A String-Theoretic Introduction to Mirror Symmetry

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Warnings

These slides serve as a motivational introduction to mirror symmetry from a physical perspective. We will go through some basic concepts aimed at mathematical audiences. The logic flow is more chronological than pedagogical.

I am new to this subject, so mistakes and missings are inevitable.

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Definition (Action Functional)

Let $\mathcal{M}$ be the "space of configurations" of a physical system. The action functional is a linear map $S\colon \mathcal{M} \to \mathbb{R}$.

- In classical mechanics, $\mathcal{M}$ is the space of piecewise smooth paths from x to y on a smooth manifold M.
- In classical field theory, $\mathcal{M}$ is the space of sections of a vector bundle E over the spacetime manifold (M, g).

We need a well-behaved measure on $\mathcal{M}$ to perform calculus of variations and integrations. This is usually not well-defined, but physicists just assume it.

Proposition (Principle of Least Action)

The trajectory of a system is the one which extremise the action: $\delta S = 0$.

In classical mechanics, the action arises as the integral of the Lagrangian:

$$S = \int_{t_0}^{t_1} L(q_a, \dot{q}_a, t)\, \mathrm{d}t.$$

The principle of least action is equivalent to the Euler–Lagrange equations ("equations of motions"):

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0,$$

where $(q_1, ..., q_n)$ is a set of coordinates on M.

Proposition (Principle of Least Action)

The trajectory of a system is the one which extremise the action: $\delta S = 0$.

In classical field theory, the action arises as the integral of the Lagrangian density:

$$S = \int_M \mathcal{L}(\varphi_\alpha, \partial_a \varphi_\alpha) \, \mathrm{d}\,\mathrm{vol}_g = \int_{\mathbb{R}^{1,d-1}} \mathcal{L} \sqrt{-\det g} \, \mathrm{d}^n x.$$

The Euler–Lagrange equations:

$$\sum_a \frac{\partial}{\partial x^a} \frac{\partial \mathcal{L}}{\partial(\partial_a \varphi_\alpha)} - \frac{\partial \mathcal{L}}{\partial \varphi_\alpha} = 0.$$

Now we move from classical to quantum! In the path integral formalism, the primary object is the partition function:

$$Z := \int_{\mathcal{M}} \mathcal{D}[\gamma]\ \mathrm{e}^{\mathrm{i}S(\gamma)/\hbar},$$

where $\mathcal{D}[\gamma]$ is a well-behaved measure on $\mathcal{M}$ (which, unfortunately, does not exist in general. This is one of the most notable mathematical difficulties of quantum field theory.)

In the classical limit $\hbar \to 0$, only the classical solution such that $\delta S = 0$ contributes to the partition function.

An observable O is an operator-valued distribution on $\mathcal{M}$ with the expectation:

$$\langle O \rangle := \int_{\mathcal{M}} \mathcal{D}[\gamma]\, O(\gamma)\, \mathrm{e}^{\mathrm{i}S(\gamma)/\hbar}\,.$$

A string is a 1-dimensional object in the space. It traces out a 2-dimensional surface (the "worldsheet") in the spacetime M. So mathematically, string theory is about (the quantisation of) the embedding $X : \Sigma \to M$ of a Lorentzian surface into a d-dimensional Lorentzian manifold.

Definition (Polyakov Action)

The classical bosonic string is described by the Polyakov action:

$$S[X,h] = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-\det h}\,h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu},$$

where h, g are the metrics on Σ and M respectively, and $X = (X^\mu)$ are the coordinates of M.

Slogan:

bosonic fields	fermionic fields
even variables	odd variables
(“c-numbers”)	(“Grassmann numbers”)
commutator $[a, b] = ab - ba$	anti-commutator $\{a, b\} = ab + ba$

ungraded Lie algebra $\rightarrow$ $\mathbb{Z}/2$-graded Lie algebra
(“supersymmetric algebra”)

Supersymmetric scalar multiplet: (scalar fields X^μ, fermions ψ^μ);
Supergravity multiplet: (frame fields e^a_α, gravitini χ_α).

Bosonic string + $N = 1$ worldsheet supersymmetry = RNS string.

Proposition (RNS String Action)

The RNS string (fixed to superconformal gauge) is described by the action

$$S = -\frac{1}{8\pi}\int_\Sigma \mathrm{d}^2\sigma \left(\frac{2}{\alpha'}\partial_\alpha X^\mu \partial^\alpha X^\nu + 2\mathrm{i}\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu\right) g_{\mu\nu}$$

The action is invariant under the supersymmetric transformation:

$$\sqrt{\frac{2}{\alpha'}}\delta_\epsilon X^\mu = \mathrm{i}\bar{\epsilon}\psi^\mu$$

$$\delta_\epsilon \psi^\mu = \sqrt{\frac{2}{\alpha'}}\frac{1}{2}\rho^\alpha \partial_\alpha X^\mu \epsilon$$

The classical string action is Weyl invariant:

$$h \mapsto \Omega^2 h$$

So string theory is a 2-dimensional conformal field theory!
The stress-energy tensor is traceless: $\operatorname{tr} T = 0$.

After quantisation, Weyl invariance must be preserved:

$$\langle \operatorname{tr} T \rangle = -\frac{c}{12} R \qquad \text{(worldsheet scalar curvature)}$$

So *central charge* $c = 0$.

After massaging operator product expansions...

$$c = d - 10 \implies d = 10$$

Conformal field theory tells us that superstrings live in 10 spacetime dimensions!!! Needs compactification of 6 extra dimensions to get real-world physics. See later.

For closed strings, the periodic boundary condition on ψ^μ:

$$\begin{aligned} \psi^\mu(\sigma) &= +\psi^\mu(\sigma+\ell), \qquad && \text{Ramond sector} \\ \psi^\mu(\sigma) &= -\psi^\mu(\sigma+\ell), \qquad && \text{Neveu–Schwarz sector} \end{aligned}$$

Two independent spinors $\psi^\mu_+(\sigma^+)$, $\psi^\mu_-(\sigma^-)$, so 4 types of closed strings: (R,R), (NS,NS), (R,NS), (NS,R).

$N = 1$ super-Virasoro algebra:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12} m\left(m^2-1\right)\delta_{m,-n} \\ [L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}\left(4r^2-1\right)\delta_{r,-s} \end{aligned}$$

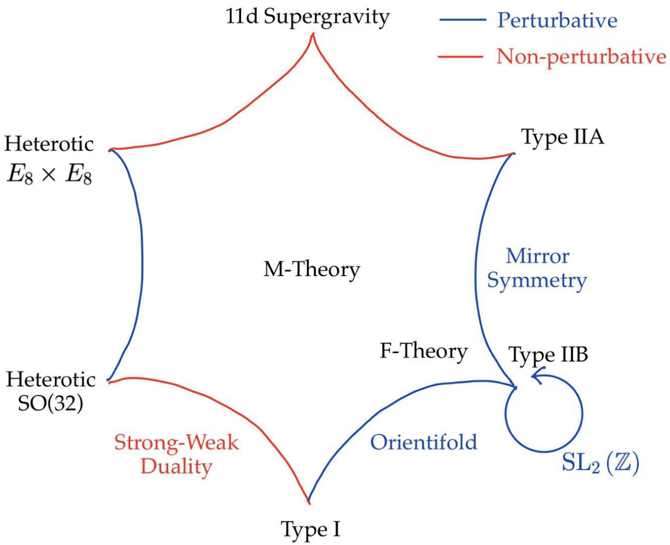
Ramond sector: $r, s \in \mathbb{Z}$; Neveu–Schwarz sector: $r, s \in \mathbb{Z} + \frac{1}{2}$.

Recall that in bosonic string theory, critial dimension $d = 26$; and the ground state is *tachyonic* ($m^2 < 0$).

For superstring with worldsheet as a Riemann surface:

$$\begin{array}{c}\text{Modular invariance \& Vanishing of one-loop partition function} \\ \Downarrow \\ \text{GSO projection } (-1)^F,\ (-1)^{\overline{F}} \\ \Downarrow \\ \text{Tachyon-free \& Spacetime supersymmetry } \mathcal{N} = 2 \\ \swarrow \qquad\qquad \searrow \\ \text{Type IIA superstring theory (Non-chiral)} \qquad \text{Type IIB superstring theory (Chiral)}\end{array}$$

The Landscape of String Dualities



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Superstring theory works in 10 dimensions, but our physical world has only 4 dimensions. We would like to factor the spacetime as

$$\mathbb{R}^{1,3} \times M_6$$

where M_6 is a 6-dimensional compact Riemannian manifold of length scale $\sim 10^{-35}$ m.

Even though we do not observe M_6, the geometry of it actually determines the physics in the Minkowski space $\mathbb{R}^{1,3}$!

Toroidal compactification: $M_6 = T^6$.
All supersymmetries preserved $\rightarrow$ phenomenologically unfavourable.

Calabi–Yau compactification of heterotic strings: $M_6 = \mathrm{CY}_3$.
$\mathcal{N} = 1$ minimal supersymmetric standard model (MSSM)!

There are simply too many choices of M_6 for string theory to make physical predictions. This is why string theory is not considered as a part of physics by many people.

Naive Example: Toroidal Compactification

Consider the compactification $M^{d+1} = M^d \times S^1$:

$$x^d \sim x^d + 2\pi R w.$$

$R \in R$ is the radius of the circle S^1; $w \in \mathbb{Z}$ is the winding number.

Slogan: Gravity in $(d+1)$-dimensions produces electromagnetism in d-dimensions.

Factorisation of metric:

$$G = \sum_{M,N=0}^{d} g_{MN} \mathrm{d}x^M \mathrm{d}x^N = \sum_{\mu,\nu=0}^{d-1} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{dd} \left(\mathrm{d}x^d + \sum_{\mu=0}^{d-1} A_\mu \mathrm{d}x^\mu \right)^2 .$$

Coordinate change $x^d \mapsto x^d + \lambda(x^\mu)$ produces gauge transformation $A_\mu \mapsto A_\mu - \partial_\mu \lambda$.

"**Kaluza–Klein Reduction**". A_μ: KK gauge boson.

Naive Example: Toroidal Compactification

Scalar field ϕ in $(d+1)$-dimensions has the mode expansion:

$$\phi(x^M) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu)\, \mathrm{e}^{\mathrm{i}nx^d/R} .$$

The momentum $p_d = n/R$ is quantised.

Mass for bosonic string:

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \widetilde{N} - 2), \qquad N - \widetilde{N} = nw.$$

T-duality:

$$R \leftrightarrow \frac{\alpha'}{R}, \qquad n \leftrightarrow w.$$

Also Type IIA $\leftrightarrow$ Type IIB.

Naive Example: Toroidal Compactification

Now consider compactification $M^{d+D} = M^d \times T^D$, $T^D := \mathbb{R}^D / 2\pi\Lambda_D$.
For modular invariance the lattice Λ_D must be self-dual: $\Lambda_D = \Lambda_D^*$.

KK reduction $\implies$ Gauge boson A_μ, 2-form B_{mn}.
B-field: "massless scalar degrees of freedom contributed by the internal 2-form"...

T-duality:

$$\frac{1}{\alpha'}(\boldsymbol{g} + \boldsymbol{b}) \leftrightarrow \alpha'(\boldsymbol{g} + \boldsymbol{b})^{-1}, \qquad \boldsymbol{n} \leftrightarrow \boldsymbol{w}$$

Moduli space:

$$\frac{\mathrm{O}(D,D)}{\mathrm{O}(D) \times \mathrm{O}(D)} \Big/ \mathrm{O}(D,D;\mathbb{Z}).$$

2-torus: $T^2 = \mathbb{R}^2/(\mathbb{Z}e_1 + \mathbb{Z}e_2)$. $g_{ij} := e_i \cdot e_j$.

Complexified Kähler modulus: $T := \dfrac{1}{\alpha'}(B + \mathrm{i}\sqrt{\det g}) = T_1 + \mathrm{i}T_2$.
Complex structure modulus:
$U := \dfrac{\|e_2\|}{\|e_1\|}\,\mathrm{e}^{\mathrm{i}\varphi(e_1,e_2)} = \dfrac{g_{12} + \mathrm{i}\sqrt{\det g}}{g_{11}} = U_1 + \mathrm{i}U_2$.

$$(g_{ij}) = \alpha' \frac{T_2}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}.$$

T-duality: $(n_1, n_2, w_1, w_2) \leftrightarrow (-w_1, n_2, -n_1, w_2)$; $\quad T \leftrightarrow U$.

This is a prototype of mirror symmetry, where the complex structure and the Kähler structure interchanges under the mirror transformation.

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Compactification: $\mathbb{R}^{1,3} \times M_6$. Why Calabi–Yau?

Ricci-flat: $R_{mn} = 0$. (*Think of vacuum solutions of Einstein equation.*)

SU(3) holonomy.

Definition (Curvature)

Let (M, g) be a Riemannian/Lorentzian manifold with Levi-Civita connection ∇. The Riemann curvature is given by $R_{abc}{}^{d}\partial_d = [\nabla_a, \nabla_b]\partial_c$. The Ricci curvature is given by $R_{ab} := R_{cab}{}^{c}$.

Definition (Holonomy Group)

Let (M, g) as above. The holonomy group $\mathrm{Hol}_x(\nabla)$ based at $x \in M$ is the group generated by the parallel transports P_γ, where γ is a loop in M based at x.

Compactification: $\mathbb{R}^{1,3} \times M_6$. Why Calabi–Yau?

Ricci-flat: $R_{mn} = 0$.

SU(3) holonomy.

Decomposition of Weyl representation under
$\mathrm{SO}(1,9) \to \mathrm{SO}(1,3) \times \mathrm{SO}(6)$:

$$\mathbf{16} = \mathbf{2}_{\boldsymbol{L}} \otimes \overline{\mathbf{4}} \oplus \mathbf{2}_{\boldsymbol{R}} \otimes \mathbf{4}.$$

Further decomposition of $\mathbf{4}$ of $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ if M_6 has a SU(3)-structure:

$$\mathbf{4}_{\mathfrak{su}(4)} = (\mathbf{1} \oplus \mathbf{3})_{\mathfrak{su}(3)}.$$

The singlet state of $\mathfrak{su}(3)$ is a covariantly constant spinor of M_6, i.e. $\nabla_m \epsilon = 0$. This produces $\mathcal{N} = 1$ supersymmetry in $d = 4$.

Side effect: M_6 is Ricci-flat. (Exercise: try to prove this$\sim$)

Slogan: A Kähler manifold is the one with compatible Riemannian, complex, and symplectic strctures, encoded by:

$$\omega(X, Y) = g(JX, Y),$$

where:

$\omega \in \Gamma(\bigwedge^2 \mathrm{T}^* M)$ is a symplectic form;

$J \in \Gamma(\mathrm{End}(\mathrm{T}M))$ is a complex structure ($J^2 = -\mathrm{id}$);

$g \in \Gamma(\mathrm{S}^2 \mathrm{T}^* M)$ is a Riemannian metric.

Any two of (ω, J, g) determines the third one.

Local expression:

$$\omega = \mathrm{i} g_{\mu\overline{\nu}} \mathrm{d}z^\mu \wedge \mathrm{d}\overline{z}^\nu, \qquad g_{\mu\nu} = g_{\overline{\mu\nu}} = 0.$$

Kähler manifolds are a special class of complex manifolds (real manifolds with an integrable almost complex structure J.)

Exterior differential splits into holomorphic & anti-holomorphic parts:

$$\mathrm{d} = \partial + \overline{\partial}.$$

de Rham cohomology $\mathrm{H}^n_{\mathrm{dR}}(M;\mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^{p,q}(M)$

Dolbeault cohomology $\mathrm{H}^{p,q}(M) = \mathrm{H}^q(M, \Omega^p_M)$.

$$\partial\colon \mathrm{H}^{p,q}(M) \to \mathrm{H}^{p+1,q}(M); \qquad \overline{\partial}\colon \mathrm{H}^{p,q}(M) \to \mathrm{H}^{p,q+1}(M).$$

Hodge numbers: $h^{p,q}(M) := \dim_{\mathbb{C}} \mathrm{H}^{p,q}(M)$.

Hodge Theory

Serre duality:

$$\mathrm{H}^{p,q}(M) \cong \mathrm{H}^{n-q,n-p}(M).$$

Hodge star:

$$\mathrm{H}^{p,q}(M) \cong \mathrm{H}^{n-p,n-q}(M)$$

Complex conjugation:

$$\mathrm{H}^{p,q}(M) \cong \mathrm{H}^{q,p}(M)$$

Hodge diamond:

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & & \vdots & & & \\
h^{n,0} & & & \curvearrowleft & & & h^{0,n} \quad \updownarrow \ \textbf{Hodge} \\
 & & & \textbf{Serre} & & & \\
 & & & \vdots & & & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & & & h^{n,n} & & & \\
 & & & \leftrightarrow & & & \\
 & & & \textbf{conjugation} & & &
\end{array}$$

There are a lot of different and inequivalent definitions of a Calabi–Yau manifold.

Definition (Calabi–Yau Manifolds)

Let M be a $2n$-dimensional compact Kähler manifold. We say that M is Calabi–Yau, if it satisfies any of the following conditions:

1. M has vanishing first Chern class $c_1(M)$;
2. M admits a Ricci-flat Kähler metric;
3. M has trivial canonical bundle $K_M = \bigwedge^n \Omega_M$;
4. M admits a Kähler metric with holonomy contained in $\mathrm{SU}(n)$.
5. M has a unique nowhere vanishing holomorphic n-form.

The conditions satisfy $(1) \iff (2) \Longleftarrow (3) \iff (4) \iff (5)$. They are all equivalent if M is simply connected.

The hard part is $(1) \implies (2)$, known as the *Calabi conjecture*, proven by Yau in 1978.

Definition (Physicists' Calabi–Yau Manifolds)

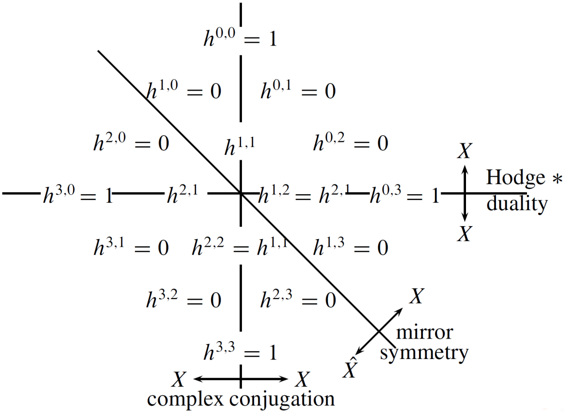
Let M be a $2n$-dimensional compact Kähler manifold. We say that M is Calabi–Yau, if M admits a Kähler metric with holonomy *exactly equal to* $\mathrm{SU}(n)$.

Hodge numbers:

$$h^{0,0} = 1, \qquad h^{n,0} = 1, \qquad h^{i,0} = 0 \text{ for } 0 < i < n.$$

So for a Calabi–Yau 3-fold, the only independent Hodge numbers are $h^{1,1}$ and $h^{1,2}$.

Euler characteristic $\chi(\mathrm{CY}_3) = 2(h^{1,1}(\mathrm{CY}_3) - h^{1,2}(\mathrm{CY}_3))$.



Mirror pair $(X, X^\vee)$:

$$h^{1,1}(X) = h^{1,2}(X^\vee), \qquad h^{1,2}(X) = h^{1,1}(X^\vee).$$

Infinitesmal deformation of the metric $g \mapsto g + \delta g$ preserving Ricci-flatness.

Lichnerowicz equation:

$$\Delta_L \delta g_{\mu\nu} := \nabla^\rho \nabla_\rho \delta g_{\mu\nu} + 2R_\mu{}^\rho{}_\nu{}^\sigma \delta g_{\rho\sigma} = 0$$

1. Mixed indices $\delta g_{i\bar{\jmath}}$:
$$(\Delta \delta g)_{i\bar{\jmath}} = 0.$$
So $\delta g_{i\bar{\jmath}}$ are components of a $(1,1)$-form.
$$\delta g_{i\bar{\jmath}} = \sum_{\alpha=1}^{h^{1,1}} \widetilde{t}^\alpha b^\alpha_{i\bar{\jmath}}$$
where $\widetilde{t}^\alpha \in \mathbb{R}$ are the Kähler moduli, which spans the *Kähler cone*.

Infinitesmal deformation of the metric $g \mapsto g + \delta g$ preserving Ricci-flatness.

Lichnerowicz equation:

$$\Delta_L \delta g_{\mu\nu} := \nabla^\rho \nabla_\rho \delta g_{\mu\nu} + 2R_\mu{}^\rho{}_\nu{}^\sigma \delta g_{\rho\sigma} = 0$$

1. Combining with the deformation of B-field:

$$(\mathrm{i}\delta g_{i\bar{\jmath}} + \delta B_{i\bar{\jmath}})\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j = \sum_{\alpha=1}^{h^{1.1}} t^\alpha b^\alpha_{i\bar{\jmath}}$$

   Now $t^\alpha \in \mathbb{C}$ are complexfied Kähler moduli.

Infinitesmal deformation of the metric $g \mapsto g + \delta g$ preserving Ricci-flatness.

Lichnerowicz equation:

$$\Delta_L \delta g_{\mu\nu} := \nabla^\rho \nabla_\rho \delta g_{\mu\nu} + 2 R_\mu{}^\rho{}_\nu{}^\sigma \delta g_{\rho\sigma} = 0$$

2. Pure indices $\delta g_{\bar{\imath}\bar{\jmath}}$:

$$\Delta_{\bar{\partial}} \delta g^i = 0, \qquad \delta g^i := g^{i\bar{k}} \delta g_{\bar{k}\bar{\jmath}} \mathrm{d}\bar{z}^{\bar{\jmath}}$$

δg^i is a $\mathrm{T}_M := \mathrm{T}^{1,0} M$-valued $(0,1)$-form.

$\mathrm{H}^{0,1}_{\bar{\partial}}(M, \mathrm{T}_M) \cong \mathrm{H}^1(M, \mathrm{T}_M) \cong \mathrm{H}^{2,1}(M)$, so

$$\Omega_{ijk} \delta g^k_{\bar{l}} = \sum_{a=1}^{h^{2,1}} t^\alpha b^\alpha_{ij\bar{l}}$$

where $t^\alpha \in \mathbb{C}$ are the complex structure moduli.

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Massless Spectra of Type IIA & Type IIB Theories

Type IIA on $X \longleftrightarrow$ Type IIB on $X^{\vee}$!

Table 14.1 Massless spectrum of the type II theories on CY_3

Multiplet	Component fields	Multiplicity
	Type IIA	
Gravity	$g_{\mu\nu},\ \psi_{\mu\alpha}\eta,\ \bar{\tilde{\psi}}_{\mu\dot{\alpha}}\eta,\ \bar{\psi}_{\mu\dot{\alpha}}\eta_{\bar{\imath}\bar{\jmath}\bar{k}},\ \tilde{\psi}_{\mu\alpha}\eta_{\bar{\imath}\bar{\jmath}\bar{k}},\ (C_1)_\mu$	1
Hyper	$\lambda_\alpha\eta,\ \bar{\tilde{\lambda}}_{\dot{\alpha}}\eta,\ \bar{\lambda}_{\dot{\alpha}}\eta_{\bar{\imath}\bar{\jmath}\bar{k}},\ \widetilde{\lambda}_\alpha\eta_{\bar{\imath}\bar{\jmath}\bar{k}},\ \Phi,\ B_{\mu\nu},\ (C_3)_{ijk},\ (C_3)_{\bar{\imath}\bar{\jmath}\bar{k}}$	1
Hyper	$\psi_\alpha\eta_{i,\bar{\jmath}\bar{k}},\ \overline{\psi}_{\dot{\alpha}}\eta_{\bar{\imath},\bar{\jmath}},\ \bar{\tilde{\psi}}_{\dot{\alpha}}\eta_{i,\bar{\jmath}\bar{k}},\ \tilde{\psi}_\alpha\eta_{\bar{\imath},\bar{\jmath}},\ g_{ij},\ g_{\bar{\imath}\bar{\jmath}},\ (C_3)_{i\bar{\jmath}\bar{k}},\ (C_3)_{\bar{\imath}jk}$	$h^{2,1}$
Vector	$(C_3)_{\mu i\bar{\jmath}},\ \bar{\psi}_{\dot{\alpha}}\eta_{i,\bar{\jmath}},\ \psi_\alpha\eta_{\bar{\imath},\bar{\jmath}\bar{k}},\ \tilde{\psi}_\alpha\eta_{i,\bar{\jmath}},\ \bar{\tilde{\psi}}_{\dot{\alpha}}\eta_{\bar{\imath},\bar{\jmath}\bar{k}},\ g_{i\bar{\jmath}},\ B_{i\bar{\jmath}}$	$h^{1,1}$
	Type IIB	
Gravity	$g_{\mu\nu},\ \bar{\psi}_{\mu\dot{\alpha}}\eta,\ \psi_{\mu\alpha}\eta_{\bar{\imath}\bar{\jmath}\bar{k}},\ \bar{\tilde{\psi}}_{\mu\dot{\alpha}}\eta,\ \tilde{\psi}_{\mu\alpha}\eta_{\bar{\imath}\bar{\jmath}\bar{k}},\ (C_4^+)_{\mu ijk}$	1
Hyper	$\lambda_\alpha\eta,\ \bar{\lambda}_{\dot{\alpha}}\eta_{\bar{\imath}\bar{\jmath}\bar{k}},\ \tilde{\lambda}_\alpha\eta,\ \bar{\tilde{\lambda}}\eta_{\bar{\imath}\bar{\jmath}\bar{k}},\ \Phi,\ a,\ B_{\mu\nu},\ (C_2)_{\mu\nu}$	1
Hyper	$\psi_\alpha\eta_{i,\bar{\jmath}},\ \bar{\psi}_{\dot{\alpha}}\eta_{\bar{\imath},\bar{\jmath}\bar{k}},\ \tilde{\psi}_\alpha\eta_{i,\bar{\jmath}},\ \bar{\tilde{\psi}}_{\dot{\alpha}}\eta_{\bar{\imath},\bar{\jmath}\bar{k}},\ g_{i\bar{\jmath}},\ B_{i\bar{\jmath}},\ (C_2)_{i\bar{\jmath}},\ (C_4^+)_{\mu\nu i\bar{\jmath}}$	$h^{1,1}$
Vector	$(C_4^+)_{\mu i\bar{\jmath}\bar{k}},\ \bar{\psi}_{\dot{\alpha}}\eta_{i,\bar{\jmath}\bar{k}},\ \psi_\alpha\eta_{\bar{\imath},\bar{\jmath}},\ \bar{\tilde{\psi}}_{\dot{\alpha}}\eta_{i,\bar{\jmath}\bar{k}},\ \tilde{\psi}_\alpha\eta_{\bar{\imath},\bar{\jmath}},\ g_{ij},\ g_{\bar{\imath}\bar{\jmath}}$	$h^{2,1}$