrum $\mathbb{E}$ over S, there is a Milnor exact sequence

$$0 \to \lim_{n\geq 0}^{1} \mathbb{E}^{2,1}(\mathbb{P}^n_S) \to \mathbb{E}^{2,1}(\mathbb{P}^\infty_S) \to \lim_{n\geq 0} \mathbb{E}^{2,1}(\mathbb{P}^n_S) \to 0$$

The following is an application of the projective bundle formula.

Lemma:

If $\mathbb{E}$ is oriented, then $\lim^1_{n\geq 0} \mathbb{E}^{2,1}(\mathbb{P}^n_S) = 0$.

Application: Vanishing of $\lim^1$-Term

Lemma:

If $\mathbb{E}$ is oriented, then $\lim^1_{n\geq 0} \mathbb{E}^{2,1}(\mathbb{P}^n_S) = 0$.

Proof: It suffices to show the system satisfies the Mittag-Leffler condition: for all k, there exists $i \geq k$ such that for all $j \geq i \geq k$,

$$\mathrm{im}(\mathbb{E}^{2,1}(\mathbb{P}^i_S) \to \mathbb{E}^{2,1}(\mathbb{P}^k_S)) = \mathrm{im}(\mathbb{E}^{2,1}(\mathbb{P}^j_S) \to \mathbb{E}^{2,1}(\mathbb{P}^k_S)).$$

Since $\mathbb{E}$ is oriented, there is a very clear description of what the cohomologies here are and the induced maps by the **projective bundle formula**, which satisfies the Mittag-Leffler condition.

Application: Criterion to Construct an Orientation

$$0 \to \lim_{n \geq 0}^{1} \mathbb{E}^{2,1}(\mathbb{P}^n_S) \to \mathbb{E}^{2,1}(\mathbb{P}^\infty_S) \to \lim_{n \geq 0} \mathbb{E}^{2,1}(\mathbb{P}^n_S) \to 0$$

Remark: By the previous lemma, specifying an orientation c on $\mathbb{E}$ is equivalent to:

- Constructing elements $c_n \in \mathbb{E}^{2,1}(\mathbb{P}^n_S)$ for $n > 0$.
- With $c_1 = 1_{\mathbb{E}}$ and $c_n = \iota_n^*(c_{n+1})$ in the limit system.

Construction of Higher Chern Classes (via Grothendieck)

Let $(\mathbb{E}, c)$ be oriented motivic ring spectrum over S and X be a smooth S-scheme, V/X be a vector bundle of rank n.

Using the projective bundle formula, there exists an unique family $c_0(V) = 1, c_1(V), ..., c_n(V)$ where $c_i(V) \in \mathbb{E}^{2i,i}(X)$ (we set $c_i(V) = 0$ if $i > n$) such that the following equality holds in $\mathbb{E}^{*,*}(\mathbb{P}(V))$:

$$0 = \sum_{i=0}^{n} p^*(c_i(V)) \cdot (-c_1(\lambda_P))^{n-i}$$

where λ_P is the canonical line bundle over $p : \mathbb{P}(V) \to X$.

Properties of (Motivic) Chern Classes

These Chern classes satisfy a variety of properties:

1. (**Isomorphism Invariance**) For V, V' isomorphic vector bundles over X, $c_i(V) = c_i(V')$.
2. (**Naturality**): Let $g : Y \to X$ be a scheme morphism and V a vector bundle over X, then $f^*(c_i(V)) = c_i(f^*(V))$.
3. (**Vanishing**) If V is a trivial vector bundle over X, then $c_i(V) = 0$ for all $i > 0$.
4. (**Nilpotence**): If S is Noetherian, and V is a vector bundle over an S-scheme X, then $c_i(V)$ is nilpotent for all $i > 0$.

(1) and (2) are clear. For (3), by naturality, it suffices to verify this over S. By the projective bundle formula, we know that $c_1(\mathcal{O}_{\mathbb{P}^n_S}(-1))^n = 0$. Since the family of c_i's is unique, choosing $c_i(V) = 0$ for $i > 0$ would satisfy the equation.

Let us now look at Thom spaces and classes.

For $X \in \mathrm{Sm}/S$, there is a category of virtual vector bundles over X - $\underline{K}(X)$ - which is the groupoid associated to Quillen K-theory space $K(X)$. The map $\mathrm{Vect}(X) \to \mathrm{SH}(S)$ with $V \mapsto \Sigma^\infty \mathrm{Th}(V)$ extends to a functor

$$\underline{K}(X) \to \mathrm{Ho}\,\mathrm{SH}(S), [v] \mapsto \Sigma^\infty \mathrm{Th}(v).$$

Note: We will use notation $\mathrm{Th}(v)$ and the stable Thom space $\Sigma^\infty \mathrm{Th}(v)$ interchangeably.

Thom Classes

Motivation: For the Thom space of V/X $\mathrm{Th}(V)$ is the homotopy cofiber V/V^*, but this is also $\mathbb{A}^1$-equivalent to $\mathbb{P}(V \oplus \mathbb{A}^1)/\mathbb{P}(V)$, ie.

$$0 \to \mathbb{P}(V) \to \mathbb{P}(V \oplus 1) \to \mathrm{Th}(V) \to 0.$$

If we apply $(\mathbb{E}, c)$ to this, we obtain a sequence

$$\mathbb{E}^{*,*}(\mathrm{Th}(V)) \to \mathbb{E}^{*,*}(\mathbb{P}(V \oplus 1)) \to \mathbb{E}^{*,*}(\mathbb{P}(V))$$

By the projective bundle formula, this in fact becomes a **split exact sequence**:

$$0 \to \mathbb{E}^{*,*}(\mathrm{Th}(V)) \to \mathbb{E}^{*,*}(\mathbb{P}(V \oplus 1)) \to \mathbb{E}^{*,*}(\mathbb{P}(V)) \to 0$$

Thom Classes

This motivates our definition of Thom classes to be constructed on $\mathbb{P}(V \oplus 1) = \mathbb{P}(V \oplus \mathbb{A}^1)$ as follows[1]

Definition

Let $(\mathbb{E}, c)$ be an oriented ring spectrum (in $\mathrm{SH}(S)$). The Thom class of V/X (vector bundle of rank n) is the element in $\mathbb{E}^{2n,n}(\mathbb{P}(V \oplus \mathbb{A}^1))$ with

$$\mathrm{th}(V) = \sum_{i=0}^{n} p^*(c_i(V)) \cdot (-c_1(\lambda))^{n-i}.$$

where $p : \mathbb{P}(V \oplus \mathbb{A}^1) \to X$ and λ is the canonical line bundle on $\mathbb{P}(V \oplus \mathbb{A}^1)$.

[1] which restricts to a trivial class over $\mathbb{P}(V)$.

The Refined Thom Class

$$0 \to \mathbb{E}^{*,*}(\mathrm{Th}(V)) \to \mathbb{E}^{*,*}(\mathbb{P}(V \oplus 1)) \to \mathbb{E}^{*,*}(\mathbb{P}(V)) \to 0$$

By construction, $\mathrm{th}(V) \in \mathbb{E}^{*,*}(\mathbb{P}(V \oplus 1))$ restricts to 0 over $\mathbb{E}^{*,*}(\mathbb{P}(V))$. Exactness implies that there is an unique class $\overline{\mathrm{th}}(V) \in \mathbb{E}^{*,*}(\mathrm{Th}(V))$ that is sent to $\mathrm{th}(V)$.

$\overline{\mathrm{th}}(V)$ is called the refined Thom class.

Calculating the Thom Class

The canonical bundle $O(-1)$ on $\mathbb{P}(V \oplus \mathbb{A}^1)$ naturally fits in $p^*(V \oplus 1)$ where $V \oplus 1$ is over $V \oplus \mathbb{A}^1$ and $p : \mathbb{P}(V \oplus \mathbb{A}^1) \to V \oplus 1$ is the natural projection.

Now consider ξ (called the **universal quotient bundle**) over $\mathbb{P}(V \oplus \mathbb{A}^1)$ given by the exact sequence

$$0 \to O(-1) \to p^*(V \oplus 1) \to \xi \to 0.$$

Lemma:

$\mathrm{th}(V) = c_n(\xi)$.

Calculating the Thom Class

Lemma:

$\mathrm{th}(V) = c_n(\xi)$.

Proof: The Whitney sum formula[2] implies that $c(p^*(V \oplus 1)) = c(O(-1))c(\xi)$.[3] On the level of c_n, we have

$$c_n(p^*(V \oplus 1)) = \sum_{p+q=n} c_p(O(-1))c_q(\xi) = c_n(\xi) + c_1(O(-1))c_{n-1}(\xi).$$

Thus, we have by naturality and Whitney-sum formula that

$$c_n(\xi) = c_n(p^*(V \oplus 1)) - c_1(O(-1))c_{n-1}(\xi)$$

$$= p^*(c_n(V)) \cdot 1 + c_{n-1}(\xi)(-c_1(O(-1))).$$

[2] which we did not state, but it is the one the reader would expect

[3] where c represents the total Chern class

Calculating the Thom Class

Now on the level of c_{n-1}, the initial equation gives us

$$p^*(c_{n-1}(V)) = c_{n-1}(p^*(V \oplus 1)) = c_{n-1}(\xi) + c_1(O(-1))c_{n-2}(\xi)$$

Plugging this in the previous question gives us that

$$\begin{aligned} c_n(\xi) &= p^*(c_n(V)) \cdot 1 + c_{n-1}(\xi)(-c_1(O(-1))) \\ &= p^*(c_n(V)) \cdot 1 + p^*(c_{n-1}(V))(-c_1(O(-1))) \\ &+ c_{n-2}(\xi)(-c_1(O(-1)))^2 \end{aligned}$$

Repeatedly the recursive relations, we will get eventually that

$$c_n(\xi) = \sum_{i=0}^{n} p^*(c_i(V)) \cdot (-c_1(O(-1)))^{n-i} = \mathrm{th}(V).$$

Motivic Thom Isomorphism

Theorem (Motivic Thom Isomorphism)

The map $\mathbb{E}^{*,*}(X) \to \mathbb{E}^{*,*}(\mathrm{Th}(V))$ given by

$$\lambda \mapsto \lambda \cdot \overline{\mathrm{th}}(V)$$

is an isomorphism. Here the map is understood as $\mathbb{E}^{*,*}(X)$ actings on $\mathbb{E}^{*,*}(\mathrm{Th}(V))$ as the rings of coefficients.

Proof: Again consider the exact sequence

$$0 \to \mathbb{E}^{*,*}(\mathrm{Th}(V)) \xrightarrow{f} \mathbb{E}^{*,*}(\mathbb{P}(V \oplus 1)) \xrightarrow{g} \mathbb{E}^{*,*}(\mathbb{P}(V)) \to 0.$$

By the projective bundle formula, the middle term is a free $\mathbb{E}^{*,*}(X)$ module of rank $n+1$ and the right term is free of rank n, so $\mathbb{E}^{*,*}(\mathrm{Th}(V))$ is a 1-dimensional $\mathbb{E}^{*,*}(X)$-module (ie. invertible), and the refined Thom class is its basis.

Motivic Thom Isomorphism, Reforumlated

There is a reforumlation of the Thom isomorphism as follows.

Theorem (Motivic Thom Isomorphism, Version 2)

Let $p : X \to S$ be the structure map with $\mathbb{E}_X = p^*\mathbb{E}$. Let V/X, then the multiplication by the Thom class gives $\overline{th}(V)$ gives a map:

$$\gamma_{\overline{th}(V)} : \mathbb{E}_X \otimes \mathrm{Th}(V) \to \mathbb{E}_X(m)[2m],$$

which is an isomorphism of $\mathbb{E}_X$-modules.

Finally, we will construct Thom classes for virtual bundles. We first note there is a splitting of Thom spaces

Lemma:

Consider an exact sequence of vector bundles over S

$$0 \to V' \to V \to V'' \to 0,$$

then we have an $\mathbb{A}^1$-equivalence of motivic spectra[4],

$$\Sigma^\infty \mathrm{Th}(V) \cong \Sigma^\infty \mathrm{Th}(V') \otimes \Sigma^\infty \mathrm{Th}(V'').$$

[4] This splitting is even true in the unstable world.

Whitney Sum Formula for Thom Classes

Lemma:

The equivalence before admits a canonical isomorphism

$$\mathbb{E}^{*,*}(\mathrm{Th}(V)) \simeq \mathbb{E}^{*,*}(\mathrm{Th}(V')) \otimes_{\mathbb{E}^{*,*}(X)} \mathbb{E}^{*,*}(\mathrm{Th}(V'')).$$

This in particular implies that $\overline{\mathrm{th}}(V) = \overline{\mathrm{th}}(V') \otimes \overline{\mathrm{th}}(V'')$.

Proof Idea: The first equality is because $\mathbb{E}^{*,*}$, as an ∞-functor between stable ∞-categories, preserves finite (co)limits.

To check the formula for refined Thom classes, we can reduce to the case of a split SES:

$$0 \to V' \xrightarrow{i} V \xrightarrow{q} V'' \to 0$$

Whitney Sum Formula for Thom Classes

Let $O(-1)$ be the canonical bundle over $\mathbb{P}(V)$ and $p : \mathbb{P}(V) \to X$ consider the following vector bundles over $\mathbb{P}(V)$ as

$$\xi = p^*(V)/O(-1), \xi' = p^*(i(V'))/O(-1), \xi'' = (q \circ p)^*(V'')/O(-1)$$

Because the sequence is split, we in fact have that

$$\xi = \xi' \oplus \xi''.$$

Write e as the top Chern class, the Whitney sum formula implies that $e(\xi) = e(\xi')e(\xi'')$, which can be used to conclude the proof.

Motivic Thom Class of Virtual Bundles

Let v be a virtual vector bundle.

By the previous lemma, $\mathbb{E}^{*,*}(\mathrm{Th}(v))$ is a free $\mathbb{E}^{*,*}(X)$-module of rank 1. The lemma gives it the choice of a canonical basis $\mathrm{th}(v)$ in $\mathbb{E}^{2r,r}(\mathrm{Th}(v))$, where r is the virtual rank of v. This is the **Thom class of a virtual vector bundle**.

Outline

Theorem (Voevodsky, Ayoub)

There are 3-pairs of adjoint functors with several properties.

1. $(\otimes_S, \underline{Hom}_S)$ on $\mathrm{SH}(S)$ (Note this is simply because $\mathrm{SH}(S)$ is a presentable symmetric monoidal ∞-category).
2. For any morphism of schemes $f : T \to S$, we get a pair of symmetric monoidal adjoint functors
$$f^* : \mathrm{SH}(S) \rightleftarrows \mathrm{SH}(T) : f_*$$
which is induced by an ∞-functor $\mathrm{SH}^* : \mathrm{Sch}^{op} \to \mathrm{Cat}_\infty^\otimes$
3. Let $p : X \to S$ be a morphism of schemes that is separated of finite type (call this s-morphism), we get another pair of adjoints
$$p_! : \mathrm{SH}(Y) \rightleftarrows \mathrm{SH}(X) : p^!$$
which comes from an ∞-functor $\mathrm{SH}_! : \mathrm{Sch} \to \mathrm{Cat}_\infty$

Six Functor Formalism Properties[5]

1. There is a natural transformation $f_! \to f_*$ which is invertible if f is furthermore proper.
2. There is a invertible natural transformation $f^* \to f^!$ whenever f is an open immersion.
3. There is a canonical isomorphism

$$\mathbb{E} \otimes p_!(\mathbb{F}) \to p_!(p^*(\mathbb{E}) \otimes \mathbb{F})$$

for any s-morphism $p : T \to S$, $\mathbb{E} \in \mathrm{SH}(S), \mathbb{F} \in \mathrm{SH}(T)$.

4. (Base Change) $f_!$ (resp. $f^!$) satisfies base change w.r.t to inverse images g^* (resp. g_*) - that is if f, g are both s-morphisms and there is a Cartesian square:

$$\begin{array}{ccc} T' & \xrightarrow{g} & S' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow[f]{} & S \end{array}$$

then $p^* f_! \xrightarrow{\sim} g_! q^*$ and $q_* g^! \xrightarrow{\sim} f^! p_*$.

[5] Skip in presentation

Smooth Purity

If $p : X \to S$ is smooth with relative tangent bundle T_p/X, then there exists an isomorphism of functors:

$$\mathfrak{p}_p : \Sigma^{T_p} p^* \to p^!.$$

where Σ^{T_p} notationally means smashing with the Thom space, ie. this means that:

$$p^! \simeq \mathrm{Th}(T_p) \otimes p^*,$$

or, equivalently by the lemma before, that

$$p^* \simeq \mathrm{Th}(-T_p) \otimes p^!.$$

Example: Let $p : X \to S$ be smooth and separated, then

$$\Sigma^\infty X_+ \simeq p_! p^1(1_S).$$

Outline

Let $\mathbb{E}$ be an motivic ring spectrum[6].

Definition

Let $n \in \mathbb{Z}$ and $v \in \underline{K}(X)$. A bi-variant theory[7] of a s-morphism $p : X \to S$ in degree n with a twist v is

$$\mathbb{E}_n(X/S, v) = [\mathrm{Th}(v)[n], p^!\mathbb{E}]$$

Note that equivalently $\mathbb{E}_n(X/S, v)$ is $\pi_n(\mathbb{E}(X/S, v))$ where $\mathbb{E}(X/S)$ is the **v-twisted bivariant spectrum** (as a mapping spectrum)

$$\mathbb{E}(X/S) = \mathrm{Maps}_{\mathrm{SH}(S)}(p_!(\mathrm{Th}(v)), \mathbb{E}).$$

[6] Much of what we discussed holds for a general motivic spectra.

[7] Also called Borel-Moore $\mathbb{E}$-homology

Examples of Bi-Variant Theory

1. If $p : X \to X$ is the identity, then
$$\mathbb{E}_n(X/X, v) = \mathbb{E}^{-n,0}(\mathrm{Th}(v)).$$
2. If we take $\mathbb{E}$ to represent Betti cohomology H_B with integral coefficients over $v = 0$ and $S = \mathrm{Spec}(k)$, then the bi-variant theory is the Borel-Moore homology of X.
3. If S is regular and $\mathbb{E} = \mathrm{KGL}_S$, then one can compute
$$\mathrm{KGL}_n(X/S, 0) \simeq G_n(X),$$
where $G_n(X)$ is Quillen's K-theory on the exact category of coherent sheaves over X. This is a theorem of Fangzhou Jin in [Jin, 2019] with a similar statement extending to the non-regular case.

Fulton-MacPherson Formalism

Fulton-MacPherson devised some formal consequences of these bi-invariant theories (without the twist v) as follows.

1. For any map $f : T \to S$, there is a base change map:

$$f^* : \mathbb{E}_n(X/S, v) \to \mathbb{E}_n(X \times_S T = X_T/T, g^*(v))$$

where g appears in the Cartesian square

$$\begin{array}{ccc} X_T & \xrightarrow{g} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow[f]{} & S \end{array}$$

2. **Proper Covariance** - Covariant with respect to proper maps $f : Y \to X$, which induces

$$f_* : \mathbb{E}_*(Y/S, f^*(v)) \to \mathbb{E}_*(X/S, v)$$

Fulton-MacPherson Formalism

3. **Étale contravariance** - let $f : X \to Y$ be an étale-s-morphism of S-schemes, there is an inverse image map
$$f^! : \mathbb{E}_n(Y/S, v) \to \mathbb{E}_n(X/S, f^*v)$$
4. **Product:** For morphisms $p : X \to S$ and S-scheme morphism $q : Y \to X$, and $v \in \underline{K}(X)$, $w \in \underline{K}(Y)$, there is a product
$$\mathbb{E}_n(Y/X, w) \otimes \mathbb{E}_m(X/S, v) \to \mathbb{E}_{n+m}(Y/S, w + q^*v).$$

Outline

Smoothable LCI

Now we first recall the following definition.

Definition

We say a morphism of schemes $f : X \to S$ is **smoothable lci** if there is a factorization

$$X \xrightarrow{i} P \xrightarrow{p} S$$

such that i is a regular closed immersion and p is smooth.

The **virtual tangent bundle**[8] of f, is $\tau_f \in \underline{K}(X)$ such that

$$\tau_f = [i^* T_p] - [N_i],$$

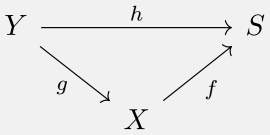
where N_i is the normal bundle of i and T_p is the tangent bundle of p.

[8] For readers interested in derived algebraic geometry, they may note that τ_f is equivalently the virtual bundle associated to the cotangent complex $\mathcal{L}_f$ of f.

Virtual Tangent Bundle

Lemma:

Consider a triangle of maps f, g, h all smoothable lci,



Then $\tau_h = \tau_g + g^*\tau_f$.

Fundamental Classes

Theorem ([Déglise et al., 2021])

Let $f : X \to S$ be a smoothable lci, there exists a class (called the (refined) **fundamental class**) $\eta_f \in \mathbb{E}_0(X/S, \tau_f)$ such that

1. η_{id} is $1_{\mathbb{E}}$, the unit of the ring spectrum.
2. Let $p : X' \to S$ be a morphism transversal to f (that is - p is transversal to f if p' is a local complete intersection and $\tau_{f'} \simeq (p')^{-1}(\tau_f)$):

$$\begin{array}{ccc} X' \times_S X & \xrightarrow{p'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{p} & S \end{array}$$

   In this case, one has $p^* \eta_f = \eta_{f'}$.[9]

[9] Note that we are using (1) of the Fulton-MacPherson formalism here.

Fundamental Classes

Theorem ([Déglise et al., 2021])

3. When $h = g \circ f$ and they are all smoothable lci, then by lemma above, we have that $\mathbb{E}_0(Y/S; \tau_h) = \mathbb{E}_0(Y/S, \tau_g + g^* \tau_f)$. In this case, we have

$$\eta_h = \eta_g \cdot \eta_f,$$

where the product is given by the Fulton-MacPherson formalism before.

Proof Sketch of Theorem

Case 1: f is a smooth morphism. In this case, the existence of such η_f can be deduced from the smooth purity isomorphism prior.

Case 2: f is a (regular) closed immersion. Write $f = i : Z \to X$. To do this we need the theory of **deformation spaces**. Write

$$D = D(X, Z) := Bl_Z(\mathbb{A}^1_X) - Bl_Z X$$

Note that D is a scheme fibered over $\mathbb{A}^1$:

- The fiber of D over $\mathbb{G}_m$ is $D|_{\mathbb{G}_m} \simeq \mathbb{G}_m \times X$.
- The fiber over 0 is $D|_0 = N := N_Z(X)$ (the normal cone of Z in X[10]).

[10] When i is regular, this is actually the normal bundle.

Then the fundamental class η_i may be constructed as the image of the unit under the following sequence of maps

$$\mathbb{E}_0(X/X,0) \xrightarrow{\gamma_t} \mathbb{E}_{-1}(\mathbb{G}_m \times X/X,0) \xrightarrow{\partial_{D,N}} \mathbb{E}_0(N/X,0) \simeq \mathbb{E}_0(Z/X,-N)$$

- γ_t is multiplication by $[t]$, which is associated to the canonical unit of $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t,t^{-1}])$.
- $\partial_{D,N}$ is a certain residue map associated with the closed immersion $N \subset D$.
- Note that $D - N \simeq \mathbb{G}_m \times X$. The isomorphism in the last step is evident, and note $\tau_i = -N$

General Case: The general step involves showing that the "composition" of the two constructions above "glue correctly" so that its properties would still hold.

Oriented Thom Isomorphism

We seek a specialized construction when we have $(\mathbb{E}, c)$ - a motivic ring spectrum with orientation. Let v be a virtual vector bundle over X, using the **bi-variant theory** above, we can obtain a version of **cap product** given by

$$\mathbb{E}^{m,j}(\mathrm{Th}(w)) \otimes \mathbb{E}_{n,i}(X/S, v) \to \mathbb{E}_{n-m,i-j}(X/S, v+w).$$

Theorem (Oriented Thom Isomorphism)

The following map is an isomorphism:

$$\mathbb{E}_0(X/S, v) \to \mathbb{E}_{2r,r}(X/S), \alpha \mapsto \mathrm{th}(-v) \cdot \alpha$$

where the map is given by the cap product above.

Definition

Let $f : X \to S$ be a smoothable lci. Suppose f has relative dimension d[11]. We define the (refined) oriented fundamental class of f in coefficients $(\mathbb{E}, c)$ as

$$\eta_i^c := \mathrm{th}(-\tau_f) \cdot \eta_f \in \mathbb{E}_{2d,d}(X/S),$$

by the map in the oriented Thom isomorphism prior.

If f is in addition proper, then we can define a **Gysin morphism without twits** as

$$f_! : \mathbb{E}^{*,*}(X) \to \mathbb{E}^{*+2d,*+d}(S), \alpha \mapsto f_*(\alpha \cdot \eta_f^c).$$

[11] this is the rank of τ_f

Outline

Let $\phi : (\mathbb{E}, c) \to (\mathbb{F}, d)$ be a morphism of ring spectra over S. This induces a map

$$\phi_* : \mathbb{E}^{*,*}(\mathbb{P}^\infty_S) \to \mathbb{F}^{*,*}(\mathbb{P}^\infty_S)$$

By the projective bundle formula, we know that this is a map

$$\phi^* : \mathbb{E}^{*,*}(S)[[c]] \to \mathbb{F}^{*,*}(S)[[d]]$$

From here we see that $\phi^*(c)$ is a power-series - denoted $\theta_\phi(d)$.

Fact: The element $\theta_\phi(d) = \phi^*(c)$ is an orientation on $\mathbb{F}$ with the form $\theta_\phi(d) = d + ...$

θ_ϕ is a Strict Isomorphism

As in the classical case, one can check that

Lemma

Let $F_{\mathbb{E}}(x, y)$ be the formal group law over $(\mathbb{E}, c)$ and consider $F_{\mathbb{E}}(x, y)|_{\mathbb{E}^{*,*}}$ be the formal group law obtained by substituting the coefficients via the map ϕ. In this case, θ_ϕ gives a strict isomorphism of FGLs with $F_{\mathbb{E}}(x, y)|_{\mathbb{F}^{*,*}}$ and $F_{\mathbb{F}}(x, y)$ - ie.

$$\theta_\phi(F_{\mathbb{E}}(x, y)|_{\mathbb{F}^{*,*}}) = F_{\mathbb{F}}(\theta_\phi(x), \theta_\phi(y)).$$

Example: The Chern Character

Let S be regular. The **Chern character** is an isomorphism of oriented motivic ring spectrum with

$$\mathrm{ch} : \mathrm{KGL}_S \otimes \mathbb{Q} \to \bigoplus_{i \in \mathbb{Z}} H_M \mathbb{Q}_S(i)[2i]$$

such that on degree 0 it corresponds to the map

$$K_0(S) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{CH}^*(S) \otimes \mathbb{Q}.$$

The group law on KGL_S is the multiplicative FGL and the group law on $H_M \mathbb{Q}_S$ is the additive FGL, so θ_{ch} is the unique strict isomorphism between them known as the **exponential**.

Motivic Todd Class

For $\phi : (\mathbb{E}, c) \to (\mathbb{F}, d)$, there exists a **Todd Class** morphism

$$\mathrm{td}_\phi : K_0(X) \to \mathbb{F}^{0,0}(X)^\times.^{12}$$

The Todd class satisfies:

1. It is natural with respect to pullbacks.
2. For a line bundle L/X,

$$\mathrm{td}_\phi(L) = \frac{t}{\theta_\phi(t)}|_{t=d_1(L)}$$

   This is the power series $\frac{t}{\theta_\phi(t)}$, with t substituted out with $d_1(L)$ (Here d_1 is the first Chern class of L, but we use d instead of c as the orientation).
3. For a vector bundle V/X of rank n,

$$d_n(V) = \mathrm{td}_\phi(V) \cdot \phi_*(c_n(V)).$$

[12] Here $\mathbb{F}^{0,0}(X)$ is a ring, so we take its units.

Example: The Chern Character

Let $\mathrm{ch} : \mathrm{KGL}_S \otimes \mathbb{Q} \to \bigoplus_{i \in \mathbb{Z}} H_M \mathbb{Q}_S(i)[2i]$ be the Chern character. Recall the map here is the exponential (ie. $\theta_\phi(t) = 1 - \exp(-t)$). Thus, we have that for a line bundle

$$\mathrm{td}(L) = \frac{c_1(L)}{1 - \exp(-c_1(L))}$$

Now we are ready to state the Grothendieck-Riemann-Roch theorem in this context.

Theorem (Motivic Grothendieck-Riemann-Roch)

For $\phi : (\mathbb{E}, c) \to (\mathbb{F}, d)$, let $f : X \to S$ be a smoothable lci morphism of dimension n. The following holds in $\mathbb{F}_{2n,n}(X/S)$

$$\phi_*(\eta_f^c) = \mathrm{td}_\phi(\tau_f) \cdot \eta_f^d.$$

This theorem can be used to recover the classical one.

Proof

Write $\eta_f^{\mathbb{E}}$ as the (unoriented) fundamental class for $\mathbb{E}$ with respect to f. From the definition of oriented fundamental class, we have that

$$\phi_*(\eta_f^c) = \phi_*(\mathrm{th}^c(-\tau_f) \cdot \eta_f^{\mathbb{E}}) = \phi_*(\mathrm{th}^c(-\tau_f)) \cdot \eta_f^{\mathbb{F}}.$$

It suffices to then prove that $\phi_*(\mathrm{th}^c(-\tau_f)) = \mathrm{td}_\phi(\tau_f) \cdot \mathrm{th}^d(-\tau_f)$.

We will indeed prove more generally that $\phi_*(\mathrm{th}^c(v)) = \mathrm{td}_\phi(-v) \cdot \mathrm{th}^d(v)$. By a version of splitting principle (which we did not mention, but it is the one you might expect), this equality reduces to when $v = [L]$ is the class by a line bundle. For a line bundle L, $\mathrm{th}(L) = c_1(L)$ and the definition fo Todd class concludes the proof.

Application: Fundamental Classes Comes From Algebraic Cobordisms

Fix $f : X \to S$ and let $(\mathbb{E}, c)$ be an oriented ring spectra. We have a classifying map

$$\phi : \mathrm{MGL} \to \mathbb{E}, \phi_*(c^{\mathrm{MGL}}) = c.$$

By the GRR theorem, we have that

$$\phi_*(\eta_f^{\mathrm{MGL}}) = \mathrm{td}_\phi(\tau_f) \cdot \eta_f^{\mathbb{E}}.$$

Now since $\phi_*(c^{\mathrm{MGL}}) = c$, there is no transition and hence $\mathrm{td}_\phi(\tau_f) = 1$. Thus, we have that

$$\phi_*(\eta_f^{\mathrm{MGL}}) = \eta_f^{\mathbb{E}}.$$

Application: Beilinson

Let $\Lambda = \mathbb{Q}, \Lambda_\ell = \mathbb{Q}_\ell$ where ℓ is prime, over all schemes. There is a morphism of motivic ring spectra

$$\rho_\ell : H\Lambda \to H_{ét}\Lambda_\ell,$$

where $H\Lambda$ is the Beilinson motivic ring spectrum and $H_{ét}\Lambda_\ell$ is the étale motivic Λ-spectrum. One can check the map $H^{*,*}(-,\Lambda) \to H^{*,*}_{ét}(-,\Lambda)$ induces the identity on the formal group laws (which is both additive). Thus, we have

$$\rho_\ell(\eta_f) = \eta_f^{ét}.$$

Remark: What About K^{MW}?

Given the fact that over a field k, $\pi_0(1_k)_* \cong K_*^{\mathrm{MW}}$, one might hope some of this technology can be applied to Milnor-Witt K-theory.

- However, we note that HK^{MW} is not orientable, so GRR is not applicable.
- There is a work-around developed, called the quadratic Riemann-Roch theorem in [Déglise and Fasel, 2024] that resolves the issue.

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