Lecture 7: Introduction to Stable Motivic Homotopy Theory

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Modern Techniques in Homotopy Theory Learning Seminar

July 9th, 2025

Outline

Reframing Motivic Homotopy Theory to Model Categories

Today, we are starting our discussions on stable motivic homotopy theory!

There is a model category treatment of the stable motivic homotopy theory, which we will start with.

- Here, we mostly follow the outline given in [Voevodsky et al., 2007] and [Hlavinka, 2021].
- A more detailed model category treatment of the unstable case is outlined in [Antieau and Elmanto, 2016].

The idea in "stable motivic homotopy theory" is to "stabilize":

$$\text{motivic spaces} \rightsquigarrow \text{motivic spectra}.$$

This perspective is useful if we, for example, want to represent motivic cohomology theories.

Two Ways to Suspend a Spectrum

Question:

What should a motivic spectra look like?

In a first class on stable homotopy theory, you might see a (sequential) spectrum E being formulated as a sequence of pointed spaces

$$E_0, E_1, E_2, \ldots$$

equipped with structure maps

$$\sigma_n : \Sigma E_n \simeq S^1 \wedge E_n \to E_{n+1}.$$

But in motivic homotopy theory, we have two family of spheres:

1. S^1 being viewed as a constant simplicial object.
2. $\mathbb{G}_m$ being viewed as a motivic space.

For our purposes, a motivic spectrum should compose of a grid of motivic spaces with two ways to suspend, satisfying some properties. In order to get there, we must answer two questions:

1. Since we are out of ∞-category land for a bit, what is a motivic space now?
2. How do we suspend things? It seems like we need a smash product of sorts for spaces.

A third question we will try to partially answer (but not definitively) is:

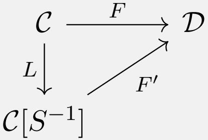
3. How does this relate to our ∞-category language?

Localization of a Category

Definition

Let $\mathcal{C}$ be a 1-category and S a class of morphisms in M. The localization of $\mathcal{C}$ by S is a 1-category $\mathcal{C}[S^{-1}]$ with a functor $L : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that:

1. For every $f \in S$, $L(f)$ is an isomorphism.
2. Pre-composition by L is a fully faithful functor $\mathrm{Fun}(\mathcal{C}[S^{-1}], \bullet) \to \mathrm{Fun}(\mathcal{C}, \bullet)$.
3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor that sends $f \in S$ to isomorphisms, then F factors uniquely through



where the diagram commutes up to natural isomorphism.

Homotopy Category

For a model category, we can build an associated homotopy category.

Theorem

Let $\mathcal{C}$ be a model category with weak equivalences W, then $\mathcal{C}[W^{-1}]$ exists.

$\mathcal{C}[W^{-1}]$ is called the homotopy category of $\mathcal{C}$.

Ex: For the standard model structure on Top_*, the associated hhomotopy category is equivalent to the category of CW complexes with morphisms bneing homotopy classes of maps.

Motivic Spaces Concretely

Definition

In this section, by a motivic space, we mean a Nisnevich sheaf with values in simplicial sets.[1]

A pointed motivic space (X, x_0) is an object in the undercategory $\mathrm{Spc}(k)_{\mathrm{Spec}(k)/}$. For a motivic space X, there is a formal procedure to admit a base point $X \to X_+$ where $X_+ = X \sqcup \mathrm{Spec}(k)$.

For $(X, x_0), (Y, y_0) \in \mathrm{Spc}_*(k)$. There is a symmetric monoidal structure $\bigwedge$ (ie. smash product) on $\mathrm{Spc}_*(k)$ given by the sheaf associated to the presheaf:

$$U \mapsto ((X, x_0) \times (Y, y_0))(U)/((X, x_0) \vee (Y, y_0))(U).$$

[1] I believe, after considering an $\mathbb{A}^1$-model structure on this, the underlying homotopy category will correspond to that of what we usually call $\mathrm{Spc}(k)$.

The Simplicial Model Structure

There is a model structure on $\mathrm{Spc}(k)$ (called the **simplicial model structure**) where:

1. Weak equivalences are simplicial weak equivalences, that is a map $f : X \to Y$ such that for any choice of base-points x_0, y_0 with $f \circ x_0 = y_0$, the map of sheaves

$$\pi_n((X, x_0)) \to \pi_n((Y, y_0))$$

is an isomorphism.

2. Cofibrations are monomorphisms (termwise monomorphisms).
3. Fibration is determined by the first two.

The simplicial homotopy category of $\mathrm{Spc}(k)$ is called $H_s\,\mathrm{Spc}(k)$. Note that **we have not said anything about $\mathbb{A}^1$-invariance yet**, this is just carrying over the usual model structure on sSet to this context.

A motivic space Z is $\mathbb{A}^1$-local if the natural map

$$\mathrm{Hom}_{H_s \mathrm{Spc}(k)}(Y, Z) \to \mathrm{Hom}_{H_s \mathrm{Spc}(k)}(Y \times \mathbb{A}^1, Z)$$

induced by projection, is a bijection for any $Y \in \mathrm{Sm}/k$.

There is a model structure on $\mathrm{Spc}(k)$ given by:

1. **Weak Equivalences** are $\mathbb{A}^1$-weak equivalences, that is, a map $f : X \to Y$ such that for any $\mathbb{A}^1$-local Z, the natural map is a bijection:
$$\mathrm{Hom}_{H_s \mathrm{Spc}(k)}(Y, Z) \to \mathrm{Hom}_{H_s \mathrm{Spc}(k)}(X, Z).$$
2. Cofibrations are monomorphisms.

The associated homotopy category $H(k)$ is called the $\mathbb{A}^1$-homotopy category.

Bi-Spectrum

A (motivic) (s,t)-bi-spectrum is composed of the data:

- $E_{n,m}$ pointed motivic spaces for $n, m \geq 0$.
- Structure Maps given by suspensions from S^1 and $\mathbb{G}_m$:

$$\sigma_s : S^1 \wedge E_{n,m} \to E_{n+1,m},$$

$$\sigma_t : \mathbb{G}_m \wedge E_{n,m+1} \to E_{n,m+1}.$$

such that the following diagram commutes:

$$\begin{array}{ccccc} S^1 \wedge \mathbb{G}_m \wedge E_{n,m} & & \xrightarrow{\tau \wedge E_{n,m}} & & \mathbb{G}_m \wedge S^1 \wedge E_{n,m} \\ \downarrow{\scriptstyle S^1 \wedge \sigma_t} & & & & \downarrow{\scriptstyle \mathbb{G}_m \wedge \sigma_s} \\ S^1 \wedge E_{n,m+1} & \xrightarrow{\sigma_s} & E_{n+1,m+1} & \xleftarrow{\sigma_t} & \mathbb{G}_m \wedge E_{n+1,m} \end{array}$$

where $\tau : S^1 \wedge \mathbb{G}_m \to \mathbb{G}_m \wedge S^1$ is the isomorphism given by the symmetry of the smash product.

The Category of Bi-Spectra

Let E, E' be two (s,t)-bispectra, a morphism $f : E \to E'$ is the data of maps

$$f_{n,m} : E_{n,m} \to E'_{n,m}, n \geq 0, m \geq 0$$

such that they commute with the structure maps, ie.

$$\begin{array}{ccc} S^1 \wedge E_{n,m} & \xrightarrow{S^1 \wedge f_{n,m}} & S^1 \wedge E'_{n,m} \\ \sigma_s \downarrow & & \downarrow \sigma'_s \\ E_{n+1,m} & \xrightarrow[f_{n+1,m}]{} & E'_{n+1,m} \end{array} \qquad \begin{array}{ccc} \mathbb{G}_m \wedge E_{n,m} & \xrightarrow{\mathbb{G}_m \wedge f_{n,m}} & \mathbb{G}_m \wedge E'_{n,m} \\ \sigma_t \downarrow & & \downarrow \sigma'_t \\ E_{n,m+1} & \xrightarrow[f_{n,m+1}]{} & E'_{n,m+1} \end{array}$$

The category of (s,t)-bispectra is denoted $\mathrm{Spt}_{s,t}(k)$.

Constructions on Bi-Spectra

Let $\mathrm{Spt}_{s,t}(k)$ be the category of (s,t)-bispectra. For $X, Y \in \mathrm{Spt}_{s,t}(k)$.

1. The categorical coproduct $X \vee Y$ is the component wise wedge product.
2. For any pointed motivic space Z, there is a suspension (bi)-spectrum given by $\Sigma^\infty Z_{m,n} := S^{m,n} \wedge Z$, where $S^{m,n}$ is (m,n)-motivic-sphere.
3. For any bi-spectra X, there is a sequence of "s-spectra" given by
$$E_i := E_{0,i}, E_{1,i}, E_{2,i}, \ldots$$
with structure maps $\sigma_s : S^1 \wedge E_{j,i} \to E_{j+1,i}$.
4. We denote the category of "s-spectra" as $\mathrm{Spt}_s(k)$.
5. For any motivic space Z, there is a similar s-spectra given by $\Sigma_s^\infty Z$ whose i-th component of $(S^1)^i \wedge Z$.

The s-simplicial model structure for s-Spectra

The category $\mathrm{Spt}_s(k)$ is supposed to mimick the notion of spectra we are more familar with previously. Similar to how the homotopy groups of spectra are defined, for $E \in \mathrm{Spt}_s(k)$, we define π_n as the sheaf associated to the presheaf of abelian groups, given by

$$\pi_n^{\mathrm{pre}}(E) \coloneqq \mathrm{colim}_{k>0}\, \pi_{n+k}(E_k).$$

Similar to how we defined the simplicial model structure for $\mathrm{Spc}(k)$, we can define a s-simplicial model structure on $\mathrm{Spt}_s(k)$:

1. **Weak equivalences** are maps inducing an equivalence on π_n for all n.
2. **Cofibrations** are morphisms such that the component-wise morphisms are $\mathbb{A}^1$-homotopical cofibration of pointed motivic spaces.

The homotopy category is denoted $\mathrm{SH}_s(k)$.

Similar to $\mathbb{A}^1$-model structure in the unstable case, we can define one for $\mathrm{Spt}_s(k)$.

- $E \in \mathrm{Spt}_s(k)$ is $\mathbb{A}^1$-local if for all $U \in \mathrm{Sm/k}$ and $n \geq 0$, there is a bijection induced by the natural projection:

$$\mathrm{Hom}_{\mathrm{SH}_s(k)}(\Sigma_s^\infty(U)_+, \Sigma_s^n E) \to \mathrm{Hom}_{\mathrm{SH}_s(k)}(\Sigma_s^\infty(U \times \mathbb{A}^1)_+, \Sigma_s^n E)$$

- An $\mathbb{A}^1$-weak equivalence is a map $f : E \to E'$ of s-spectrum such that for all $\mathbb{A}^1$-local Z and all n, the induced maps

$$\mathrm{Hom}_{\mathrm{SH}_s(k)}(E', Z) \to \mathrm{Hom}_{\mathrm{SH}_s(k)}(E, Z)$$

  is a bijection.

The s-stable $\mathbb{A}^1$-model structure on $\mathrm{Spt}_s(k)$ has weak equivalences being $\mathbb{A}^1$-weak equivalences and cofibrations being pointwise cofibrations. The associated homotopy category is denoted $\mathrm{SH}_s^{\mathbb{A}^1}(k)$.

The $\mathbb{A}^1$-model Structure for bi-Spectra

For any bi-spectra X, there is bigraded homotopy group sheaf $\pi_{p,q}$ that is the sheaf associated to the presheaf:

$$U \to \operatorname{colim}_m \operatorname{Hom}_{\mathrm{SH}_s^{\mathbb{A}^1}(k)}(\Sigma_s^\infty(S_s^{p-q} \wedge S_t^{q+m} \wedge U_+), E_m).$$

We define the $\mathbb{A}^1$-model structure on $\mathrm{Spt}_{s,t}(k)$ as:

1. A weak equivalence is a map $f : E \to E'$ inducing an isomorphism on all $\pi_{p,q}$.
2. A cofibration is a componentwise cofibration of pointed motivic spaces.

The associated homotopy category is called $\mathrm{SH}(k)$, the $\mathbb{A}^1$-stable homotopy category.

Theorem

For $X \in Sm/k$, $X \times \mathbb{A}^1 \to X$ induces an equivalence $\Sigma^\infty(X \times \mathbb{A}^1)$ and $\Sigma^\infty(X)$ in $\mathrm{SH}(k)$. There is a good smash product structure for bi-spectra that becomes symmetric monoidal in $\mathrm{SH}(k)$.

Outline

We will now give an ∞-categoric perspective of what we discussed in the previous section[2].

- Here we mainly follow the discussions in [Bachmann, 2021].
- Aided by much of the very helpful resources in [Lurie, 2017] and [Rischel, 2018].
- The presenter listened to a relevant lecture by Julie Bannwart at the European Talbot workshop earlier, which are helpful for some parts of this section.

[2]**Warning:** I am not sure what exactly the connection between this section's definition of $\mathrm{SH}(k)$ is with the previous one. My understanding is that the homotopy category of what will be defined in the this section is what we have defined in the last section.

Let $\mathcal{C}$ be a category with finite products, a ∞-commutative monoid is a functor $\underline{M} : N(\mathrm{Fin}_*) \to \mathcal{C}$[3] such that for any map $\rho_i : \langle m \rangle = \{*, 1, ..., m\} \to \langle 1 \rangle = \{*, 1\}$ sending everything to 1 except for $*$, the induced map

$$\underline{M}(\langle m \rangle) \to \prod_{i=1}^{m} \underline{M}(\langle 1 \rangle)$$

is an equivalence.

- The underlying object is $\underline{M}(\langle 1 \rangle) \in \mathcal{C}$, denoted M.
- The multiplication structure is given by

  $$\underline{M}(\langle 1 \rangle) \times \underline{M}(\langle 1 \rangle) \simeq \underline{M}(\langle 2 \rangle) \xrightarrow{f_*} \underline{M}(\langle 1 \rangle)$$

  where the first map is induced by the definition and the second map is induced by $f : \{*, 1, 2\} \to \{*, 1\}$ with $f(*) = *$ and $f(1) = f(2) = 1$.
- The monoidal unit is given by $\underline{M}(\langle 0 \rangle)$.

[3] Here Fin_* is the category pointed finite sets with morphisms being base-point preserving set functions.

Symmetric Monoidal ∞-Categories

The full subcategory of $\mathrm{Fun}(N(\mathrm{Fin}_*), \mathcal{C})$ spanned by ∞-commutative monoids is denoted $\mathrm{CMon}(\mathcal{C})$.

Example: A monoid $\underline{M} \in \mathrm{CMon}(\mathrm{Spc})$ is **group-like** if $\pi_0 M$ is a group. By the recognition principle, group-like commutative monoids over $\underline{M}$ correspond exactly to connective spectra.

Definition

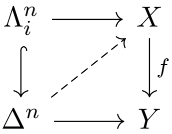
A (small) symmetric monoidal ∞-category is an object of $\mathrm{CMon}(\mathrm{Cat}_\infty)$ (ie. $\underline{\mathcal{C}} : N(\mathrm{Fin}_*) \to \mathrm{Cat}_\infty$).

Reformulation with co-Cartesian Fibration

Our definition of symmetric monoidal ∞-categories matches with our intuition, but it can be very hard to work with!

An alternative formulation is possible with co-Cartesian fibrations. Let $f : X \to Y$ be a morphism of simplicial sets, f is a coCartesian fibration if:

1. f is an inner fibration, that is it has the right lifting property w.r.t to all inclusions of inner horns:



2. For every edge $\underline{e} : y \to y'$ in Y and every vertex $x \in X$ such that $f(x) = y$, there exists a f-coCartesian edge $e : x \to x'$ of X such that $q(e) = \overline{e}$.

f-co-Cartesian Edge

Note that given $f : X \to Y$, we say $e : x \to x'$ in X is an f-coCartesian edge if for any map

$$\Delta^1 \simeq N_\bullet(\{0 < 1\}) \hookrightarrow \Lambda^n_0 \xrightarrow{\sigma_0} X$$

that corresponds to the edge e, the following diagram has a lift

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{\sigma_0} & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta^n & \xrightarrow[\overline{\sigma}]{} & Y \end{array} .$$

Lurie's insight in this is the following theorem.

- Let $\mathcal{K}$ be an ∞-category.
- Let $(\mathrm{Cat}_\infty)^{\mathrm{coCart}}_{/\mathcal{K}}$ be the full subcategory of the slice category $(\mathrm{Cat}_\infty)_{/\mathcal{K}}$ spanned by morphisms $\mathcal{C} \to \mathcal{K}$ that are co-Cartesian fibrations.

Theorem (Lurie)

There is an equivalence of ∞-categories between $\mathrm{Fun}(\mathcal{K}, \mathrm{Cat}_\infty)$ and $(\mathrm{Cat}_\infty)^{\mathrm{coCart}}_{/\mathcal{K}}$, called the straightening equivalence.

Thus, a symmetric monoidal category is equivalently some coCarteisna fibration $f : \mathcal{C}^\otimes \to N(\mathrm{Fin}_*)$ for some ∞-category $\mathcal{C}^\otimes$, satisfying certain conditions.

Conditions to Satisfy

If we unwind the condition, it is exactly specifying the following:

Definition (See 2.0.0.7 of [Lurie, 2017])

A coCartesian fibration $f : \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ is a symmetric monoidal ∞-category if for each map $f : \langle n \rangle \to \langle 1 \rangle$ in Fin_* that sends everything except for $*$ to 1, there is an equivalence

$$\mathcal{C}^{\otimes}_{\langle m \rangle} \cong \prod_{i=1}^{m} \mathcal{C}^{\otimes}_{\langle 1 \rangle},$$

where $\mathcal{C}^{\otimes}_{\langle i \rangle}$ denotes the fiber over $\langle i \rangle$.

Commutative Algebra over Symmetric Monoidal ∞-Categories

A morphism $f : (X, *) \to (Y, *)$ in Fin_* is inert if $f^{-1}(y)$ is a singleton for all $y \neq * \in Y$.

Definition

Let $f : \mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$ be a symmetric monoidal ∞-category, a commutative algebra object over $\mathcal{C}$ is a section $N(\mathrm{Fin}_*) \to \mathcal{C}^{\otimes}$ that sends inert morphisms to f-co-Cartesian edges.

Theorem (Section 2.4.1 of [Lurie, 2017])

Let $\mathcal{C}$ be an ∞-category with finite products, then $\mathcal{C}$ admits an essentially unique symmetric monoidal structure $\mathcal{C}^{\times}$ with $\times$ being the tensor product with a few other axioms.

Ex: Let $\mathcal{C}$ be an ∞-category with finite products, then $\mathrm{CMon}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C}^{\times})$.

Presentable ∞-Categories

Let κ be a regular cardinal, and $\mathrm{Ind}_\kappa(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Spc})$[4]. An ∞-category $\mathcal{C}$ is κ-accessible if there exists a small ∞-category $\mathcal{C}'$ such that $\mathrm{Ind}_\kappa(\mathcal{C}') \simeq \mathcal{C}$.

Definition

Let $\mathcal{C}$ be an ∞-category, we say $\mathcal{C}$ is presentable if:

1. $\mathcal{C}$ has all small colimits.
2. $\mathcal{C}$ is κ-accessible for some regular cardinal κ.

1. The presheaf category of spaces on any small ∞-category is presentable.
2. Let $\mathcal{D}$ be a diagram of presentable ∞-categories whose functors either all preserve colimits or all preserve limits, then $\lim \mathcal{D}$ is also presentable.
3. Spc and Sp are both presentable.

[4] Here we omit the definition of $\mathrm{Ind}_\kappa(\mathcal{C})$ but the reader is encouraged to think about the 1-categorical analogy.

Let $\mathcal{C}$ be a presentable ∞-category with a final object $*$, the pointed ∞-category is

$$\mathcal{C}_* := \{*\} \times_{\mathcal{C}} \mathrm{Fun}([1], \mathcal{C}).$$

An object of $\mathcal{C}_*$ consists of $c \in \mathcal{C}$ with a map $* \to c$, we denote coproduct of $c, d \in \mathcal{C}_*$ as $c \vee d$.

Theorem (Probably Also By Lurie)

Let $\mathcal{C}$ be a presentable ∞-category, there is a symmetric monoidal structure $\wedge$ on $\mathcal{C}_*$ given by

$$X \wedge Y := X \times Y / X \vee Y.$$

Ex: This endows the smash product structure on Sp with unit $\mathbb{S}$.

Presentably Symmetric Monoidal Categories

We use Pr^L to denote the subcategory of $\widehat{\mathrm{Cat}_\infty}$ composing of presentable categories whose morphisms are left adjoint functors.

Theorem (See 4.8.15 of [Lurie, 2017])

There is a symmetric monoidal structure on Pr^L such that:

1. Spc is the unit.
2. $C \otimes \mathrm{Sp} \simeq \mathrm{Sp}(\mathcal{C})$.
3. Pr^L_{st} (full subcategory of stable ones) has an induced symmetric monoidal structure such that Sp is the unit.
4. and more properties not mentioned.

A presentably symmetric monoidal ∞-category is an object $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$.

Inverting Objects and Stabilization

Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category and X be a set of objects in $\mathcal{C}$.

Theorem

There exists a presentably symmetric monoidal ∞-category $\mathcal{C}[X^{-1}]$ and map $L : \mathcal{C} \to \mathcal{C}[X^{-1}]$ such that for any $f : \mathcal{C} \to \mathcal{D}$ in $\mathrm{CAlg}(\mathrm{Pr}^L)$ with $f(x)$ invertible for all $x \in X$, f factors uniquely through L.

Example: Sp is $\mathrm{Spc}_*[(S^1)^{-1}]$.

The Stable Motivic ∞-Category

We are finally ready to define $\mathrm{SH}(k)$ and more generally $\mathrm{SH}(S)$.

Definition

Let S be a scheme, $\mathrm{SH}(S)$ is $\mathrm{Spc}(S)_*[(\mathbb{P}^1)^{-1}]$, called the stable motivic ∞-category.

It is equipped with a smash product $\wedge$ and a suspension spectrum functor $\Sigma^\infty_{\mathbb{P}^1} : \mathrm{Spc}(S)_* \to \mathrm{SH}(S)$.

The Spectrum Description of $\mathrm{SH}(S)$

Theorem

Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr})$ and $X \in \mathcal{C}$. If there exists $n \geq 2$ such that the cycle permutation on $X^{\otimes n}$ is homotopic to the identity, then $\mathcal{C}[X^{-1}]$ to the spectra category $\mathrm{Sp}^{\mathbb{N}}(\mathcal{C}, X)$ whose objects are collections $(c_0, c_1, \ldots)$ equipped with equivalences $c_i \simeq \Omega_X c_{i+1}$.

Application to $\mathrm{SH}(S)$:[5] Observe that $\mathbb{P}^1$ is the cofiber $\mathbb{A}^1/(\mathbb{A}^1 - 0)$ and we can identify $(\mathbb{P}^1)^{\otimes 3} \simeq \mathbb{A}^3/(\mathbb{A}^3 - 0)$. The permutation (123) corresponds to the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

which is a product of elementary matrices that are each $\mathbb{A}^1$-homotopic to the identity.

[5] Proposition 3.19.5.1 of [Brazelton, 2024]

The Spectrum Description of $\mathrm{SH}(S)$

Since $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$, we have that the following two suspensions are also invertible in $\mathrm{SH}(S)$:

$$\Sigma^{1,1} := (-) \wedge \Sigma^\infty \mathbb{G}_m \text{ and } \Sigma^{1,0} := (-) \wedge \Sigma^\infty S^1.$$

From here, we write $\Sigma^{p,q} := (\Sigma^{1,1})^{\circ q}(\Sigma^{1,0})^{\circ p-q}$.

We define the bi-graded homotopy groups as

$$\pi_{i,j}(E) = [\Sigma^{i,j} 1, E]_{\mathrm{SH}(S)},$$

where 1 denotes the symmetric monoidal unit. This can be enhanced to a sheaf by considering the sheaf associated to the presheaf

$$U \mapsto [\Sigma^{i,j} U_+, E]_{\mathrm{SH}(S)},$$

but for our purposes we will mainly stick to 1.

Let $E \in \mathrm{SH}(S)$ and $X \in \mathrm{Sm}/S$, we can define the motivic homology and cohomology of X with respect to E as:

1. $E_{p,q}(X) := [S^{p,q}, \Sigma^{\infty}_{\mathbb{P}^1} X_+ \wedge E]$
2. $E^{p,q}(X) := [\Sigma^{\infty}_{\mathbb{P}^1} X_+, \Sigma^{p,q} E]$.

Note that over $S = \mathrm{Spec}(k)$, $\pi_{p,q} E \cong E^{-p,-q}(\mathrm{Spec}\, k)$.

Sanity Check: Computation Example

Question:

What is the graded ring $\pi_{-*,-*}(1)$?

Well, unwinding the definition, for each $\pi_{-i,-i}(1)$, it is given by

$$\pi_{-i,-i}(1) = [\Sigma^{-i,-i}1, 1] = [1, \Sigma^{i,i}1] = \pi_0(S^{i,i})^6.$$

Since $S^1 = S^{1,0}$, we have that $\pi_0(S^{i,i})$ is the colimit

$$\operatorname{colim}_n \pi_n(S^{i+n,i}) = K_i^{\mathrm{MW}},$$

which we showed in Lecture 5 is the i-th Milnor Witt K-theory.

Conclusion: $\pi_{-*,-*}(1) \cong K_*^{\mathrm{MW}}$.

[6] This homotopy group is taken in the context of s-spectra.

For completeness: Milnor-Witt K-Theory

We might as well introduce what Milnor-Witt K-theory is[7].

Definition

For any field F, the Milnor Witt K-theory of F is the graded ring $K_*^{\mathrm{MW}}(F)$ that is the quotient of the free non-cmmutative algebra on **generators**

$$[a] \in K_1^{\mathrm{MW}}(F) \forall a \in F^\times \text{ and a formal symbol } \eta \in K_{-1}^{\mathrm{MW}}(F),$$

with relations imposed as:

1. $\eta[a] = [a]\eta$
2. $[a][1-a] = 0$ for all $a \neq 0, 1$ in F.
3. $[ab] = [a] + [b] + \eta[a][b]$.
4. $\eta(2 + [-1]) = 0$.

Note Milnor-K theory $K_*^{\mathrm{M}}(F)$ is $K_*^{\mathrm{MW}}(F)$ mod η.

[7] Note we did assume some familiarity with algebraic K-theory coming in, which implicitly included some of Milnor K-theory.

Outline

The Eilenberg-Maclane Spectrum (Following [Brazelton, 2024])

Question:

Let A be a sheaf of abelian groups over Sm/k, how would be build a motivic spectrum out of this?

Well in good cases, we do have a sequence of spaces $K(A,0), K(A,1), \ldots$, but do we have an equivalence

$$K(A,0) \to \Omega^{2,1}K(A,1), \text{ and so on } \ldots?$$

There are a few problems already.

1. $\Omega^{2,1}K(A,1)$ is not usually $K(A,0)$.
2. This is because we are dealing with two spheres here.

Instead, we need to introduce an adjustment known as contractions.

Contractions

Let $\mathcal{F}$ be a sheaf of pointed sets, the contraction of F, denoted F_{-1}, is the sheaf associated to the presheaf F_{-1}^{pre} where

$$F_{-1}^{\text{pre}}(U) \coloneqq \ker(F(U \times \mathbb{G}_m) \to F(U)),$$

where the map is induced by $id \times 1 : U \to U \times \mathbb{G}_m$.

1. **Ex:** $(K_n^{\text{MW}})_{-1} \cong K_{n-1}^{\text{MW}}$.
2. If A is strictly $\mathbb{A}^1$-invariant abelian sheaf, then

$$\Omega_{\mathbb{G}_m} K(A, n) \simeq K(A_{-1}, n).$$

Thus, we have that

$$\Omega^{2,1} K(A, n) \simeq K(A_{-1}, n-1).$$

Thus, we see that A admits an Eilenberg-Maclane spectrum HA if it satisfies the following two conditions:

1. Successive applications of $(-)_{-1}$ always exists for A (ie. A admits infinite de-looping).
2. Each term in the sequence is strictly $\mathbb{A}^1$-invariant abelian sheaf.

These two conditions together have a name - they are called homotopy modules $\mathrm{HM}(k)$!

Ex: Milnor-Witt K-theory K_n^{MW}, Milnor K-theory K_n^M, the ideal I^n in Milnor K-theory are all homotopy modules.

The Algebraic K-theory Spectrum (Following [Bachmann, 2021])

We already have the motivic space K given by algebraic K-theory. For convenience it will be useful for us to construct $\mathrm{KVect}(-) \in \mathrm{PShv}(\mathrm{Sm}_S)$ such that

$$K \simeq L_{mot} \mathrm{KVect}(-).$$

We construct $\mathrm{KVect}(-)$ as follows:

1. For $X \in \mathrm{Sm}_S$, $\mathrm{Vect}(X)$ is a symmetric monoidal 1-category under Whitney sum.
2. Let us restrict to only invertible morphisms $\mathrm{Vect}(X)^{\simeq}$. Take the group completion of the classifying space of $\mathrm{Vect}(X)^{\simeq}$ produces a (group-like) space for which we call $\mathrm{KVect}(X)$ - the direct sum K-theory.

The Bott Element

Given the tautological line bundle $\gamma \in \mathrm{Vect}(\mathbb{P}^1_S)$, there is a natural additive functor induced by γ

$$\mathrm{Vect}(X) \to (X \times \mathbb{P}^1)$$

which induces a map

$$\gamma : \mathrm{KVect}(X) \to \mathrm{KVect}(X \times \mathbb{P}^1).$$

We also have a map $1 : \mathrm{KVect}(X) \to \mathrm{KVect}(X \times \mathbb{P}^1)$ given by pulling back along the natural map. Since these are group-like spaces, we also get a map -1, for which we can use to define a map

$$\gamma - 1 : \mathrm{KVect}(X) \to \mathrm{KVect}(X \times \mathbb{P}^1).$$

Periodicity

The morphism of presheaves $\gamma - 1 : \mathrm{KVect}(-) \to \Omega_{\mathbb{P}^1} \mathrm{KVect}(-)$ given above induces a natural map through the motivic localization, which we also denote

$$\gamma - 1 : K \simeq L_{mot}(\mathrm{KVect}(-)) \to \Omega_{\mathbb{P}^1} K \in \mathrm{Spc}(S)_*.$$

Theorem (Motivic Bott Periodicity)

The map $\gamma - 1 : K \simeq L_{mot}(\mathrm{KVect}(-)) \to \Omega_{\mathbb{P}^1} K \in \mathrm{Spc}(S)_*$ is an equivalence.

The proof for the special case when S is Noetherian, regular, and finite dimensional should be of interests, because it involves some motivic concepts, the Thomason-Trobaugh K-theory, and the projective bundle formula.

With respect to the base scheme S, we define the algebraic K-theory spectrum KGL as follows.

Definition

KGL (also written as KGL_S) is the sequence $K, K, K, \ldots$ with structure maps given by $\gamma - 1$.

One can check that $\Sigma^{2n,n}\mathrm{KGL} \simeq \mathrm{KGL}$.

Theorem

Let S be regular, Noetherian, finite-dimensional, then

$$\pi_{p,q}\mathrm{KGL}_S \simeq \begin{cases} K_{p-2q}(S), p \geq 2q \\ 0, \text{otherwise} \end{cases}.$$

Outline

The Cellular Motivic Category

We conclude this lecture by remarking that, in special circumstances, there is a purely topological reformulation of motivic spectra. This reformulation was established by Piotr Pstragowski in [Pstragowski, 2023].

Definition

Let $\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ denote the smallest subcategory of complex motivic spectra containing the motivic spheres and closed under colimits. This is called the cellular motivic category over $\mathbb{C}$.

Piotr Pstragowski's theorem was that, after p-completing, $\mathrm{Sp}_{\mathbb{C}}$ can be refomulated in the language of what are called synthetic spectra.

What is a Synthetic Spectrum?

A synthetic spectrum should be thought of as a categorification of an E-based Adams spectral sequence. Due to the limited time in the lecture, we will not really be able to shed light on this intuition.

1. Let $\mathrm{Sp}_{\mathrm{MU}}^{fp}$ denote the full subcategory spanned by MU_*-projective finite spectra E. By MU_*-projective, we mean $MU_*(E)$ is projective.
2. A presheaf of spectra on $\mathrm{Sp}_{\mathrm{MU}}^{fp}$ is spherical if it sends coproducts to products.
3. The ∞-category of MU-based synthetic spectra[8] Syn_{MU} is the full sub-category of spherical presheaves of spectra X such that

$$A \to B \to C \text{ an } MU_*\text{-SES}$$
$$\Updownarrow$$
$$X(C) \to X(B) \to X(A) \text{ a fiber sequence.}$$

[8] They are actually sheaves given an appropriate Grothendieck topology.

Synthetic Spectra to Cellular Motivic Category

Similarly, if we only restrict to those E with $MU_*(E)$ projective and concentrated in even degrees, we can similarly build the even synthetic spectra on $(\mathrm{Sp}_{MU}^{fp})^{ev}$ to obtain the category

$$\mathrm{Syn}_{MU}^{\mathrm{even}} .$$

Theorem ([Pstragowski, 2023])

After p-completion, there is an equivalence between $\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ and $\mathrm{Syn}_{MU}^{\mathrm{even}}$.

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