Lecture 5: Unstable Motivic Homotopy Theory, Continued

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Outline

Vector Bundle Torsor

In the previous lecture, we saw two (building blocks) localization given by L_{Nis} and $L_{\mathbb{A}^1}$. Here, we introduce another using the notion of strong homotopy invariance.

Definition:

An (algebraic) vector bundle torsor over X is an affine morphism $\phi : Y \to X$ that is Zariski locally trivial and the fibers of the maps are isomorphic to affine spaces (ie. $\mathbb{A}^n$).

An (algebraic) vector bundle is a vector bundle torsor. The difference between them is that for a vector bundle torsor, the patching maps need not be linear, just affine.

Vector Bundle Torsor

More generally, a vector bundle torsor $W \to X$ with respect to some vector bundle $E \to X$ is given by an action of E on W and W is locally isomorphic to E with affine patching maps.

In particular, the usage of the word torsor suggests that they are classified exactly by

$$H^1(X; E).$$

In particular, if X is affine, $H^1(X; E) = 0$, so W is the same as E. This means that every vector bundle torsor over affine schemes is a vector bundle.

Vector Bundle Torsor that is Not a Vector Bundle

Example: Consider the diagonal map $\delta : \mathbb{P}^1_k \to \mathbb{P}^1_k \times_k \mathbb{P}^1_k$. This is a closed immersion since $\mathbb{P}^1_k$ is separated, so we can consider an open immersion of the diagonal complement given by

$$X \to \mathbb{P}^1_k \times_k \mathbb{P}^1_k$$

Let $\pi : X \to \mathbb{P}^1_k$ be projection to one of its factors. The fibers are $\mathbb{A}^1_k$ since removing the diagonal removes a point off the projective line, but this is not a vector bundle!

Strong Homotopy Invariance

Recall by Sm_S we mean smooth S-schemes of finite type. A presheaf $F \in \mathrm{PShv}(\mathrm{Sm}/S)$ is strongly homotopy invariant if for all vector bundle torsors $Y \to X$, the corresponding map

$F(X) \to F(Y)$ is an equivalence.

We refer to such presheaves as $\mathrm{PShv}_{htp}(\mathrm{Sm}/S)$. Note that clearly every strongly homotopy invariant sheaf is $\mathbb{A}^1$-invariant.

Now we denote the left adjoint of the inclusion functor $\mathrm{PShv}_{htp}(\mathrm{Sm}/S) \to \mathrm{PShv}(\mathrm{Sm}/S)$ as

$$L_{\mathrm{htp}} : \mathrm{PShv}(\mathrm{Sm}/S) \to \mathrm{PShv}_{htp}(\mathrm{Sm}/S).$$

L_{htp} preserves finite products and is locally Cartesian!

Equivalence of Motivic Spaces

Once we require the presheaves to be Nisnevich sheaves, we actually have that

Theorem

$\mathrm{PShv}_{\mathrm{htp}}(\mathrm{Sm}\,/S) \cap \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}\,/S) =$
$\mathrm{PShv}_{\mathbb{A}^1}(\mathrm{Sm}\,/S) \cap \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}\,/S).$

The intuition is that being $\mathbb{A}^1$-invariant and strongly homotopy invariant becomes a local condition if they are both sheaves, and locally they are equivalent.

The Jouanolou–Thomason Trick

There is a good reason for why we are interested in vector bundle torsors.

Theorem

Let $\pi : Y \to X$ be a vector bundle torsor, then π induces a motivic equivalence.

Idea: Check this equivalence over a trivializing open cover, then this reduces to the previous lecture.

Theorem (Jouanolou-Thomason)

Let S be a qcqs scheme such that either:

1. S is affine, or
2. S is Noetherian separated and regular[1]

Let $X \in \mathrm{Sch}_S$ be quasi-projective, then there exists a vector bundle torsor $Y \to X$ where Y is affine.

[1] Remark: This is also the setting where G-theory and K-theory of S are the same.

Example of The Jouanolou-Thomason Trick

When k is algebraically closed and $X = \mathbb{P}^n_k$, consider a variety Y in $\mathbb{A}^{n+1}_k \times \mathbb{A}^{n+1}_k$ where Y is composed of $(n+1) \times (n+1)$-matrices M such that:

1. M is idempotent, ie. $M^2 = M$.
2. M has rank 1.

Question:

Why is Y affine?

1. Idempotence is a polynomial relations.
2. Having ≤ 1 is equivalent to all higher minors vanishing.
3. Having $\neq 0$ means the zero matrix is not here.

Example of The Jouanolou-Thomason Trick

Question:

Why is Y affine?

1. Idempotence is a polynomial relations.
2. Having $\mathrm{rank} \leq 1$ is equivalent to all higher minors vanishing.
3. X Having $\neq 0$ means the zero matrix is not here. This is an open condition.
4. An idempotent matrix can only have eigenvalues 0 or 1. We use the condition $\det(I - M) = 0$. This implies that 1 is an eigenvalue of M, so M is not the zero matrix.

Example of The Jouanolou-Thomason Trick

Note that the image of a rank 1 idempotent matrix is exactly a line. Now consider the map

$$\phi : Y \to \mathbb{P}^n_k, M \in Y \mapsto \mathrm{im}(M).$$

For each closed point in $\mathbb{P}^n_k$ - the fiber is exactly $\mathbb{A}^n_k$!

The idea is to a basis of k^n with one vector v being the generator of $\mathrm{im}(M)$, then an idempotent matrix has to fix v, but we are free to choose where the other n vectors get sent to in the line, so this gives $\mathbb{A}^n_k$.

The Jouanolou-Thomason Trick

Proof Sketch for the Affine Case ($S = \operatorname{Spec} A$):

- If X is projective, then it has a closed immersion to $\mathbb{P}^n_A$ for $n >> 0$. The previous construction works fine over a ring A with more details, and we pull back the example constructed.
- If X is quasi-projective then it is a quasicompact open subscheme of a projective scheme Z. We can do an operation called "blow up" on the complement of X such that the inclusion $i : X \to \mathrm{Bl}_{Z-X} Z$ is affine and the blow up is projective. Now pull back the example constructed again.

This was, allegedly, how Jouanolou originally prove this.

Thomason's contribution was the second case (Noetherian, separated, regular) of the theorem with a different approach.

Proof Sketch of Second Version: Such X satisfying the hypothesis would admit an ample family of line bundles $\mathcal{L}_0, ..., \mathcal{L}_n$ - that is, the collection induces a morphism

$$s : \mathcal{O}_X \to \mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{L}_i.$$

One can check the cokernel of s is actually locally free, and construct

$$Y = \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathrm{coker}(s))$$

$$Y \hookrightarrow \mathbb{P}(\mathcal{E}) \to X.$$

Before we move on, we note one good criterion for checking motivic equivalences.

Theorem:

Let $f : F \to G$ be a morphism in $\mathrm{PShv}(\mathrm{Sm}/S)$. If $F(U) \to G(U)$ is an equivalence for every affine $U \in \mathrm{Sm}_S$, then f is a motivic equivalence.

Outline

Generalizing EG

Let $\mathcal{C}$ be a category and $G \in \mathcal{C}$ be a group object, we can generalize the definition of EG we know in topology to a simplicial object $E_\bullet G$ given by

$$E_\bullet G : \Delta^{op} \to \mathcal{C}, [n] \mapsto \prod_{i=1}^{n+1} G$$

$$G \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; G \times G \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; G \times G \times G \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \ldots$$

whose face maps $d_i : \prod_{j=1}^{n} G \to \prod_{j=1}^{n-1} G$ are

$$d_0(g_1, ..., g_n) = (g_2, ..., g_n)$$

$$d_i(g_1, ..., g_i, g_{i+1}, ..., g_n) = (g_1, ..., g_i g_{i+1}, ..., g_n), i > 0$$

Degeneracy maps are insertions of identity.

EG in the Motivic Setting

In the previous slide we wrote elements g_i's for notational ease, but the same exact construction works for any group object in any category.

Now, let $X \in \operatorname{Sm}/S$ and $\mathcal{G}$ be a τ-sheaf of groups, we define a presheaf $E_\bullet \mathcal{G}$ such that for each cover $U \in \operatorname{Sm}/S$,

$$E_\bullet \mathcal{G}(U) := E_\bullet(\mathcal{G}(U))$$

(ie. it produces a simplicial group for each U). Now $E_\bullet \mathcal{G}$ is in $\operatorname{PShv}(\operatorname{Sm}/S)$!

We define $E_\tau \mathcal{G}$ to be the τ-localization of $E_\bullet \mathcal{G}$.

EG in the Motivic Setting

Similarly to how in topology, EG is contractible.

Prop:

$E_\tau \mathcal{G}$ is contractible.

Proof: There is an extra degeneracy at each step in the diagram for $G = \mathcal{G}(U)$:

$$G \;\begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix}\; G \times G \;\begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix}\; G \times G \times G \;\begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix}\; \ldots$$

Notice at each step, there is an extra coordinate for G to insert its identity! This implies $|G|$ is contractible by standard simplicial homotopy theory, and it induces a weak equivalence between $\mathcal{E}_\bullet \mathcal{G}$ and $*$. The functor L_τ preserves weak equivalences.

The Bar Construction

Let $\mathcal{C}$ be a category and $G \in \mathcal{C}$ be a group object. Let G act on $E_\bullet G$ on the right in the last coordinate, the quotient $B_\bullet G$ is called the bar construction of G. More explicitly, we have

$$* \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; G \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \; G \times G \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \ldots$$

whose face maps $d_i : \prod_{j=1}^{n} G \to \prod_{j=1}^{n-1} G$ are

$$d_0(g_1, ..., g_n) = (g_2, ..., g_n) \text{ and } d_n(g_1, ..., g_n) = (g_1, ..., g_{n-1})$$

$$d_i(g_1, ..., g_i, g_{i+1}, ..., g_n) = (g_1, ..., g_i g_{i+1}, ..., g_n).$$

Degeneracy maps are insertions of identity.

Now, let $X \in \operatorname{Sm}/S$ and $\mathcal{G}$ be a τ-sheaf of groups. Define $B_\bullet \mathcal{G}$ by

$$U \mapsto B_\bullet(\mathcal{G}(U))$$

and then localize to define the bar construction as

$$B_\tau \mathcal{G} := L_\tau B_\bullet \mathcal{G}.$$

When $\tau = \mathrm{Nis}$, we call this the classifying space and drop τ.

Classifying Spaces in the Motivic Setting

Similar to how BG classifies principal G-bundles (ie. torsors) in topology, we have a similar result in the motivic setting.

A $\tau - \mathcal{G}$-torsor on X is a τ-sheaf P of sets equipped with compatible maps

$$\alpha : P \times \mathcal{G} \to P, \pi : P \to X$$

where π is $\mathcal{G}$-equivariant with trivial action on X.

The $\mathcal{G}$-torsors over $U \in \mathrm{Sm}/S$ form a groupoid, we define $B\,\mathrm{Tors}_\tau(\mathcal{G})(U)$ to be the nerve of this groupoid. Note that this is already τ-local!

Theorem

There is a natural isomorphism[2]

$$[X, B_\tau \mathcal{G}]_{Shv_\tau} \simeq H^1_\tau(X, \mathcal{G}) \simeq \pi_0(B\,\mathrm{Tors}_\tau(\mathcal{G})(X)).$$

[2] If not abelian, take the right side as the definition of $H^1_\tau(\bullet, \mathcal{G})$.

Let GL_n be the general linear group scheme and SL_n be the special linear group scheme. We define

$$GL = \mathrm{colim}_n \, GL_n \text{ and } SL = \mathrm{colim}_n \, SL_n.$$[3]

and BGL, BSL as their associated classifying spaces.

Prop:

Let $\mathcal{G}$ be a Nisnevich sheaf of groups, then $\pi_0^{\mathbb{A}^1}(B\mathcal{G}) = 0$.

Proof: From last lecture, we know we only need to compute $\pi_0^{\mathrm{Nis}}(B\mathcal{G})$. Now $\mathcal{G}$-torsors are Nisnevich locally trivial, and the last theorem tells us $\pi_0^{\mathbb{A}^1}$ is the sheafification of $U \mapsto H^1(U; \mathcal{G})$.

[3] Colimit considered as presheaves

Lifting to Higher Spaces

So far, we have representations on the level of H^1.

Question:

What about $H^2, H^3, ...$?

Theorem (Dold-Kan Correspondence)

Over perfect field k, there is an equivalence of categories between

$$\mathrm{Ch}_{\geq 0}(\mathrm{Ab}_{\mathrm{Nis}}(k)) \cong \mathrm{Fun}(\Delta^{op}, \mathrm{Ab}_{Nis}(k))$$

The chain complex is actually given by alternating sums of certain face maps!

$$\mathrm{Ch}_{\geq 0}(\mathrm{Ab}_{\mathrm{Nis}}(k)) \cong \mathrm{Fun}(\Delta^{op}, \mathrm{Ab}_{Nis}(k))$$

Given a chain complex of abelian sheaves,

1. Send to its corresponding simplicial object on the right.
2. A simplicial object in abelian sheaves is a simplicial abelian group by forgetting the sheaf structure.
3. Every simplicial abelian group is a Kan complex.

This defines a map

$$\mathrm{DK} : \mathrm{Ch}_{\geq 0}(\mathrm{Ab}_{\mathrm{Nis}}(k)) \to \mathrm{PShv}(\mathrm{Sm}_k).$$

Fact:[4] Let $\mathcal{A} \in \mathrm{Ch}_{\geq 0}(\mathrm{Ab}_{\mathrm{Nis}}(k))$, then

$$H_n(\mathcal{A}) \cong \pi_n(L_{Nis}\,\mathrm{DK}(\mathcal{A})).$$

[4] This is really a formal consequence of examining the DK correspondence.

Let $A \in \mathrm{Ab}_{\mathrm{Nis}}(k)$. View $A[n]$ as the chain complex with only one A sitting at degree n. The motivic Eilenberg-Maclane Space for A is

$$K(A, n) := DK(A[n]).$$

Note that $K(A, n)$ is a Nisnevich sheaf.

Corollary: $K(A, n) \in \mathrm{Shv}_{\mathrm{Nis}}(Sm_k)$ satisfies

$$\pi_i(K(A, n)) \cong \begin{cases} A, i = n \\ 0, i \neq n \end{cases}.$$

Representability

Recall in topology, we have that

$$[X, K(G,n)]_* \cong H^n(X;G).$$

We have an analogous result in the motivic world.

Theorem (Motivic Eilenberg Representability)

$H^n_{\mathrm{Nis}}(-;A) \cong \pi_0(\mathrm{Map}_{\mathrm{Shv}_{Nis}}(-, K(A,n)).$

Fiber and Cofiber Sequences

For purposes that will be clear later, we fix k as a perfect field. Let $X \to Y \to Z$ in $\mathrm{Spc}(k)_*$, we say

1. This is a fiber sequence if the following diagram is a pullback:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & Z \end{array}$$

2. This is a cofiber sequence if the following diagram is a pushout:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & Z \end{array}$$

Note: What is discussed in this topic applies to the generality of $L_{\mathbb{A}^1} L_{Nis} \mathrm{PShv}(Sm/k)$ (See [Antieau and Elmanto, 2016]).

Let $F \to X \to Y$ be a fiber sequence in $\mathrm{Spc}(k)_*$, then we have a long exact sequence of homotopy groups:

$$... \to \pi_{n+1} F \to \pi_{n+1} X \to \pi_{n+1} Y \to \pi_n F \to ...$$

Cohomology and Co-Fiber Sequences

We say a Nisnevich sheaf of groups $\mathcal{G}$ is strongly invariant[5] if

$$H^i_{Nis}(X;\mathcal{G}) \to H^i_{Nis}(X \times \mathbb{A}^1;\mathcal{G})$$

is an isomorphism for $i = 0, 1$.

Theorem

$\mathcal{G}$ is strongly invariant if and only if $B_{Nis}\mathcal{G}$ is $\mathbb{A}^1$-invariant.

Let $X \to Y \to C$ be a cofiber sequence in $\mathrm{Spc}(k)_*$ and $\mathcal{G}$ be strongly invariant, then we have a long exact sequence of cohomology:

$$0 \to H^0_{Nis}(C;\mathcal{G}) \to H^0_{Nis}(Y;\mathcal{G}) \to H^0_{Nis}(X;\mathcal{G}) \to H^1_{Nis}(C;\mathcal{G}) \to \ldots$$

that terminates at level 1.

[5] ex. $\mathbb{G}_m$

Cohomology and Co-Fiber Sequence

Let $\mathcal{A}$ be a Nisnevich abelian sheaf, we say $\mathcal{A}$ is strictly invariant if

$$H^i_{Nis}(X;\mathcal{A}) \to H^i_{Nis}(X \times \mathbb{A}^1;\mathcal{A})$$

is an isomorphism for all i.

Theorem

$\mathcal{A}$ is strictly invariant if and only if $K(\mathcal{A},n)$ is $\mathbb{A}^1$-invariant for all $n \geq 0$.

Let $X \to Y \to C$ be a cofiber sequence in $\mathrm{Spc}(k)_*$ and $\mathcal{A}$ be strictly invariant, then we have a long exact sequence of cohomology:

$$0 \to H^0_{Nis}(C;\mathcal{G}) \to H^0_{Nis}(Y;\mathcal{G}) \to H^0_{Nis}(X;\mathcal{G}) \to H^1_{Nis}(C;\mathcal{G}) \to ...$$

that continues for all i.

Strong and Strict Invariance

Question:

Is $\pi_0^{\mathbb{A}^1}(X)$ always $\mathbb{A}^1$-invariant?

There is a counter-example! Apparently due to Ayoub in 2023.

Fortunately, we do have the following results.

Theorem (Morel)

Let X be a motivic space over a field, then $\pi_1^{\mathbb{A}^1}(X)$ is strongly $\mathbb{A}^1$-invariant.

Theorem (Morel)

If k is a perfect field and $\mathcal{A}$ is a sheaf of abelian groups, then $\mathcal{A}$ is strongly invariant if and only if it is strictly invariant.

Strong and Strict Invariance

Corollary:

Let k be a perfect field and $X \in \mathrm{Spc}(k)_*$, then $\pi_n^{\mathbb{A}^1}(X)$ is strictly invariant for $n \geq 2$ and strongly invariant for $n = 1$.

Proof: Consider the loop space functor Ω and observe

$$\pi_n(X) = \pi_1(\Omega^{n-1} X).$$

At least in this lecture, we will assume k is perfect from now on. We use $\mathrm{HI}(k) \subset \mathrm{Ab}_{\mathrm{Shv}}(k)$ to denote the full subcategory of strongly ($=$ strictly) invariant sheaves.

Theorem (Morel)

$\mathrm{HI}(k)$ is an abelian category with exact inclusions.

Fiber Sequence for Classifying Spaces

Theorem (Asok, Hoyois, Wendt, 2015)

Let $X \to Y \to Z$ be a fiber sequence of $\mathrm{PShv}(\mathrm{Sm}/S)$ such that Z satisfies affine Nisnevich excision and $\pi_0(Z)$ has affine $\mathbb{A}^1$-invariance, then $X \to Y \to Z$ is admits a fiber sequence that gives the aforrementioned LES of homotopy groups.

Corollary: Let $\mathcal{G}$ be a sheaf of groups, the $G \to EG \to BG$ is a fiber sequence that gives the LES of homotopy groups if $H^1_{\mathrm{Nis}}(\bullet, \mathcal{G})$ is $\mathbb{A}^1$-invariant.

Outline

1. Strong Homotopy Invariance and Vector Bundle Torsors
2. Classifying Spaces and Motivic Eilenberg-MacLane Spaces
3. Suspension Theorems
4. Some Computations

(Motivic) S^1-Freudenthal Suspension Theorem

A (pointed) motivic space X is $\mathbb{A}^1$-n-connected if $\pi_i^{\mathbb{A}^1}(X) = 0$ for $i \leq n$.

Theorem (S^1-Freudenthal Suspension Theorem, Morel, Asok-Bachmann-Hopkins)

Let X be $\mathbb{A}^1$-n-connected, then the natural map

$$\pi_i^{\mathbb{A}^1}(X) \to \pi_{i+1}^{\mathbb{A}^1}(\Sigma X)$$

is an isomorphism for $i \leq 2n$ and an epimorphism for $i = 2n+1$.

This theorem follows directly by applying the LES of homotopy groups to the following theorem.

Theorem (Morel, Asok-Bachmann-Hopkins)

Let $n \geq 0$ and X be $\mathbb{A}^1$-n-connected, for $i \geq 1$, the natural map

$$X \to \Omega^i \Sigma^i X$$

has $\mathbb{A}^1$-$2n$-connected fibers.

Proof Sketch: Let F be the fiber of the map $\phi : X \to \Omega^i \Sigma^i X$ taken in the level of sheaves, this fiber always exists on the level of sheaves - it is just not clear whether it would exist in the level of pointed motivic spaces.

The S^1-Freudenthal Suspension Theorem

Theorem (Morel, Asok-Bachmann-Hopkins)

Let $n \geq 0$ and X be $\mathbb{A}^1$-n-connected, for $i \geq 1$, the natural map

$$X \to \Omega^i \Sigma^i X$$

has $\mathbb{A}^1$-$2n$-connected fibers.

Proof Sketch (cont'd): The motivic localization L_{mot} sends the fiber sequence $F \to X \to \Omega^i \Sigma^i X$ to a fiber sequence

$$L_{mot} F \to L_{mot} X = X \to L_{mot}(\Omega^i \Sigma^i X) = \Omega^i \Sigma^i X.$$

This in particular implies $L_{mot} F = F$, so F is motivic. F is $2n$-connected before localization by classical theorems.

By Morel's unstable connectivity theorem, L_{mot} actually preserves connectivity in the $\mathbb{A}^1$-sense too!

Remark: The S^1-Freudenthal Suspension theorem can be used to show that:

Theorem

For $n \geq 2$ and $i \geq 1$, $\pi_n(S^{n+i,i}) \cong K_i^{\mathrm{MW}}$ where K_i^{MW} denotes the Milnor-Witt K-theory.

In an ideal world, we would have loved to cover more about this and the interesting relationships between K-theory, Chow groups, Milnor-Witt K-theory, with the motivic spheres. This may be covered in a future lecture (Lecture 10 tentatively).

Weakly A-Cellular Spaces

Question:

But isn't there another family of spheres in algebraic geometry with $\mathbb{G}_m$?

A suitable version of this extension was given by Asok, Bachmann, Hopkins in 2023, formulated using a notion of weakly A-cellular spaces with respect to some (compact, pointed) motivic space A.

Definition

Let A be a (pointed, compact) motivic space and consider the set $\{A \times U \to U\}$ for $U \in \mathrm{Sm}_k$. The left Bousfield localization of $\mathrm{Spc}(k)$ w.r.t to this set yields an endofunctor

$$L_A : \mathrm{Spc}(k) \to \mathrm{Spc}(k)$$

Let X be a motivic space, X is weakly A-cellular if $L_A X \simeq *$.

The $\mathbb{P}^1$-Freudenthal Suspension Theorem

Let X be a motivic space, X is weakly A-cellular if $L_A X \simeq *$. We use $O(A)$ to denote the class of weakly A-cellular spaces.

Theorem

Let $n \geq 2$ and $X \in O(S^{2n,n})$ and k be a characteristic zero, the fiber of

$$X \to \Omega^{2,1}\Sigma^{2,1}X$$

is in $O(S^{4n-1,2n})$.

Here the index $(2,1)$ indicates the suspension and loop are w.r.t to $\mathbb{P}^1 = S^{2,1}$.

Outline

Homotopy Groups of $B\mathbb{G}_m$

For simplicity we work over a perfect field k,

Theorem

We have that

$$\pi_i(B\mathbb{G}_m) = \begin{cases} *, i = 0 \\ \mathbb{G}_m, i = 1 \\ 0, i > 1 \end{cases} .$$

- Note we have shown that $i = 0$ holds in more generality.
- $i = 1$ follows from how classifying spaces are constructed.
- We really need to check $i > 1$!

$\mathbb{A}^1$-Rigidity

Observe that $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid in the sense that $h_{\mathbb{G}_m}$ is $\mathbb{A}^1$-invariant already.

Theorem

Let X be $\mathbb{A}^1$-rigid k-scheme, then $\pi_i^{\mathbb{A}^1}(X) = 0$ for $i > 0$ and $\pi_0^{\mathbb{A}^1}(X) = X$.

Now consider the sequence

$$B\mathbb{G}_m \to E\mathbb{G}_m \to \mathbb{G}_m.$$

This is a fiber sequence since $H^1(\bullet, \mathbb{G}_m)$ is $\mathbb{A}^1$-invariant (ie. Hilbert 90).

The LES of homotopy groups now shows that $\pi_i(B\mathbb{G}_m) = 0$ for $i > 1$.

Comparing BGL and BSL

Observe we have the following exact sequence

$$SL \to GL \xrightarrow{\det} \mathbb{G}_m$$

Applying the classifying space construction yields a fiber sequence

$$BSL \to BGL \to B\mathbb{G}_m$$

which gives a long exact sequence of homotopy groups.

Since $B\mathbb{G}_m$ has no non-trivial homotopy groups above 1, we conclude that

Theorem:

The natural map $SL \to GL$ induces an isomorphism $\pi_i^{\mathbb{A}^1}(BSL) \to \pi_i^{\mathbb{A}^1}(BGL)$ for $i > 1$.

The theorem works just as well for $SL_r \to GL_r \to \mathbb{G}_m$!

Range of Stability Result

Consider the fiber sequence

$$SL_{n+1}/SL_n \to BSL_n \to BSL_{n+1}.$$

By the unstable connectivity theorem of Morel, SL_{n+1}/SL_n is actually $\mathbb{A}^1$-$(n-1)$-connected! Thus, we have that:

Theorem

For $i > 0$ and $n \geq 1$, the natural map

$$\pi_i^{\mathbb{A}^1}(\mathrm{BSL}_n) \to \pi_i^{\mathbb{A}^1}(\mathrm{BSL}_{n+1})$$

is an epimorphism for $i \leq n$ and an isomorphism for $i \leq n-1$.

This can also be done similarly with GL instead of SL.

Algebraic K-theory Space

Recall for a ring R, Quillen's Plus Construction defines that the homotopy groups of

$$\mathbb{Z} \times \mathrm{BGL}(R)^+$$

is equal to the K-groups of K.

The notations we have set-up in the motivic setting look similar enough that one might wonder - can this be done in the motivic world?

Theorem

There is a motivic equivalence between $\mathbb{Z} \times \mathrm{BGL}$ and algebraic K-theory K.

Corollary: For $1 < i \leq n-1$ and $n \geq 1$, we have that

$$\pi_i^{\mathbb{A}^1} \mathrm{BSL}_n \cong \pi_i^{\mathbb{A}^1} \mathrm{BGL}_n \cong K_i.$$

References I

Antieau, B. and Elmanto, E. (2016).
A primer for unstable motivic homotopy theory.