Topological Cyclic Homology of Local Fields

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Nov 2024

Outline

1. Motivation: Computation of Algebraic K-Theory
2. Topological Hochschild Homology and Topological Cyclic Homology
3. Descent Spectral Sequences
4. TC of Local Fields

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Write e_F to be the ramification index of $F \mid \mathbb{Q}_p$, and f_K to be the inertia degree. Write $E_F(z) \in \mathcal{O}_F[z]$ to be the Eisenstein polynomial.

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- K-theory of F verifies the Quillen-Lichtenbaum conjecture if F does not contain certain roots of unity. (Hesselholt-Madsen, 2003)
- If $F = W(\mathbb{F}_p)$, $K_*(F/p^n)$ will recover the calculation of $K_*(\mathbb{Z}/p^n)$. (Antieau-Krause-Scholze, 2024)

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Theorem (Dundas-Goodwillie-McCarthy)

Assume that I is a nilpotent ideal in R. Then there is a natural pullback square

$$\begin{array}{ccc} K(A) & \longrightarrow & K(A/I) \\ {\scriptstyle \text{cyclotomic trace}}\downarrow & & \downarrow{\scriptstyle \text{cyclotomic trace}} \\ \mathrm{TC}(A) & \longrightarrow & \mathrm{TC}(A/I) \end{array}$$

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Topological Hochschild Homology (THH)

Let R be a commutative ring spectrum, i.e. an $\mathbb{E}_\infty$-ring. Write $\mathrm{CAlg} := \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$, and $\mathcal{S}$ is the ∞-category of spaces.

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The topological Hochschild homology of R is defined to be $\mathrm{THH}(R) = R \otimes_{R \otimes_{\mathbb{S}} R^{\mathrm{op}}} R$.

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We will use these definitions interchangeably.

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- The topological periodic cyclic homology of R is defined to be $\mathrm{TP}(R) = \mathrm{THH}(R)^{tS^1}$.
- The topological cyclic homology of R, denoted TC(R), is defined to be the equalizer of the canonical map

$$\mathrm{can} : \mathrm{TC}^-(R) \to \mathrm{TP}(R),$$

and the cyclotomic Frobenius

$$\varphi : \mathrm{TC}^-(R) \to \mathrm{TP}(R).$$

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Moreover, $\mathbb{S}_{W(k)}$ is uniquely characterized by the above properties.

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We can define the related version of $\mathrm{THH}, \mathrm{TC}^{-}, \mathrm{TP}, \mathrm{TC}$:

Definition

Let R be a $\mathbb{S}_{W(k)}[z_0, \cdots, z_n]$-algebra. Then

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- Similarly can define the relative $\mathrm{TC}(R/\mathbb{S}_{W(k)}[z_0, \cdots, z_n])$.

Change-of-base formula:

$$\mathrm{THH}(R/S) \simeq \mathrm{THH}(R) \otimes_{\mathrm{THH}(S)} S.$$

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Adams Resolution

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Every E-Adams resolution will give rise to an E-Adams tower, and thus an E-Adams spectral sequence.

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$$E^2 = \mathrm{Ext}^{s,t}_{E_* E}(\pi_* E, E_*(X)) \Rightarrow \pi_{s-t}(\widehat{X}).$$

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The canonical E-Adams resolution (augmented cosimplicial spectrum) is

$$0 \to X \to E \wedge X \xrightarrow{i_0} E \wedge E \wedge X \xrightarrow{i_1} E^{\wedge 3} \wedge X \xrightarrow{i_2} \cdots,$$

where $I_n = E^{\wedge(n+1)} \wedge X$, and $i_n = \sum_{i=0}^{n+1} \mathrm{id}^{\wedge i} \wedge e \wedge \mathrm{id}^{\wedge n+1-i} \wedge \mathrm{id}_X$.

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E^1-page of the Adams spectral sequence is the cobar complex

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The cohomology of this cobar complex is E^2-page, i.e.

$$E^2 = H^*(E^1) = \mathrm{Ext}^{s,t}_{E_*E}(\pi_* E, E_*(X))$$

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There is a canonical $\mathbb{S}_{W(k)}$-Adams resolution of $\mathbb{S}$ by

$$0 \to \mathbb{S} \to \mathbb{S}_{W(k)}[z] \to \mathbb{S}_{W(k)}[z_0, z_1] \to \mathbb{S}_{W(k)}[z]^{\otimes[2]} \to \mathbb{S}_{W(k)}[z]^{\otimes[3]} \to \cdots .$$

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Regard $\mathcal{O}_K$ as an $\mathbb{E}_\infty$-algebra over $\mathbb{S}_{W(k)}[z]^{\otimes[-]}$ via

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By the change-of-base formula, we get an augmented cosimplicial $\mathbb{E}_\infty$-cyclotomic spectrum:

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(k)}) \to \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]^{\otimes[-]}).$$

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The same story happens if we replace THH_* by TP_0.

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- $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z_0, z_1]) = \mathcal{O}_K[u_0] \otimes_{\mathcal{O}_K} \mathcal{O}_K \langle t_{z_0 - z_1} \rangle$, where $u_0 = \eta_L(u)$, and $t_{z_0 - z_1}$ is obtained by a variant of Hochschild-Konstant-Rosenberg theorem applying to $z_0 - z_1 \in \mathrm{HH}_2(\mathcal{O}_K/W(k)[z_0, z_1])$.

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- Antipode: $c(u_0) = u_1$, $c(u_1) = u_0$, $c(t_{z_0 - z_1}) = t_{z_1 - z_0}$.

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where $D_0 : t_{z_0 - z_1}^{[i]} \mapsto 1$ iff $i = 0$, and 0 else.

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Corollary

$$\mathrm{THH}_m(\mathcal{O}_K/\mathbb{S}_{W(k)}) = \begin{cases} \mathcal{O}_K, & m = 0; \\ \mathcal{O}_K/(nE'_K(\Pi_K)), & m = 2n-1; \\ 0, & \text{else.} \end{cases}$$

Outline

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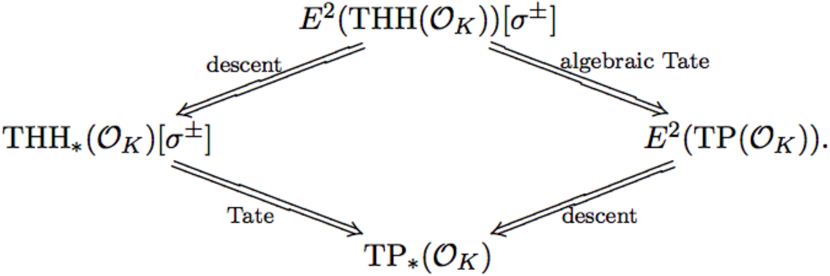

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In fact, in $\mathbb{F}_p$-coefficient, the associated graded of the (refined) Nygaard filtration on the Hopf algebroid above is given by

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The E^1-term of the algebraic HFPSS/Tate SS can be identified with the cobar complex for $k[z][\sigma^{\pm}]$ with respect to the Hopf algebroid as above. Note that $E^1(\mathrm{TC}^-)$ is just a truncation of $E^1(\mathrm{TP})$.

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Main Result

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Theorem (Liu-Wang, 2022)

Write $d = [K(\zeta_p) : K]$. As a $\mathbb{F}_p[\beta]$-module, where $\beta \in E^2_{0,2d}(\mathrm{TC})$ detects the Bott element, one has

$$\begin{aligned} \mathrm{TC}_*(\mathcal{O}_K; \mathbb{F}_p) = & \mathbb{F}_p[\beta]\{1, \gamma, \lambda, \lambda\gamma\} \oplus \\ & \mathbb{F}_p[\beta]\{\alpha^{(j)}_{i,\ell} \mid 1 \leq i \leq e_K, 1 \leq j \leq d, 1 \leq \ell \leq f_K\}, \end{aligned}$$

with $|\beta| = 2d$, $|\lambda| = -1$, $|\gamma| = 2d+1$, $|\alpha^{(j)}_{i,\ell}| = 2j-1$.

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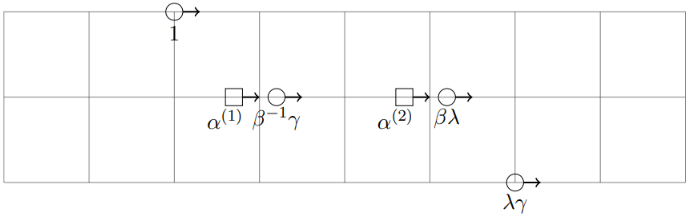

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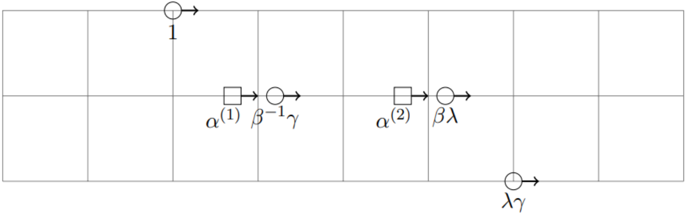
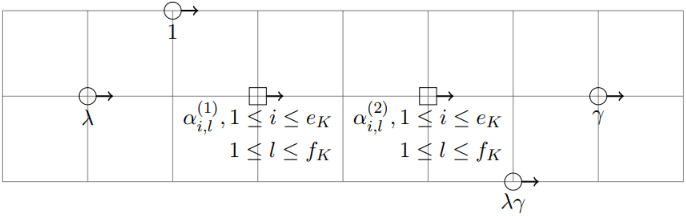


Descent SS for $\mathrm{TC}(\mathcal{O}_K; \mathbb{F}_p)$:



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Thank you!