SIMPLICIAL CATEGORIES, KAN EXTENSIONS AND HOMOTOPICAL ALGEBRA

JINGHUI YANG

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1. Simplicial categories

Write Δ to be a category, whose objects consist of sets $[n] = \{0, 1, \cdots, n\}$ with finite total order for any $n \in \mathbb{N}$, and morphisms are order-preserving maps between sets.

Definition 1.1. A **simplicial object** in category $\mathcal{C}$ is a contravariant functor $X : \Delta \to \mathcal{C}$.

If $\mathcal{C} = \mathsf{Set}$, then a simplicial object is called a **simplicial set**. Write $\mathrm{Fun}(\Delta^{op}, \mathcal{C}) = s\mathcal{C}$. In particular, $\mathrm{Fun}(\Delta^{op}, \mathsf{Set}) = s\mathsf{Set}$. One can easily check from definition that Set is a full subcategory of $s\mathsf{Set}$.

Remark 1.2. Dually, one can define what is called the **cosimplicial objects** by replacing "contravariant" with "covariant".

There are two collection of morphisms in Δ, called **face maps** and **degeneracy maps**, defined as follows:

Definition 1.3. Let $0 \leq i, j \leq n$. **Face maps** $d^i : [n-1] \hookrightarrow [n]$ sends k to k when $k < i$, and sends k to $k+1$ when $k \geq i$. In other words, d^i skips i. **Degeneracy maps** $s^j : [n+1] \to [n]$ sends k to k when $k \leq j$, and sends k to $k-1$ when $k > j$. In other words, s^j doubles j.

We get the following theorem which is highly combinatorial:

Theorem 1.4. *For any $f \in \hom_\Delta([n], [m])$, f can be uniquely decomposed into $f = d^{i_1} \cdots d^{i_r} s^{j_1} \cdots s^{j_s}$, where $m = n - s + r$, $i_1 < \cdots < i_r$, $j_1 < \cdots < j_s$, up to linear order.*

Example 1.5. Let $f : [4] \to [2]$. Then $f = s^0 \circ s^2$ because s^0 doubles 0 and s^2 doubles 2.

It is easy to check the face maps and the degeneracy maps satisfy the relation stated as below:

Corollary 1.6.

$$d^j d^i = d^i d^{j-1}, \quad i < j; \tag{1.7}$$

$$s^j s^i = s^i s^{j+1}, \quad i \le j; \tag{1.8}$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & i < j; \\ \mathrm{id} & i = j, j+1; \\ d^{i-1} s^j & i > j+1. \end{cases} \tag{1.9}$$

Let $X_* : \Delta^{op} \to \mathcal{C}$ be a simplicial object in $\mathcal{C}$. Denote $X_n = X_*([n])$, $d_i = X_*(d^i)$, $s_j = X_*(s^j)$. Corollary 1.6 can be rewritten in the form:

Corollary 1.10.

$$d_i d_j = d_{j-1} d_i, \quad i < j; \tag{1.11}$$

$$s_j s_i = s_{i+1} s_j, \quad j \le i; \tag{1.12}$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j; \\ \mathrm{id} & i = j, j+1; \\ s_j d_{i-1} & i > j+1. \end{cases} \tag{1.13}$$

Example 1.14 (Standard simplex)**.** The most important example of simplicial sets is the standard simplices. Consider the category Δ. By Yoneda embedding, any $[n] \in \Delta$ associates to $\hom_\Delta(-, [n])$. Write $\Delta[n]_* = \hom_\Delta(-, [n]) \in s\mathsf{Set}$, with $\Delta[n]_k = \hom_\Delta([k], [n])$. This is called a **standard n-simplex**. Observe that, from Theorem 1.4,

$$\Delta[n]_k \cong \{(j_0, j_1, \cdots, j_k) : 0 \le j_0 \le \cdots \le j_k \le n\}.$$

The first two terms goes $\Delta[0]_k = \{\underbrace{(0, \cdots, 0)}_{k}\}$ and $\Delta[1]_k = \{(\underbrace{0, \cdots, 0}_{i}, \underbrace{1, \cdots, 1}_{k+1-i}) : 0 \le i \le k+1\}$. Informally speaking, there are $k+2$ simplices in $\Delta[1]_k$.

By Yoneda lemma, any simplicial set X_* associates to $\hom_{s\mathsf{Set}}(-, X_*)$. In particular,

$$\hom_{s\mathsf{Set}}(\Delta[n]_*, X_*) \cong X_*([n]) = X_n.$$

So standard n-simplices recover the information in simplicial sets. Generally, since $\Delta \to s\mathsf{Set}$ sending $[n] \mapsto \Delta[n]_*$ is a fully faithful functor, $\Delta[-]_*$ is a cosimplicial object in $s\mathsf{Set}$.

Example 1.15 (Δ-complexes)**.** Recall that in classical algebraic topology,

$$\Delta^n = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1}_{\ge 0} : \sum x_i = 1\}.$$

In our setting, the Δ-complex Δ^* builds a cosimplicial set $\Delta \to \mathsf{Top}$.

Remark 1.16. We often work in category of CGWH (compactly generated and weak Hausdorff) spaces instead of Top since the latter is not Cartesian closed, i.e. there is no natural mapping

$$\underline{\hom}(-,-) : \mathsf{Top}^{op} \times \mathsf{Top} \to \mathsf{Top}$$

such that

$$\hom(Z \times X, Y) \cong \hom(Z, \underline{\hom}(X, Y)),$$

which makes it hard to discuss the right adjoint of the product functor.

Definition 1.17. Let $x \in X_n$. x is called **degenerate** if $x \in \operatorname{im}(s_j : X_{n-1} \to X_n)$ for some j. The set of degenerate n-simplices is given by

$$\bigcup_{j=0}^{n-1} s_j(X_{n-1}) \subset X_n.$$

Example 1.18 (Simplicial spheres)**.** Given a standard n-simplex $\Delta[n]_*$. The **boundary** of $\Delta[n]_*$ is

$$\partial\Delta[n]_* = \bigcup_{0 \le i \le n} d^i(\Delta[n-1]_*) \subset \Delta[n]_*.$$

The **simplicial n-sphere** is $S^n_* = \Delta[n]_*/\partial\Delta[n]_*$. When $n = 1$,

$$\begin{aligned}
\partial\Delta[1]_k &= d^0(\Delta[0]_k) \cup d^1(\Delta[0]_k) \\
&= d^0(\{(0, \cdots, 0)\}) \cup d^1(\{(0, \cdots, 0)\}) \\
&= \{(1, \cdots, 1)\} \cup \{(0, \cdots, 0)\} \\
&= \{(0, \cdots, 0), (1, \cdots, 1)\}.
\end{aligned}$$

By definition,

$$S^1_k = \frac{\{\underbrace{(0, \cdots, 0)}_{i}, \underbrace{(1, \cdots, 1)}_{k+1-i} : i = 0, 1, \cdots, k+1\}}{\{(0, \cdots, 0), (1, \cdots, 1)\}}.$$

Non-degenerate simplices are those such that there is no $y \in \Delta[n+1]_*$ such that $s^j(y) \in \Delta[n]_*$ for some j. So only possible candidate for j to make y degenerate is $j = 0$ or 1. Note that s^j doubles j, and $(0), (1), (0, 0), (1, 1)$ are zero in S^1_*. Hence $(0), (0, 1)$ are non-degenerate simplices in S^1_*. Geometrically, this corresponds to the fact that $S^1 = e^0 \cup e^1$.

S^n_* can also be given by the pushout diagram:

$$\begin{tikzcd}
\partial\Delta[n]_* \arrow[r, hook] \arrow[d] & \Delta[n]_* \arrow[d] \\
* \arrow[r, hook] & S^n_*
\end{tikzcd}$$

where $*$ is the discrete simplicial set associated to the singleton $\{*\}$.

2. Kan extension and geometric realization

Definition 2.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be functors. A **left Kan extension** of F along G is a functor $\mathrm{Lan}_G F : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta : F \Rightarrow \mathrm{Lan}_G F \circ G$ that is universal from F to $\mathrm{Lan}_G F \circ G$. That is, for any $\eta' : F \Rightarrow S \circ G$, there exist a unique natural transformation $\varphi : \mathrm{Lan}_G F \Rightarrow S$

making the diagram commute:

$$\begin{tikzcd} F \arrow[r, Mapsto, "\eta'"] \arrow[d, Rightarrow, "\eta"] & S \circ G \\ \mathrm{Lan}_G F \arrow[ur, Mapsto, "\varphi\circ\mathrm{id}"] & \end{tikzcd}$$

Intuitively, a left Kan extension is a map such that the diagram commutes at each object and morphism:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "F"] & \mathcal{D} \arrow[dl, dashed, "\mathrm{Lan}_G F"] \\ \mathcal{E} & \end{tikzcd}$$

Dually, one can write down the **right Kan extensions** simply by reversing the arrow in the definition of left Kan extensions. We use the notation $Ran_G F$ to denote a right Kan extension of F along G.

Corollary 2.2. *There are two adjoint pairs*

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C},\mathcal{E}) \arrow[r, bend left, "\mathrm{Lan}_G F"] \arrow[r, phantom, "\top"] & \mathrm{Fun}(\mathcal{D},\mathcal{E}) \arrow[l, bend left, "(-)\circ G"] \arrow[r, bend left, "(-)\circ G"] \arrow[r, phantom, "\top"] & \mathrm{Fun}(\mathcal{C},\mathcal{E}) \arrow[l, bend left, "\mathrm{Ran}_G F"] \end{tikzcd}$$

Definition 2.3. Let $Y : \Delta \to s\mathsf{Set}$ be the Yoneda functor (i.e. sending $[n]$ to $\Delta[n]_*$), $\Delta^* : \Delta \to \mathsf{Top}$ be the Δ-complex functor (i.e. sending $[n]$ to Δ^n). The left Kan extension of Δ^* along Y is then called the **geometric realization**, denoted by $|-| \coloneqq \mathrm{Lan}_Y \Delta^*$. One can visualize it as the following diagram:

$$\begin{tikzcd} \Delta \arrow[r, "Y"] \arrow[d, "\Delta^*"] & s\mathsf{Set} \arrow[dl, dashed, "\mathrm{Lan}_Y \Delta^* = |-|"] \\ \mathsf{Top} & \end{tikzcd}$$

Remark 2.4. Here we implicitly assume such a left Kan extension always exists. We will prove that this is the case in Theorem 2.9.

Classically, there are three ways to define a geometric realization functor. The most topology-intimate one goes: for X_* a simplicial set,

$$|X_*| = \left(\coprod_{n\geq 0} X_n \times \Delta^n \right) / \sim, \tag{Defn 1}$$

where $(f_*(x), t) \sim (x, f^*(t))$ for any $x \in X_n$, $t \in \Delta^n$, and $f_* = X_*(f)$, $f^* = \Delta^*(f)$ are induced by $f : [m] \to [n]$ in Δ. Equivalently, this can be described as a coequalizer

$$|X_*| = \mathrm{colim} \left(\coprod_{f:[n]\to[m]} X_m \times \Delta^n \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{[n]} X_n \times \Delta^n \right). \tag{Defn 2}$$

A fancier way to say this is through the coend:

$$|X_*| = \int^{\Delta} X_n \times \Delta^n. \tag{Defn 3}$$

Proposition 2.5. *Defn 1 $\sim$ 3 provided above give the same data, which is functorial.*

Theorem 2.6. *For any $X_* \in s\mathsf{Set}$, $|X_*|$ is a CW complex with n-skeleton $\mathrm{sk}_n(X_*) = \langle X_k \mid k \leq n \rangle$.*

Proof. By definition, we have a skeleton filtration

$$\mathrm{sk}_0(X_*) \subset \mathrm{sk}_1(X_*) \subset \cdots \subset \mathrm{sk}_n(X_*) \subset \cdots \subset X_*,$$

and

$$X_* = \bigcup_{n \geq 0} \mathrm{sk}_n(X_*).$$

Recall that the boundary of $\Delta[n]_*$ can be written as

$$\partial\Delta[n]_* = \langle \Delta[n]_k : k < n \rangle .$$

So we have pushout squares

$$\begin{tikzcd}
\sqcup \partial\Delta[n]_* \arrow[r, "\sqcup f_x|_{\partial\Delta[n]}"] \arrow[d, hook] & \mathrm{sk}_{n-1}(X_*) \arrow[d, hook] \\
\sqcup \Delta[n]_* \arrow[r, "\sqcup f_x"] & \mathrm{sk}_n(X_*)
\end{tikzcd}$$

where the disjoint unions are taken over all non-degenerate simplices $x \in X_n$, and f_x are the representing maps for such $x \in X_n$, i.e. $f_x : \Delta[n]_* \to X_*$ is the map corresponding to $x \in X_n$ under the isomorphism $\hom_{s\mathsf{Set}}(\Delta[n]_*, X) \cong X_n$. Since the geometric realization functor commutes with colimits, the previous pushout diagram is preserved:

$$\begin{tikzcd}
\sqcup |\partial\Delta[n]_*| \arrow[r, "\sqcup |f_x||_{\partial\Delta[n]}"] \arrow[d, hook] & |\mathrm{sk}_{n-1}(X_*)| \arrow[d, hook] \\
\sqcup |\Delta[n]_*| \arrow[r, "\sqcup |f_x|"] & |\mathrm{sk}_n(X_*)|
\end{tikzcd}$$

Thus, we construct $|X_*|$ inductively, by attaching cells one by one. Hence, $|X_*|$ is a cell complex. To see it is a CW complex, one only need to check the intersection of any simplex with $\mathrm{sk}_n(X_*)$ for $n \geq 0$. We leave it to readers. $\square$

It can be seen from previous theorem that only non-degenerate simplices contributes to the cell structure of $|X_*|$. This is why they get their names.

Corollary 2.7. $|S^n_*| = S^n$.

Proof. By Corollary 2.2, $|-| = \mathrm{Lan}_Y \Delta^* \dashv \Delta^* \circ Y$. Since left adjoint functors preserve colimits, we have the diagram

$$\begin{tikzcd}
|\partial\Delta[n]_*| \arrow[r, hook] \arrow[d] & |\Delta[n]_*| \arrow[d] \\
|*| \arrow[r, hook] & |S^n_*|
\end{tikzcd}$$

From the fact $|\Delta[n]_*| = \Delta^n$ (see Corollary 2.11) we get the desired result. $\square$

A natural question is how to concretely describe the left (resp. right) Kan extension; that is, how does it perform on each object and morphism? To answer the question, we need the notion of comma categories. Recall that in a left comma category F/d for a functor $F : \mathcal{C} \to \mathcal{D}$ and an object $d \in \mathcal{D}$, the objects are defined to be the collection $\{(c, f) : c \in \mathcal{C}, f \in \hom_{\mathcal{D}}(F(c), d)\}$, and the morphisms are given by

$$\hom_{F/d}((c, f), (c', f')) = \{h \in \hom_{\mathcal{C}}(c, c') : f = f' \circ F(h)\}.$$

One can similarly write down the definition of a right comma category d/F by inverting the arrows. There are two natural functors associated to F/d:

- forgetful functor $\mathrm{Forget} : F/d \to \mathcal{C}$ sending (c, f) to c;
- constant functor concentrated at d $\mathrm{const}_d : F/d \to \mathcal{D}$ sending $(c, f) \mapsto d$ and $f \mapsto \mathrm{id}_d$.

A natural transformation $\eta : F \circ \mathrm{Forget} \Rightarrow \mathrm{const}_d$ exists because for any $(c, f), (c', f') \in F/d$, $h : c \to c'$ compatible with f, f', the diagram commutes:

$$\begin{tikzcd}
d = \mathrm{const}_d(c, f) \arrow[d, "\mathrm{id}=\mathrm{const}_d(h)"] & F(c) = F \circ \mathrm{Forget}(c, f) \arrow[l, "f=\eta_{(c,f)}"'] \arrow[d, "F(h)"] \\
d = \mathrm{const}_d(c', f') & F(c') = F \circ \mathrm{Forget}(c', f') \arrow[l, "f'=\eta_{(c',f')}"]
\end{tikzcd}$$

Fix $e \in \mathcal{E}$. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{C} \to \mathcal{E}$ be functors. Consider the composition

$$\begin{tikzcd}
G/e \arrow[r, "\mathrm{Forget}"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \\
(c, f) \arrow[r, mapsto] & c \arrow[r, mapsto] & F(c)
\end{tikzcd}$$

Assume $\mathrm{colim}(F \circ \mathrm{Forget})$ exists. Define

$$L_G F(e) \coloneqq \mathrm{colim}_{G/e}(F \circ \mathrm{Forget}) = \mathrm{colim}\left(G/e \to \mathcal{C} \to \mathcal{D}\right).$$

Proposition 2.8. *$L_G F$ is a well-defined functor from $\mathcal{E}$ to $\mathcal{D}$.*

Proof. For any $\varphi : e \to e'$, φ induces a functor $\varphi_* : G/e \to G/e'$ sending (c, f) to $(c, \varphi \circ f)$. It is obvious that the diagram commute:

$$\begin{tikzcd}
G/e \arrow[dd, "\varphi_*"] \arrow[dr, "\mathrm{Forget}"] & & \\
& \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \\
G/e \arrow[ur, "\mathrm{Forget}"'] & &
\end{tikzcd}$$

The task is to describe $L_G F(\varphi)$. The universal property of colimit gives

$$\begin{tikzcd}
F \circ \mathrm{Forget}(i_1) \arrow[rr, "F\circ\mathrm{Forget}(\xi)"] \arrow[dr, "p_{i_1}"] \arrow[ddr, "p_{i_1}\circ\varphi_*"'] & & F \circ \mathrm{Forget}(i_2) \arrow[dl, "p_{i_2}"'] \arrow[ddl, "p_{i_2}\circ\varphi_*"] \\
& \mathrm{colim}(F \circ \mathrm{Forget}) \arrow[d, dashed, "\exists!"] & \\
& \mathrm{colim}(F \circ \mathrm{Forget} \circ \varphi_*) &
\end{tikzcd}$$

where $\xi : i_1 \to i_2$ is any morphism in some small diagram I. So $L_G F(\varphi) = \operatorname{colim}(F \circ \text{Forget}) \to \operatorname{colim}(F \circ \text{Forget} \circ \varphi_*)$ is well-defined. This is the desired construction. Associativity can be easily checked (exercise). $\square$

Theorem 2.9. *Let $\mathcal{C}$ be a small category and $\mathcal{D}$ be a cocomplete category. Every functor $F : \mathcal{C} \to \mathcal{D}$ has a left Kan extension along an arbitrary functor G.*

Proof. Check $L_G F$ satisfies the universal property of $\operatorname{Lan}_G F$. We leave it as an exercise. $\square$

Corollary 2.10. *If G is fully faithful, then $\eta : F \Rightarrow \operatorname{Lan}_G F \circ G$ is an natural isomorphism.*

Proof. Since G is fully faithful, for any $f : Gc \to Gc'$, there exists a unique morphism $h : c \to c'$ such that $f = Gh$. So (c, id_d) is a terminal object in G/Gc. Note that for any small diagram I with a terminal object $*$, and a functor $J : I \to \mathcal{C}$, we have $\operatorname{colim}_I J = J(*)$. This implies that

$$\begin{aligned} \operatorname{colim}_{G/Gc} F \circ \text{Forget} &\cong \operatorname{Lan}_G F(Gc) = (\operatorname{Lan}_G F \circ G)(c) \\ &= F \circ \text{Forget}(c, \mathrm{id}_c) = F(c). \end{aligned}$$

Hence η is a natural isomorphism. $\square$

Corollary 2.11. $|\Delta[n]_*| = \Delta^n$.

Proof. Let Y be Yoneda embedding $Y : \Delta \to s\mathsf{Set}$, $Y([n]) = \Delta[n]_*$. Y is fully faithful, so $\eta : \Delta^* \Rightarrow \operatorname{Lan}_Y \Delta^* \circ Y$ is a natural isomorphism. In particular, $\eta([n]) : \Delta^n \xrightarrow{\cong} |\Delta[n]_*|$. $\square$

3. Homotopy theory of categories

Before we proceed, we need the notion of nerve. From now on, we will assume the underlying category $\mathcal{C}$ is small.

Definition 3.1. The **nerve** of $\mathcal{C}$, denoted $B_*\mathcal{C}$, is a simplicial set with

$$\begin{aligned} &B_0\mathcal{C} = \operatorname{Obj}\mathcal{C}, \\ &B_1\mathcal{C} = \operatorname{Mor}\mathcal{C}, \\ &B_2\mathcal{C} = \{\text{composable morphisms } c_0 \to c_1 \to c_2\}, \\ &\cdots \\ &B_n\mathcal{C} = \{\text{composable morphisms } c_0 \to c_1 \to \cdots \to c_n\}, \end{aligned}$$

with face maps

$$d_i : [c_0 \to \cdots \to c_n] \mapsto [c_0 \to \cdots \to c_{i-1} \to \widehat{c_i} \to c_{i+1} \to \cdots \to c_n]$$

and degeneracy maps

$$s_j : [c_0 \to \cdots \to c_n] \mapsto [c_0 \to \cdots \to c_j \to c_j \to \cdots \to c_n].$$

The nerve of a category $\mathcal{C}$ encodes every "critical" morphism that is not isolated in $\mathcal{C}$. Another way to see the n-cells in $B_*\mathcal{C}$ is through the pullback. Consider

$$\begin{tikzcd}
\operatorname{Mor}\mathcal{C} \times_{\mathcal{C}} \operatorname{Mor}\mathcal{C} \arrow[r] \arrow[d, "s"] & \operatorname{Mor}\mathcal{C} \arrow[d] \\
\operatorname{Mor}\mathcal{C} \arrow[r, "t"] & \mathcal{C}
\end{tikzcd}$$

where t, s are the target functor and source functor, respectively. Explicitly, $t(x \to y) = y$ and $s(x \to y) = x$ for any morphism $x \to y$. The pullback $\operatorname{Mor}\mathcal{C} \times_{\mathcal{C}} \operatorname{Mor}\mathcal{C} = \{(f, g) \in \operatorname{Mor}\mathcal{C} \times \operatorname{Mor}\mathcal{C} : sf = tg\}$ contains exactly those morphisms that are composable, i.e. $f \circ g \in B_2\mathcal{C}$. So $B_2\mathcal{C} \cong \operatorname{Mor}\mathcal{C} \times_{\mathcal{C}} \operatorname{Mor}\mathcal{C}$. This isomorphism generalize to n-cells:

$$B_n\mathcal{C} \cong \underbrace{\operatorname{Mor}\mathcal{C} \times_{\mathcal{C}} \cdots \times_{\mathcal{C}} \operatorname{Mor}\mathcal{C}}_{n}$$

.

Let I be a poset. Write $\overrightarrow{I}$ for the associated category whose objects are I itself, and morphisms are given by

$$\hom_{\overrightarrow{I}}(i, j) = \begin{cases} i \to j & , i < j, \\ \varnothing & , \text{ else.} \end{cases}$$

Denote Cats by the category of small categories. Let $F : \Delta \to \mathsf{Cats}$ be a functor sending $[n]$ to $\overrightarrow{[n]} = [0 \to 1 \to \cdots \to n]$. It is straightforward to check that F is fully faithful. Let $\mathcal{C}$ be any small category. $\mathcal{C}$ represents a functor

$$\begin{aligned} \mathsf{Cats}^{op} &\xrightarrow{h_{\mathcal{C}}} \mathsf{Set} \\ \mathcal{D} &\longmapsto \hom_{\mathsf{Cats}}(\mathcal{D}, \mathcal{C}). \end{aligned}$$

So

$$\begin{aligned} B_*\mathcal{C} = \Delta^{op} &\xrightarrow{F^{op}} \mathsf{Cats}^{op} \xrightarrow{h_{\mathcal{C}}} \mathsf{Set} \\ [n] &\longmapsto \overrightarrow{[n]} \longmapsto \hom_{\mathsf{Cats}}(\overrightarrow{[n]}, \mathcal{C}). \end{aligned}$$

If we take $\mathcal{C} = \overrightarrow{[n]}$, then $B_k\overrightarrow{[n]} = \hom_{\mathsf{Cats}}(\overrightarrow{[k]}, \overrightarrow{[n]}) \cong \hom_{\Delta}([k], [n]) = \Delta[n]_k$. This implies $B_*\overrightarrow{[n]} = \Delta[n]_*$, as we expected.

Example 3.2. Let $X \in \mathsf{Top}$. Assume X has a open cover $\{X_\alpha\}_{\alpha \in I}$. Write X_I to be the category whose objects are $\{(x, X_\alpha) : x \in X_\alpha\} = \sqcup_{\alpha \in I} X_\alpha$, and

$$\hom_{X_I}((x, X_\alpha), (y, X_\beta)) = \begin{cases} \varnothing & , x \neq y, \\ x \mapsto y & , x = y \text{ in } X_\alpha \cap X_\beta. \end{cases}$$

So $\operatorname{Mor} X_I = \bigsqcup_{\alpha,\beta} X_\alpha \cap X_\beta = B_1 X_I$. Moreover, it is not hard to deduce that

$$B_n X_I = \bigsqcup X_{\alpha_0} \cap \cdots \cap X_{\alpha_n}.$$

Example 3.3. Let G be a discrete group. Write $\underline{G}$ to be the one-point category with morphism being the original group G. This is clearly a groupoid. It is immediate that $B_n\underline{G} = G^n$.

Example 3.4 (Bar construction)**.** Let X be some reasonable category like Top, Grp, $\mathsf{SmoothMfd}$, Set, etc. Define EX to be the category with object X and morphism $X \times X$. We denote B_*EX by E_*X. So $E_nX = X^{n+1}$. If $X = G$, then $EX =$

$EG \neq \underline{G}$ (why?). One can justify that E_*G is a simplicial set with right G-action. Furthermore, there exists a map $E_*G \to B_*\underline{G}$, which is a fibration with fiber G.

Example 3.5. Let X be a set with left G-action. Define $\mathcal{C} = G \ltimes X$ to be the category whose objects are in X, and $\hom_{\mathcal{C}}(x, y) = \{g \in G : gx = y\}$. This implies Mor$\mathcal{C} = \{(g, x) : g \in G, x \in X\} = G \times X$ since any $f \in$ Mor$\mathcal{C}$ defines a map from x to $x' = gx$. So $\mathcal{C}$ is a groupoid. It is straightforward to check that $B_n\mathcal{C} = G^n \times X$ with face map being $d_n(g_1, g_2, \cdots, x) = (g_1, \cdots, g_n, x)$. Now $B_*\mathcal{C} = B_*(G \ltimes X) = E_*G \times_G X$ is called the **simplicial Borel construction**.

Definition 3.6 (Bousfield-Kan construction)**.** Let $F : \mathcal{C} \to \mathsf{Set}$ be a functor. Define the **translation category** $\mathcal{C}_F$ with objects $(c, x) : c \in \mathcal{C}, x \in F(c)$, and

$$\hom_{\mathcal{C}_F}((c, x), (c', x')) = \{h : c \to c' : F(h) : F(c) \to F(c') \text{ sends } x \mapsto x'\}.$$

The **homotopy colimit** of F is then defined to be

$$\mathrm{hocolim} F \coloneqq B_*\mathcal{C}_F \in s\mathsf{Set}.$$

The geometric realization of nerves reveals fruitful properties and is the key to the homotopical algebra.

Definition 3.7. Let $\mathcal{C}$ be a category. We denote the geometric realization of nerve of $\mathcal{C}$ by $B\mathcal{C} = \|B_*\mathcal{C}\|$, called the **classifying space** of $\mathcal{C}$. Then $B\mathcal{C} \in \mathsf{Top}$ (or CGWH). We say $\mathcal{C}$ is contractible, connected, etc. if $B\mathcal{C}$ is. For any functor $F : \mathcal{C} \to \mathcal{D}$, we say it is a covering, homotopy equivalence, fibration, etc. if the induced map $BF : B\mathcal{C} \to B\mathcal{D}$ is.

In fact, $B\mathcal{C}$ can be characterized by the following axioms:

$B\mathcal{C}1$ **Naturality**
$\mathcal{C} \mapsto B\mathcal{C}$ extends to a functor $B : \mathsf{Cats} \to \mathsf{Top}$.

$B\mathcal{C}2$ **Normalization**
Let $F : \Delta \to \mathsf{Cats}$ sending $[n] \to \overrightarrow{[n]}$, and $B\,|_F = \Delta \xrightarrow{F} \mathsf{Cats} \xrightarrow{B} \mathsf{Top}$.

$B\mathcal{C}3$ **Gluing**
There is a natural isomorphism $B\mathcal{C} \cong \mathrm{colim}_{([n], f) \in F/\mathcal{C}} B\overrightarrow{[n]}$, i.e. $B\mathcal{C}$ is obtained as the left Kan extension. That is,

$$B\mathcal{C} = \mathrm{colim}\{F/\mathcal{C} \xrightarrow{\text{Forget}} \Delta \xrightarrow{B|_F} \mathsf{Top}\}.$$

Corollary 3.8. *Additional, the following axiom of B can be deduced from $B\mathcal{C}1$ to $B\mathcal{C}3$:*

$B\mathcal{C}4$ *If $\mathcal{C} \subset \mathcal{D}$ as a subcategory, then $B\mathcal{C} \subset B\mathcal{D}$ as a subcomplex.*
$B\mathcal{C}5$ *B preserves coproduct in* Cats*.*
$B\mathcal{C}6$ $B\mathcal{C} \times B\mathcal{D} \cong B(\mathcal{C} \times \mathcal{D})$ *in* CGWH*. Note that this does NOT hold in* Top *(except either $B\mathcal{C}$ or $B\mathcal{D}$ is finite)!*

Remark 3.9. $B\mathcal{C}3$ tells us that $B\mathcal{C}$ is built from $B\overrightarrow{[n]}$. Indeed, by $B\mathcal{C}2$, $B\overrightarrow{[0]} \cong \Delta^0 = \{*\}$, corresponding to Obj$(\mathcal{C})$. Similarly, $B\overrightarrow{[1]} \cong \Delta^1 = [0, 1]$, corresponding to morphism $0 \to 1$ in $\mathcal{C}$.

$B\overrightarrow{[2]} \cong \Delta^2$, corresponding to composable morphisms $0 \to 1 \to 2$ in $\mathcal{C}$.

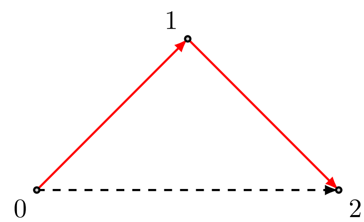

FIGURE 1. $B\overrightarrow{[2]}$, which is Δ^2

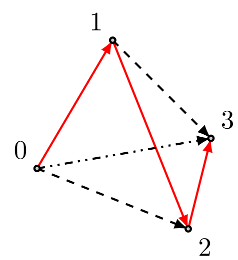

FIGURE 2. $B\overrightarrow{[3]}$, which is Δ^3

Hence, the skeletons of $B\mathcal{C}$ are given by

$$\begin{aligned} sk_0 B\mathcal{C} &= *, \\ \overline{sk}_n B\mathcal{C} &= sk_n B\mathcal{C} - sk_{n-1} B\mathcal{C} \\ &= \{ \text{ composable } f_n \circ \cdots \circ f_0 \} - \{ \text{ composable } f_n \circ \cdots \circ \widehat{f_i} \circ \cdots \circ f_0 : 0 \le i \le n \}. \end{aligned}$$

Example 3.10. Let $\mathcal{C} = \underline{\mathbb{Z}/2}$. Then $B\mathcal{C} \cong (\mathbb{Z}/2)^n$. Looking at its 0- and 1-skeleton, we find

$$\begin{aligned} sk_0 B\mathcal{C} &= *, \\ \overline{sk}_1 B\mathcal{C} &= \{ \text{ composable } * \to * \} - \{ * \xrightarrow{\text{id}} * \}. \end{aligned}$$

This indicates that $sk_1 B\mathcal{C} \cong \mathbb{RP}^1$. In fact, we can show that $sk_n B\mathcal{C} \cong \mathbb{RP}^n$ for all $n \ge 1$. Thus, $B\mathcal{C} = \mathbb{RP}^\infty$. The result corresponds to the ordinary classifying space of $\mathbb{Z}/2$.

3.1. **Homotopy.** To define a homotopy between functors, we need the following lemma:

Lemma 3.11. *Let $h : F_0 \Rightarrow F_1$ be a natural transformation of functors $F_0, F_1 : \mathcal{C} \to \mathcal{D}$. Then h defines a homotopy $B\mathcal{C} \times [0,1] \to B\mathcal{D}$.*

Proof. Define functor $H : \mathcal{C} \times \overline{[1]} \to \mathcal{D}$ by

$$H(c, 0) = F_0(c), \quad H(c, 1) = F_1(c).$$

This is well-defined: on each morphism $(f : c \to c', 0 \to 1)$, $H(f, 0 \to 1) = h_{c'} \circ F_0(f) = F_1(f) \circ h_c : F_0(c) \to F_1(c')$. The diagram reads

$$\begin{tikzcd} F_0(c) \arrow[r, "h_c"] \arrow[d, "F_0(f)"'] & F_1(c) \arrow[d, "F_1(f)"] \\ F_0(c') \arrow[r, "h_{c'}"] & F_1(c') \end{tikzcd}$$

By definition of natural transformation, it is easy to check that H satisfies associativity, so H is indeed a functor. Now consider $BH : B(\mathcal{C} \times [0,1]) \to B\mathcal{D}$. By $B\mathcal{C}6$, $B(\mathcal{C} \times [0,1]) = B\mathcal{C} \times B\overline{1} = B\mathcal{C} \times \Delta^1$. Hence, $BH : B\mathcal{C} \times [0,1] \to B\mathcal{D}$ is the desired homotopy, with

$$BH\mid_{B\mathcal{C}\times\{0\}} = BF_0, \quad BH\mid_{B\mathcal{C}\times\{1\}} = BF_1.$$

□

Corollary 3.12. *Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors. Then $B\mathcal{C} \simeq B\mathcal{D}$. In particular, if $\mathcal{C} \cong \mathcal{D}$, then $B\mathcal{C} \simeq B\mathcal{D}$.*

Proof. Consider the unit and the counit, and apply Lemma 3.11. □

Corollary 3.13. *If $\mathcal{C}$ has initial or terminal object, then $B\mathcal{C}$ is contractible.*

Proof. Consider the constant functor and the inclusion functor, and apply Lemma 3.11. □

With the concept of homotopy established, we are able to talk about the homotopy groups. Starting with π_0.

Definition 3.14. For a category $\mathcal{C}$, we define its zeroth homotopy group to be $\pi_0\mathcal{C} = \pi_0 B\mathcal{C}$.

This definition makes sense. Indeed, if X is a CW complex, then $\pi_0 X = sk_0 X/\sim$ such that x_0 and x_1 being identifies if there is an 1-cell e connecting them. So $\pi_0 B\mathcal{C} = sk_0 B\mathcal{C}/\sim = \mathrm{Obj}(\mathcal{C})/\sim$ with c and c' being identifies if there exists an arrow $c \to c'$ or its inverse.

Example 3.15. Consider the category $G \ltimes X$ in the Example 3.5. Simple observation gives that $\pi_0(G \ltimes X) = X/G$ (Exercise).

Lemma 3.16. *Consider the translation category $\mathcal{C}_F$ for $F : \mathcal{C} \to \mathsf{Set}$. Then*

$$\pi_0(\mathcal{C}_F) \cong \mathrm{colim}_{\mathcal{C}} F.$$

Proof. Let $i \xrightarrow{f} j$ be any arrow in $\mathcal{C}$. The definition of colimit gives

$$\begin{tikzcd} F(i) \arrow[rr, "F(f)"] \arrow[dr, "\varphi_i"'] & & F(j) \arrow[dl, "\varphi_i"] \\ & \mathrm{colim}F & \end{tikzcd}$$

Let $\varphi : \mathrm{Obj}\,(\mathcal{C}_F) \to \mathrm{colim}F$ sending (i, x) to $\varphi_i(x)$. Now if $(i, x) \sim (j, y)$, then there exists $f : i \to j$ such that $F(f)(x) = y$, where $x \in F(i)$ and $y \in F(j)$. So

$$\begin{aligned} \varphi_j(y) &= \varphi(j, y) = \varphi(f(i), F(f)(x)) \\ &= \varphi_j(F(f)(x)) = \varphi_i(x). \end{aligned}$$

Hence φ induces a map $\widetilde{\varphi} : \pi_0(\mathcal{C}_F) \to \mathrm{colim}F$. On the other hand, the inverse map $\widetilde{\phi} : \mathrm{colim}F \to \pi_0(\mathcal{C}_F)$ is induced by ϕ, which is defined to be $(j, y) = \phi_j(y) = \phi_j(F(f)(x)) = \phi_i(x) = (i, x)$ for $(i, x) \sim (j, y)$. Instinctly, $\widetilde{\phi}$ is the unique morphism in the following diagram:

$$\begin{tikzcd}
F(i) \arrow[rr, "F(f)"] \arrow[dr, "\varphi_i"] \arrow[ddr, "\phi_i"'] & & F(j) \arrow[dl, "\varphi_i"'] \arrow[ddl, "\phi_i"] \\
& \mathrm{colim}F \arrow[d, dashed, "\widetilde{\phi}"] & \\
& \mathcal{C}_F &
\end{tikzcd}$$

□

Corollary 3.17. *Let $F : \mathcal{C} \to \mathsf{Set}$ be a functor. Then*

$$|\mathrm{hocolim}F| = |B_*\mathcal{C}_F| = B\mathcal{C}_F.$$

Let $\mathcal{C}$ be a category, $p : E \to B\mathcal{C}$ be a covering space. The **fiber functor** $E : \mathcal{C} \to \mathsf{Set}$ is defined by

$$E(c) = E_c \coloneqq p^{-1}(c).$$

Now for any $f : c \to c'$, $f \in B_1\mathcal{C} \cong \hom_{s\mathsf{Set}}(\Delta_*[1], B_*\mathcal{C})$. So f corresponds to $f : \Delta_*[1] \to B_*\mathcal{C}$, and under the action of geometric realization functor, $|f| : \Delta \to B\mathcal{C}$ is a path in $B\mathcal{C}$. It lifts to $E(f) : E_c \to E_{c'}$ sending $e \to e' = \tilde{f}(e)$, where $\tilde{f} : \Delta^1 \to E$ is a lift of f with $\tilde{f}(0) = e$.

$$\begin{tikzcd}
& E \arrow[d, "p"] \\
\Delta^1 \arrow[r, "|f|"] \arrow[ur, "\tilde{f}"] & B\mathcal{C}
\end{tikzcd}$$

E is called **morphism-invertible**, if it maps all morphisms in $\mathcal{C}$ to isomorphisms in Set.

Proposition 3.18. *Consider the forgetful functor $Forget : \mathcal{C}_F \to \mathcal{C}$ sending (i, x) to i, where F is a functor $F : \mathcal{C} \to \mathsf{Set}$. Then $BF : B\mathcal{C}_F \to B\mathcal{C}$ is a covering space if F is morphism-invertible.*

We end this section with some important definitions.

Definition 3.19. The **fundamental groupoid** of $\mathcal{C}$, denoted by $\prod(\mathcal{C})$, is the localization $\mathcal{C}[(\mathrm{Mor}\,(\mathcal{C}))^{-1}]$.

Definition 3.20. The **n-th homotopy group** of $\mathcal{C}$ is $\pi_n(\mathcal{C}) \coloneqq \pi_n(B\mathcal{C})$.

3.2. **Homology.** Let $C_\bullet(\mathcal{C})$ be the complex whose n-th group is $C_n(\mathcal{C}) = \mathbb{Z}[B_n\mathcal{C}]$, with the differential $\partial : C_n \to C_{n-1}$ given by

$$\partial = \sum_{i=0}^{n} d_i,$$

where d_i is the i-th face map of the simplicial set $B_*\mathcal{C}$.

Proposition 3.21. $\partial^2 = 0$.

Proof. For $\sigma \in B_n\mathcal{C}$,

$$\begin{aligned}
\partial^2\sigma &= \partial\left(\sum_{i=0}^{n} d_i\sigma\right) = \sum_{j=0}^{n-1}\sum_{i=0}^{n}(-1)^{i+j}d_jd_i\sigma \\
&= \sum_{0\le i\le j\le n}(-1)^{i+j}d_jd_i\sigma + \sum_{0\le j<i\le n}(-1)^{i+j}d_jd_i\sigma \\
&= \sum_{0\le i\le j\le n}(-1)^{i+j}d_jd_i\sigma + \sum_{0\le j<i\le n}(-1)^{i+j}d_{i-1}d_j\sigma \\
&= \sum_{0\le i\le j\le n}(-1)^{i+j}d_jd_i\sigma + (-1)\cdot\sum_{0\le j\le i\le n}(-1)^{i+j}d_id_j\sigma \\
&= \sum_{0\le i\le j\le n}(-1)^{i+j}d_jd_i\sigma + (-1)\cdot\sum_{0\le i\le j\le n}(-1)^{i+j}d_jd_i\sigma = 0.
\end{aligned}$$

Note that we used the formula in Corollary 1.6. □

The homology of $C_\bullet(\mathcal{C})$ is said to be the **homology of category** $\mathcal{C}$, which is isomorphic to $H_\bullet(B\mathcal{C};\mathbb{Z})$. If the coefficient of the homology is other than $\mathbb{Z}$, then we need the local system to make a shift.

Definition 3.22. A **local system** $A : \prod(\mathcal{C}) \to \mathsf{Ab}$ is a covariant functor from fundamental groupoid of $\mathcal{C}$ to the category of abelian groups Ab. Equivalently, A can also be regarded as a morphism-invertible functor from $\mathcal{C}$ to Ab.

In the new complex $C_\bullet(\mathcal{C}, A)$ when the coefficient being the local systems A instead of $\mathbb{Z}$, we ask that

$$\begin{aligned}
C_0(\mathcal{C}, A) &= \prod_{c\in \mathrm{Obj}(\mathcal{C})} A(c), \\
C_1(\mathcal{C}, A) &= \prod_{f:c\to c'\in \mathrm{Mor}(\mathcal{C})} A(c), \\
&\cdots\cdots \\
C_n(\mathcal{C}, A) &= \prod_{f:c_0\to c_1\to\cdots\to c_n} A(c_0).
\end{aligned}$$

The differential $\partial_n^A : \sum_{i=0}^{n}(-1)^iA(d_i) : C_n(\mathcal{C}, A) \to C_{n-1}(\mathcal{C}, A)$.

Example 3.23. When $n = 1$, $\partial_1^A : \prod_{f:c_0\to c_1} A(c_0) \to \prod_c A(c)$ restricts to

$$\partial_1^A\mid_{f:c_0\to c_1} = (A(d_0) - A(d_1))\mid_{f:c_0\to c_1}.$$

Note that $d_0(c_0 \to c_1) = c_1$ and $d_1(c_0 \to c_1) = c_0$. We have that

$$\partial_1^A\mid_f : A(c_0) \mapsto A(c_0) \oplus A(c_1),$$

sending x to $(x, A(f)(x))$. Hence,

$$H_0(\mathcal{C}, A) = \prod_{c\in \mathrm{Obj}(\mathcal{C})} A(c)/\mathrm{im}\,\partial_1^A = \mathrm{coker}\,\partial_1^A.$$

Lemma 3.24. *Let A be a local system. Then there is a natural isomorphism $H_0(\mathcal{C}, A) \cong \mathrm{colim} A$.*

Proof. We know from the definition that $\pi_0(\mathcal{C}_A) = \mathrm{Obj}(\mathcal{C}_A)/\sim = \{(c, x) : c \in \mathcal{C}, x \in A(c)\}/\sim$, where $(c_0, x_0) \sim (c_1, x_1)$ iff there exists $f : c_0 \to c_1$ such that $A(f)(x_0) = x_1$. Thus, we obtain that

$$\begin{aligned}\pi_0(\mathcal{C}_A) &= \{(c, x) : c \in \mathcal{C}, x \in A(c)\}/\langle(x, A(f)(x)) : x \in A(c_0), f : c_0 \to c_1\rangle \\ &= \operatorname{coker} \partial_1^A = H_0(\mathcal{C}, A).\end{aligned}$$

So we proved our desired result. □

The previous result generalizes naturally:

Theorem 3.25 (Quillen)**.** *For $n \geq 0$, $H_n(\mathcal{C}, A) \cong \mathrm{colim}_n A$. Here $\mathrm{colim}_n = L_n(\mathrm{colim})(-)$ is the n-th left derived functor of* colim*. This is well-defined because* $\mathrm{colim} : \mathrm{Fun}(\prod \mathcal{C}, \mathsf{Ab}) \to \mathsf{Ab}$ *can be proved to be an additive right exact functor, and* $\mathrm{Fun}(\prod \mathcal{C}, \mathsf{Ab})$ *has enough projectives and injectives.*

3.3. **Quillen's theorem A.** We first state the theorem.

Theorem 3.26 (Quillen's theorem A)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that an arbitrary comma category F/d or d/F is contractible, for any object $d \in \mathcal{D}$. Then $BF : B\mathcal{C} \to B\mathcal{D}$ is homotopy equivalent.*

Example 3.27. Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors. The unit e and the counit η are

$$\begin{aligned} e &: \mathrm{id}_{\mathcal{C}} \Rightarrow RL, \\ \eta &: LR \Rightarrow \mathrm{id}_{\mathcal{D}}. \end{aligned}$$

So we obtain a functor $\mathcal{L} : L/d \to \mathrm{id}_{\mathcal{C}}/Rd$ sending $f : Lc \to d$ to $RLc \to Rd$. The inverse of $\mathcal{L}$ can be easily defined, denoted $\mathcal{L}^{-1} : \mathrm{id}_{\mathcal{C}}/Rd \to L/d$. It sends $g : c \to Rd$ to $LRd \to Lc$. $\mathcal{L}$ and $\mathcal{L}^{-1}$ are indeed mutual inverse because we have $LRL \xRightarrow{L\circ\eta} L$ and $RLR \xRightarrow{e\circ L} L$. Now $\mathrm{id}_{\mathcal{C}}/Rd$ is contractible because it has a terminal object (Rd, id_{Rd}). Therefore, by Quillen's theorem A, $BF : B\mathcal{C} \to B\mathcal{D}$ is a homotopy equivalence.

Example 3.28. Let $\imath : \underline{\mathbb{N}} \hookrightarrow \underline{\mathbb{Z}}$. Consider the comma category $*/\imath$. Its objects are in the form $\{(*, f) : f : * \to * \in \mathbb{Z}\}$, and

$$\hom_{*/\imath}((*, f_1), (*, f_2)) = \left\{ h \in \mathbb{N} : \begin{tikzcd} * \arrow[rr, "h\in\mathbb{Z}"] & & * \\ & * \arrow[ul, "f_1\in\mathbb{Z}"] \arrow[ur, "f_2\in\mathbb{Z}"'] & \end{tikzcd} \text{ commutative} \right\}.$$

Looking at the nerve of $*/\imath$, we can show that it is contractible (Exercise). By Quillen's theorem A, $B\imath : B\underline{\mathbb{N}} \to B\underline{\mathbb{Z}} = S^1$ is a homotopy equivalence.

Before presenting the proof of theorem 3.26, we need some technical constructions.

Definition 3.29. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The **left global comma category** $F/\mathcal{D}$ is a category whose objects are in the form $\{(c, d, f) : c \in \mathcal{C}, d \in \mathcal{D}, f : Fc \to d\}$, and $\hom_{F/\mathcal{D}}((c, d, f), (c', d', f'))$ is the set $\{(h, g) : h \in \hom_{\mathcal{C}}(c, c'), g \in$

$\hom_{\mathcal{D}}(d, d')\}$ such that the diagram commutative:

$$\begin{array}{ccc} F(c) & \xrightarrow{F(h)} & F(c') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ d & \xrightarrow{\quad g \quad} & d' \end{array}$$

Similarly we can define the **right global comma category** $\mathcal{D}/F$. We omit the details for simplicity.

Definition 3.30. A **bisimplicial object** $X_{*,*}$ is a simplicial object in $s\mathcal{C}$. In other words, $X_{*,*} : \Delta^{op} \times \Delta^{op} \to \mathcal{C}$. Write $X_{p,q} = X_{*,*}([p],[q])$. It obsesses a pair of horizontal face/degeneracy maps (corresponding to $X_{*,q}$, denoted d_i^h, s_j^h) and a pair of vertical face/degeneracy maps (corresponding to $X_{p,*}$, denoted d_i^v, s_j^v). We use the notation $ss\mathsf{C}$ to denote the category of bisimplicial objects in $\mathcal{C}$.

There is a natural map

$$\begin{aligned} d : \Delta^{op} &\xrightarrow{\mathrm{id}\times\mathrm{id}} \Delta^{op} \times \Delta^{op} \xrightarrow{\quad X_{*,*} \quad} \mathcal{C} \\ [n] &\longmapsto [n] \times [n] \longmapsto X_{n,n}. \end{aligned}$$

d is called the **diagonalization**. It is functorial, sending elements in $ss\mathsf{Set}$ to ones in $s\mathsf{Set}$. Let $X = X_{*,*}$ be a bisimplicial object. Then $d(X)_n = X_{n,n}$, whose face maps and degeneracy maps are given by

$$d_i = d_i^h \circ d_i^v = d_i^v \circ d_i^h, \qquad\qquad s_j = s_j^h \circ s_j^v = s_j^v \circ s_j^h.$$

Notice that horizontal and vertical maps are independent, so they are free to commute.

Proposition 3.31. *There exists a coequalizer*

$$\coprod_{f:[m]\to[n]} X_n \times \Delta^m \overset{\coprod}{\rightrightarrows}_{n\geq 0} X_n \times \Delta^n \xrightarrow{\gamma} dX,$$

where $\gamma_n : X_n \times \Delta^n \to dX$*. Its action on* r*-simplices yields* $(x, \tau : [r] \to [n]) \mapsto \gamma^*(x) \in X_{r,r}$.

Definition 3.32. The geometric realization of $X = X_{*,*}$ is

$$BX \coloneqq \coprod_{p,q\geq 0} X_{p,q} \times \Delta^p \times \Delta^q \big/ \sim,$$

where $\sim$ is the same as the one in the equation Defn 1, but given as p and q respectively.

Proposition 3.33. *d induces a homotopy equivalence* $BX \xrightarrow{\simeq} B(dX)$*.*

Let $f = f_{*,*} : X_{*,*} \to Y_{*,*}$ be a map of bisimplicial objects in $\mathcal{C}$ (i.e. compatible with face and degeneracy maps). For any $c \in \mathcal{C}$, $c \in B_0\mathcal{C}$. From the fact

$$\hom_{s\mathsf{Set}}(\Delta[0]_*, B_*\mathcal{C}) \cong B_0\mathcal{C}$$

we deduce that $s_0^p = \underbrace{s_0 \circ \cdots \circ s_0}_{p}(c) \in B_p\mathcal{C}$, $p \geq 0$. The **fiber of** f at c is

$$f^{-1}(c) = \{f_{p,q}^{-1}(c) \subset X_{p,q}\}_{p,q}.$$

Any map $\alpha : c \to c'$ yields a map of bisimplicial sets $f^{-1}(c) \xrightarrow{\alpha_*} f^{-1}(c')$. The following lemma is important in our setting:

Lemma 3.34. *Let $X = X_{*,*}$ be a bisimplicial object in $\mathcal{C}$.*

(1) If $p \geq 0$ and $f_{p,*} : X_{p,*} \to Y_{p,*}$ is a homotopy equivalence, then $Bf : BX \to BY$ is a homotopy equivalence.
(2) If for any map $\alpha : c \to c'$, $Bf^{-1}(c) \xrightarrow{\alpha_*} Bf^{-1}(c')$ is a homotopy equivalence, then $f^{-1}(c) \hookrightarrow X_{*,*}$ fits into a homotopy fibration sequence:

$$Bf^{-1}(c) \to BX \to B\mathcal{C}. \tag{$\sharp$}$$

We will discuss the homotopy fibration sequence in the next section. For now, we leave it as a black box with one important result kept in mind: if $Bf^{-1}(c)$ in the homotopy fibration sequence ($\sharp$) is contractible, then $BX \simeq B\mathcal{C}$.

Lemma 3.35. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then the forgetful functor*

$$\begin{aligned} \text{Forget} : \mathcal{D}/F &\longrightarrow \mathcal{C} \\ (c, d, f) &\longmapsto c \end{aligned}$$

is a homotopy equivalence.

Proof. Define $X = \{X_{p,q}\}_{p,q}$ with

$$X_{p,q} = \{d_p \to \cdots \to d_0 \to F(c_0),\, c_0 \to \cdots \to c_q\}_{p,q},$$

where $c_0, d_0, c_1, d_1, \cdots \in \mathcal{D}/F$. This is the same data as the triple

$$\left(d_p \to \cdots \to d_0 \to F(c_0),\, c_0 \to \cdots \to c_q,\; \begin{array}{ccccccc} d_p & \longrightarrow & d_{p-1} & \longrightarrow & \cdots & \longrightarrow & d_0 \\ \downarrow & & \downarrow & & & & \downarrow \\ F(c_p) & \longleftarrow & F(c_{p-1}) & \longleftarrow & \cdots & \longleftarrow & F(c_0) \end{array} \right).$$

Note $BX \simeq BdX$. On the other hand, $B_*dX = X_{*,*} = B_*(\mathcal{D}/F)$ by the data. So $BX \simeq B(\mathcal{D}/F)$. Consider the natural projection $f = f_{*,*'} : X_{*,*'} \to B_{*'}\mathcal{C}$. On (p, q)-simplex,

$$\begin{aligned} f_{p,q} : X_{p,q} &\longrightarrow B_q\mathcal{C} \\ \{d_p \to \cdots \to d_0 \to F(c_0),\, c_0 \to \cdots \to c_q\}_{p,q} &\longmapsto (c_0 \to \cdots \to c_q), \end{aligned}$$

and $s_0^q(c_0) = \underbrace{c_0 \to \cdots \to c_0}_{q} \in B_q\mathcal{C}$. So

$$\begin{aligned} f^{-1}(c_0) &= \{f_{p,q}^{-1}(s_0^q(c_0)) \subset X_{p,q}\}_{p,q} \\ &= \{d_p \to \cdots \to d_0 \to F(c_0),\, \underbrace{c_0 \to \cdots \to c_0}_{q}\}_{p,q} \\ &\cong B_*(\mathcal{D}/F(c_0)). \end{aligned}$$

Since $\mathcal{D}/F(c_0)$ has an initial object $(F(c_0), \mathrm{id}_{F(c_0)})$, it is contractible. Hence, $Bf^{-1}(c_0)$ is contractible. By (2) of Lemma 3.34, $BX \simeq B\mathcal{C} \simeq B(\mathcal{D}/F)$. □

Now we are ready to prove theorem 3.26.

Proof of theorem 3.26. Consider the following functors:

$$\begin{array}{ccccc} \mathcal{C} & \xleftarrow{\text{Forget}_{\mathcal{C}}} & \mathcal{D}/F & \xrightarrow{\text{Forget}_{\mathcal{D}}} & \mathcal{D}^{op} \\ c & \longleftarrow & (c,d,f) & \longmapsto & d. \end{array}$$

By Lemma 3.35, $\text{Forget}_{\mathcal{C}}$ is a homotopy equivalence. It suffices to check $\text{Forget}_{\mathcal{D}}$ is a homotopy equivalence. Write $\text{Mor}\,(\mathcal{D})$ to be a category whose objects are morphisms in $\mathcal{D}$, and

$$(\flat) \qquad \hom_{\text{Mor}\,(\mathcal{D})}(a \xrightarrow{f} b, c \xrightarrow{g} d) = \left\{ (\phi,\psi) : \begin{tikzcd} a \arrow[r, "f"] \arrow[d, "\psi"'] & b \arrow[d, "\phi"] \\ c \arrow[r, "g"] & d \end{tikzcd} \right\}.$$

Let t, s be the target, and the source functors, respectively. $t : \text{Mor}\,(\mathcal{D}) \to \mathcal{D}$ sends $(a \xrightarrow{f} b)$ to b, and sends the commutative diagram in $(\flat)$ to $b \xrightarrow{\phi} d$. Similarly, $s : \text{Mor}\,(\mathcal{D}) \to \mathcal{D}^{op}$ sends $(a \xrightarrow{f} b)$ to a, and sends the commutative diagram in $(\flat)$ to $a \xrightarrow{\psi} c$. Clearly, $\text{Mor}\,(\mathcal{D}) = \mathcal{D}/\text{id}_{\mathcal{D}}$. So Lemma 3.35 tells us that t is a homotopy equivalence. Moreover, with slight amendation, s is also a homotopy equivalence. Working on the diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "F"] & \mathcal{D}/F \arrow[l, "\cong"'] \arrow[r] \arrow[d, "\text{Forget}_{\text{Mor}\,(\mathcal{D})}"] & \mathcal{D}^{op} \arrow[d, equal] \\ \mathcal{D} & \text{Mor}\,(\mathcal{D}) \arrow[l, "t", "\cong"'] \arrow[r, "s", "\cong"'] & \mathcal{D}^{op} \end{tikzcd}$$

It suffices to show that the functor $\mathcal{D}/F \to \mathcal{D}^{op}$ on the top right is a homotopy equivalence.

Let $X = X_{*,*}$ be a bisimplicial object, with

$$X_{p,q} = \{d_p \to \cdots \to d_0 \to F(c_0),\, c_0 \to \cdots \to c_q\}_{p,q}.$$

Let $P : X_{*,*'} \to B_*\mathcal{D}^{op}$ be the projection onto the d-factor. By a similar argument in the proof of Lemma 3.35, $P^{-1}(d_0) \cong B_*(d_0/F)$, which is contractible. $B(\mathcal{D}/F) \to B\mathcal{D}^{op}$ factors through $B(\mathcal{D}/F) \xrightarrow{\simeq} BX \xrightarrow{BP} B\mathcal{D}^{op}$. Hence, $B(\mathcal{D}/F) \simeq BX \simeq B\mathcal{D}^{op}$. We get our desired result. □

3.4. **Quillen's theorem B.**

Before we proceed, we first pick up some basic knowledge of homotopy theory. Most of propositions in this section will not be proved.

3.4.1. *Homotopy fibration sequence.*

Let $\mathcal{C}$ be a locally small category. Recall that for any $i : A \to B, p : X \to Y \in \text{Mor}(\mathcal{C})$, i is said to have **left lifting property** (**LLP**) w.r.t. p if there is a map $h : B \to X$ such that $f = h \circ i$; p is said to have **right lifting property** (**RLP**) w.r.t. i if there is a map $h : B \to X$ such that $g = p \circ h$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

Let $p : E \to B$ be a surjective map. It is called a **fibration** if for any $i : D^n \hookrightarrow D^n \times I$ $(n \geq 0)$, i has LLP w.r.t. p. That is, there exists a map $h : D^n \times I \to E$ such that the diagram commutes:

$$\begin{tikzcd}
D^n \arrow[r] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\
D^n \times I \arrow[r] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

Proposition 3.36. *Pullback of a fibration is again a fibration.*

Proof. Let $p : E \to B$ be a fibration, and $f : A \to B$ be any map. We need to prove the pullback $\tilde{p} : A \times_f E \to A$ is a fibration. Look at the diagram:

$$\begin{tikzcd}
D^n \arrow[r, "g"] \arrow[d, hook, "i_0"'] & A \times_f E \arrow[r] \arrow[d, "\tilde{p}"'] & E \arrow[d, "p"] \\
D^n \times I \arrow[r, "h"] \arrow[urr, dashed, "\phi"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

By definition of fibration, there exists a map $\phi : D^n \times I \to E$ such that two big triangles with diagonal from $D^n \times I$ to E are commutative. Note that we already have $p \circ \phi = f \circ h$. The universal property of pullback yields that there exists a unique map $\psi : D^n \times I \to A \times_f E$, which is exactly the desired morphism. $\square$

As we would expect from classical homotopy theory, we have the following proposition:

Proposition 3.37. *Let $E \to B$ be a fibration with fiber F. Then there exists a long exact sequence associated to it:*

$$\cdots \to \pi_{n+1}(B) \to \pi_{n+1}(E) \to \pi_{n+1}(F) \to \pi_n(B) \to \cdots$$

Let X be a path-connected space. The **path space** of X, denoted X^I, is $\mathrm{Map}(I, X)$ with compact-open topology (i.e. generated by U^C of paths mapping a fixed compact subset $C \subset I$ into a fixed open subset $U \subset X$). Write $PX = \{\gamma \in X^I : \gamma(0) = x\}$, the space of paths based at $x \in X$.

Proposition 3.38. *There is a fibration $PX \xrightarrow{p} X$ sending γ to $\gamma(1)$. The fiber of this fibration is $\Omega X = \{\gamma \in X^I : \gamma(0) = \gamma(1)\}$, called the **loop space** of X.*

Proposition 3.39. *(1) PX is contractible.*
(2) If X is homotopy equivalent to a CW complex, then so is ΩX.

Definition 3.40. Let $f : X \to Y$ be any morphism, with Y path-connected. The **mapping path space** Nf is the pullback

$$\begin{tikzcd}
Nf \arrow[r, "g"] \arrow[d, "\pi"'] & PY \arrow[d, "p"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where p sends γ to $\gamma(1)$. In other word,

$$Nf = X \times_f PY = \{(x, \gamma) \in X \times Y^I : f(x) = \gamma(1)\}.$$

Proposition 3.41 (Example of fibrant replacement)**.** *Any morphism $f : X \to Y$ in* Top *can be written as a composite of a homotopy equivalence and a fibration.*

Definition 3.42. Let $f : X \to Y$ be any morphism, with Y path-connected. Suppose we have $Nf \xrightarrow{P} Y$, where $P = f \circ \pi = p \circ g$ in the Definition 3.40. $P(x, \gamma) = \gamma(1)$. The **homotopy fiber of f over** $y \in Y$ is $P^{-1}(y) = \{(x, \gamma) \in X \times Y^I : \gamma(1) = f(x), \gamma(0) = y\}$. When the choice of y is specified or unimportant, then we denote $P^{-1}(y)$ by Ff.

Equivalently, we see Ff as the pullback

$$\begin{tikzcd} Ff \arrow[r] \arrow[d] & Nf \arrow[d, "P"] \\ \{y\} \arrow[r, hook] & Y \end{tikzcd}$$

Let $F \xrightarrow{j} X \xrightarrow{f} Y$ be any morphism in Top such that $f \circ j$ is constant. The universal property of Ff gives a canonical map $g : F \to Ff$, sending x to $(j(x), \gamma_{f \circ j(x)})$:

$$\begin{tikzcd} F \arrow[rr, "j"] \arrow[dr, dashed, "g"] \arrow[ddr] & & X \arrow[d, hook, "i"] \arrow[dd, bend left, "f"] \\ & Ff \arrow[r] \arrow[d] & NF \arrow[d] \\ & \{y\} \arrow[r, hook] & Y \end{tikzcd}$$

Definition 3.43. The sequence $F \xrightarrow{j} X \xrightarrow{f} Y$ is called a **homotopy fibration sequence** if the induced map g is a homotopy equivalence.

Proposition 3.44. *Let $F \xrightarrow{j} X \xrightarrow{f} Y$ be a homotopy fibration sequence. Then there exists a long exact sequence associated to it:*

$$\cdots \to \pi_{n+1}(F) \to \pi_{n+1}(X) \to \pi_{n+1}(Y) \to \pi_n(F) \to \cdots$$

3.4.2. *Quillen's theorem B.* Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Fix an object $d \in \mathcal{D}$.

Definition 3.45. The **fiber of F over** d is the category $F^{-1}(d)$, whose objects and morphisms consist of $\{c \in \mathcal{C} : F(c) = d\}$ and $\{f \in \mathrm{Mor}(\mathcal{C}) : F(f) = \mathrm{id}_d\}$, respectively.

There are natural functors

$$\begin{aligned} i_* : F^{-1}(d) &\hookrightarrow d/F \\ c &\longmapsto (d \mapsto Fc, c) = (c, \mathrm{id}_d), \end{aligned}$$

and

$$\begin{aligned} i^* : F^{-1}(d) &\hookrightarrow F/d \\ c &\longmapsto (Fc \mapsto d, c) = (c, \mathrm{id}_d). \end{aligned}$$

However, i_* and i^* are not homotopy equivalences in general.

Definition 3.46. F is called **pre-cofibered** if for any object $d \in \mathcal{D}$, i_* has a right adjoint, denoted by $i^! : d/F \to F^{-1}(d)$. Dually, F is called **pre-fibered** if for any object $d \in \mathcal{D}$, i^* has a left adjoint, denoted by $i_! : F/d \to F^{-1}(d)$.

Corollary 3.47. *If F is pre-cofibered, then $B(d/F) \simeq BF^{-1}(d)$. If F is pre-fibered, then $B(F/d) \simeq BF^{-1}(d)$.*

Definition 3.48. Let F be pre-fibered. Fix a morphism $f : d \to d'$ in $\mathcal{D}$. The **base change functor** $f^* : F^{-1}(d') \to F^{-1}(d)$ is given by

$$f^* : F^{-1}(d') \xrightarrow{\quad i'_* \quad} d'/F \xrightarrow{\qquad f \qquad} d/F \xrightarrow{\qquad i^! \qquad} F^{-1}(d)$$

$$(c, d' \xrightarrow{g} Fc) \longmapsto (c, d \xrightarrow{f} d' \xrightarrow{g} Fc)$$

Let $d \xrightarrow{f} d' \xrightarrow{g} d''$ be a chain of morphism in $\mathcal{D}$. There exists a natural transformation $\alpha = f^*g^* \Rightarrow (g \circ f)^*$, induced by the counit $\varepsilon : i'_* \circ (i')^! \Rightarrow \mathrm{id}_{d/F}$:

$$\begin{tikzcd}
F^{-1}(d'') \arrow[r, "i''_*"] & d''/F \arrow[r, "g"] & d'/F \arrow[r, "(i')^!"] \arrow[dr] & F^{-1}(d') \arrow[d, "i'_*"] \arrow[dl, phantom, "\overset{\varepsilon}{\Leftarrow}"] \\
& & & d'/F \arrow[d, "f"] \\
& & & d/F \arrow[d, "i^!"] \\
& & & F^{-1}(d)
\end{tikzcd}$$

Dually, we can present the previous constructions with the assumption that F is pre-cofibered.

Definition 3.49. Let F be pre-fibered. F is **fibered** if any composable pair $f, g \in \mathrm{Mor}(\mathcal{D})$ induces the natural isomorphism $\alpha = f^*g^* \Rightarrow (g \circ f)^*$ defined as above. Dually, let F be pre-cofibered. F is **cofibered** if any composable pair $f, g \in \mathrm{Mor}(\mathcal{D})$ induces the natural isomorphism $\alpha = f^*g^* \Rightarrow (g \circ f)^*$.

The following is an easy corollary of Quillen's theorem A:

Corollary 3.50. *Let F be cofibered (resp. fibered). If $F^{-1}(d)$ is contractible for any object $d \in \mathcal{D}$, then $BF : B\mathcal{C} \to B\mathcal{D}$ is a homotopy equivalence.*

Example 3.51 (Grothendieck)**.** There is an one-to-one correspondence:

$$\{\text{cofibered } \mathcal{C} \to \mathcal{D}\} \overset{[}{\leftrightarrow}]\{\text{functors } \mathcal{D} \to \mathsf{Cats}\}.$$

To see why it is true, one can take any cofibered functor $F : \mathcal{C} \to \mathcal{D}$, and then define $F^{-1} : \mathcal{D} \to \mathsf{Cats}$ sending $d \mapsto F^{-1}(d)$. Conversely, for any $G : \mathcal{D} \to \mathsf{Cats}$, one can associate it to $G' : \mathcal{D}_G \to \mathcal{D}$, which is cofibered.

Now we come to another meta-theorem of the context:

Theorem 3.52 (Quillen's theorem B)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that any $(f : d \to d') \in \mathsf{Mor}(\mathcal{D})$ induces a homotopy equivalence in the associated base change functor f^*:*

$$F^{-1}(d') \xrightarrow{i'_*} d'/F \underset{\simeq}{\xrightarrow{f}} d/F \xrightarrow{i^!} F^{-1}(d).$$

Then, for any object $d \in \mathcal{D}$, there is a homotopy fibration sequence:

$$B(d/F) \xrightarrow{B\circ\mathrm{Forget}} B\mathcal{C} \xrightarrow{BF} B\mathcal{D}.$$

Proof. Again, we use the same technique as proving Quillen's theorem A. Let $X = \{X_{p,q}\}_{p,q}$ be a bisimplicial object in $\mathcal{C}$. $X_{p,q} = \{d_p \to \cdots \to d_0 \to F(c_0), c_0 \to \cdots \to c_q\}$. Let $\pi : X_{*,*'} \to B_*\mathcal{D}^{op}$ be as in the proof of Quillen's theorem A (see Theorem 3.26). We know

$$\pi^{-1}(d_0) \cong B_*(d_0/F).$$

From the assumption, $d'/F \xrightarrow[\simeq]{f} d/F$. (2) of Lemma 3.34 tells us that there exists a homotopy fibration sequence

$$B\pi^{-1}(d) \cong B(d/F) \to BX \to B\mathcal{C}.$$

On the other hand, since $BX \simeq BdX$ and $B_*(\mathcal{D}/F) = B_*dX$, $BX \simeq B(\mathcal{D}/F)$, and so by Lemma 3.35,

$$B \circ \mathrm{Forget} : B(d/F) \simeq BX \simeq B(\mathcal{D}/F) \xrightarrow{\simeq} B\mathcal{C}.$$

We obtain the following diagram:

$$\begin{tikzcd}
B(d/F) \arrow[r] \arrow[d, equal] & B(\mathcal{D}/F) \simeq BX \arrow[r, "B\pi"] \arrow[d, "\simeq"] & B\mathcal{D}^{op} \arrow[d, "\simeq"] \\
B(d/F) \arrow[r, "B\circ\mathrm{Forget}"] & B\mathcal{C} \arrow[r, "BF"] & B\mathcal{D}
\end{tikzcd}$$

Note that $B\pi$ is a homotopy equivalence by factoring through

$$B(\mathcal{D}/F) \simeq BX \xrightarrow{B\pi} B\mathcal{D}^{op}.$$

Hence, the upper row of the diagram is a homotopy fibration sequence. Therefore, the upper row of the diagram is also a homotopy fibration sequence. ☐

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