THICK SUBCATEGORY THEORY

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Dec 10, 2022

Contents

1. Overview

The goal of the lectures is to state and prove the thick subcategory theorem, which is very crucial in chromatic homotopy theory.

Let X be a p-local finite spectrum. Recall that the type of X is the smallest integer such that $K(n)_*(X) \neq 0$, where $K(n)$ is the Morava K-theory. For $n \geq 0$, $K(n)$ satisfies some important properties that will be used throughout the note:

- $K(n)_*(X) = 0$ implies $K(n-1)_*(X) = 0$;
- $K(n)_*(X) = K(n)_* \otimes H_*(X;\mathbb{Z}/p)$;
- $K(n)_*(X) = \mathbb{Z}/p[v_n^{\pm}]$ and $K(0)_* = \mathbb{Q}$;
- Künneth isomorphism: $K(n)_*(X \times Y) = K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y)$.

Let $E_p = \bigvee_{n\geq 0} K(n)$ and $E = \bigvee_{p\in\mathcal{P}} E_p$, where $\mathcal{P}$ is the set of all primes. A spectrum is said to be *harmonic*, if it is E_*-local. It is *dissonant*, if it is E_*-acyclic. From the definition, it suffices to discuss only the $(E_p)_*$-locality and acyclicity for prime p. A standard result in [1, 4] showed that BP is harmonic. Moreover, Ravenel showed in [1, 4] that if X is connective spectrum of finite type, and $\operatorname{hom}\dim_{MU_*} MU_*X$ (the minimal length of a resolution of MU_*X by projective graded MU_*-modules) is finite, then X is harmonic. This implied that if X is finite, then X is harmonic. A direct consequence is that all non-trivial finite p-local spectrum X have a type. For example, the sphere spectrum $\mathbb{S}$ has type 0, and the mod p Moore spectrum $\mathbb{S}/p$ has type 1.

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Recall that the spectra in the previous example admit self maps (called v_n-self map). Explicitly, for odd prime p, we have

$$v_1 : \Sigma^{2(p-1)}\mathbb{S}/p \to \mathbb{S}/p.$$

It induces an equivalence in $K(1)_*$-homology, and $K(1)_*(v_1)$ is a multiplication by v_1. Denote the cofiber of v_1 by $\mathbb{S}/(p, v_1)$. When $p \geq 5$, there is a similar result by Smith and Adams:

$$v_2 : \Sigma^{2(p^2-1)}\mathbb{S}/(p, v_1) \to \mathbb{S}/(p, v_1).$$

This procedure leads us to

Proposition 1.1 (Periodicity). *Suppose X has type n. Then there exists an v_n-self map $v_n : \Sigma^N X \to X$ for some number $N = N(p)$, which induces a $K(n)_*$-homology, given by some multiplication by some p-th power of v_n.*

The periodicity proposition allows us to construct a considerate amount of families of elements in $\pi_*\mathbb{S}$. See Lecture 3 of my notes.

If we carefully study the previous process, we cannot ignore the strategy about building new homotopy types; that is, quotienting out $I_n = (p, v_1, v_2, cdots, v_n)$. Inspired by this, we have

Proposition 1.2 (Realizability). *We can realize any ideal $I \subset BP_*$ by following the strategy that quotienting out I_n from the sphere spectrum $\mathbb{S}$. By doing that, there is an associated v_n-self map as in the Periodicity proposition.*

Two immediate examples are the following:

Example 1.3. When $p = 3$, Behrens and Pemmaraju showed in [10] that

$$v_2^9 : \Sigma^{144}\mathbb{S}/(3, v_1) \to \mathbb{S}/(3, v_1),$$

with the associated v_2-self map inducing a $K(2)_*$-homology by multiplying v_2^9. When $p = 2$, Behrens, Hill, Hopkins, and Mahowald showed in [11] that

$$v_2^{32} : \Sigma^{192}\mathbb{S}/(2, v_1) \to \mathbb{S}/(2, v_1),$$

with the associated v_2-self map inducing a $K(2)_*$-homology by multiplying v_2^{32}.

In general, it is not clear that $\mathbb{S}/(p, v_1)$ admits an v_2-self map at any prime p that is nilpotent. A natural question is if there is some spectrum X that could detect such nilpotence, i.e. its kernel of its Hurewicz map $\pi_*X \to E_*X$ consists of nilpotent elements.

Proposition 1.4 (Nilpotence). *$X = MU$.*

A remarkable corollary of the nilpotence proposition is the Nishida's theorem [6], which said that any element $\alpha \in \pi_k\mathbb{S}$, $k > 0$, is nilpotent. Applying MU to $\mathbb{S}$ yields $MU_* = \mathbb{Z}[b_1, b_2, \cdots]$. This is torsion-free. However, $\pi_k\mathbb{S}$ has torsion by Serre finiteness theorem. One showed that the Hurewicz map is zero when $k > 0$. So, every element of positive degree is nilpotent.

Let's switch to the geometric viewpoint. Let $W \to X \to Y$ be a cofiber sequence, and $f : Y \to \Sigma W$ be the connecting map which is null in MU_*-homology. By a result in Bousfield class, $\langle X\rangle = \langle W\rangle \vee \langle Y\rangle$. Informally speaking, $\langle X\rangle$, the Bousfield class of X, is the set $\{X \in \mathsf{Sp} : E \wedge X = 0\}$. The famous class invariance theorem says

Proposition 1.5 (Class invariance)**.** *Let X, Y be p-local finite spectra of type m, n, respectively. Then $\langle X \rangle = \langle Y \rangle$ iff $m = n$.*

Class invariance proposition immediately leads to the telescope conjecture. Let X be a spectrum of type n, $f : X \to \Sigma^{-k} X$ be a self map, and $f^{-1}X = \operatorname{colim}(X \xrightarrow{f} \Sigma^{-k} X \xrightarrow{f} \Sigma^{-2k} X \xrightarrow{f} \cdots)$ be the invert of X w.r.t. f.

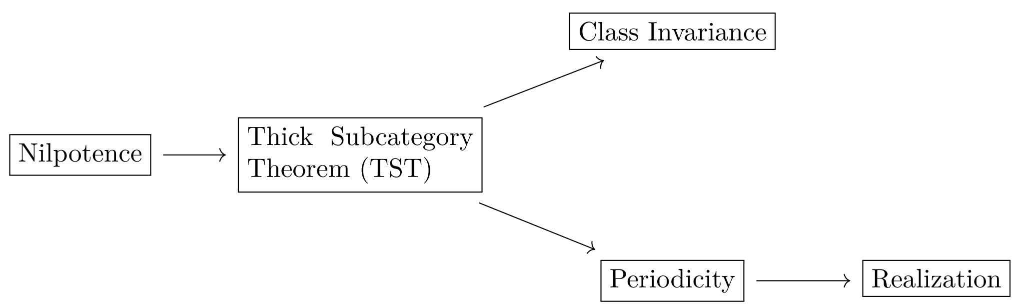
Conjecture 1.6 (Telescope)**.** $\langle f^{-1}X \rangle$ *depends only on n, and*

$$\left\langle f^{-1}X \right\rangle = \bigvee \langle K(n) \rangle .$$

The telescope conjecture plays an important role in stating the smashing conjecture; that is, $L_n X \simeq X \wedge L_n \mathbb{S}$, where L_n is the localization w.r.t. $E(n) = v_n^{-1} BP/(v_{n+1}, v_{n+2}, \cdots)$, the Johnson-Wilson theory. Unfortunately, this conjecture still remains open today.

The main references for this section are [12, 1] and [1, 10].

The work of Devinatz-Hopkins-Smith [13] finished the proof of the nilpotence proposition (which now called the nilpotence theorem), the heart of the chromatic homotopy theory. One can use a picture to show the logical sequence of the propositions mentioned above (except for the telescope conjecture):



Our task for this note is to prove the thick subcategory theorem (TST). This is an extremely important theorem that will be the key to prove the periodicity theorem and the class invariance theorem. It enables us to focus only on some much manageable subcategories of category of p-local spectra (called *thick*) by ruling out the spectra of type $\geq n$. The structure of the note goes as follows:

(1) Setting the base: two important categories of interest $\mathcal{C}\Gamma$ and FH, and definition of thick subcategories.
(2) Algebraic TST:
   - A generalization to $\mathcal{P}(n)$, category of all $P(n)_* P(n)$-comodule finitely presented as $P(n)_*$-module.
   - Proof of algebraic TST.
(3) Geometric TST:
   - Geometric background: Sp, Spanier-Whitehead duality.
   - Statement of nilpotence theorem, smash product form, and its connection to the classical nilpotence theorem.
   - Statement of the key corollary.
   - Bousfield class theory and class invariance theorem.
   - Proof of the key corollary and the TST.

2. Setting the base

In this section, we will introduce two main categories of interest, and illustrate what thickness means for their subcategories. Before we start the discussion, we need to introduce an action on the Lazard ring L.

Definition 2.1. Let Γ be the group of power series

$$\Gamma = \{\gamma = x + b_1 x^2 + b_2 x^3 + \cdots : b_i \in \mathbb{Z}, i \geq 1\},$$

where the group operation is the composition.

Let $G(x, y)$ be the universal formal group law (abbr. fgl) over L. By a theorem of Mischenko,

$$\log_G(x) = \sum_{i \geq 0} m_i x^{i+1},$$

where $m_n = [\mathbb{CP}^n]/(n+1) \in \pi_{2n}(MU) \otimes \mathbb{Q}$, and $[\mathbb{CP}^n]$ is the cobordism class represented by $\mathbb{CP}^n$.

Let $\gamma \in \Gamma$. $\gamma^{-1}(G(\gamma(x)), \gamma(y))$ is then another fgl over L, inducing by an endomorphism ϕ of L. Since γ is invertible in $\mathbb{Z}[[x]]$, ϕ is an automorphism. So there is a natural Γ-action on L. One can compare this action to the Steenrod algebra in ordinary cohomology. Namely, it is analogous to the total Steenrod square $\sum \mathrm{Sq}^n$ in mod 2 case.

Write $\mathcal{C}\Gamma$ and FH to denote the category of finitely presented graded L-modules with compatible Γ-action and category of finite CW complexes and homotopy classes of maps between them, respectively. After p-localization, we denote the output of these categories by $\mathcal{C}\Gamma_{(p)}$ and $\mathsf{FH}_{(p)}$, respectively.

Let $v_n \in L$ be the coefficient of x^{p^n} in the p-series for $G(x, y)$. It serves as a polynomial generator of L in dimension $2(p^n - 1)$. Once we localize at p, v_n's only polynomial generators of $L = MU_*$ that matter. Therefore, we can drop the "redundant elements" by tensoring $\mathbb{Z}_{(p)}$ and work in BP instead of MU. However, one should be warned that there is no analogue of Γ-action on $BP = L \otimes \mathbb{Z}_p$. One must replace Γ by certain groupoid and transplant the whole story onto Hopf algebroid, see [3, B.5].

Now we introduce the thickness and give a description of TST.

Definition 2.2. A full subcategory $\mathcal{C}$ of $\mathcal{C}\Gamma$ (or $\mathcal{C}\Gamma_{(p)}$) is **thick** if it satisfies: if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence in $\mathcal{C}$ of $\mathcal{C}\Gamma$ (or $\mathcal{C}\Gamma_{(p)}$), then $M \in \mathcal{C}$ iff $M', M'' \in \mathcal{C}$.

Correspondingly,

Definition 2.3. A full subcategory $\mathcal{D}$ of FH (or $\mathsf{FH}_{(p)}$) is **thick** if it satisfies:

(1) If $X \xrightarrow{f} Y \to C_f$ is a cofiber sequence in FH (or $\mathsf{FH}_{(p)}$), and any two of them are in $\mathcal{D}$, then the rest is also in $\mathcal{D}$.
(2) If $X \vee Y \in \mathcal{D}$, then both X and Y are in $\mathcal{D}$.

They leads to the two main theorems central to the note. Fix a prime p.

Theorem 2.4 (Algebraic TST)**.** *Denote $\mathcal{C}_n$ as the full subcategory of $\mathcal{C}\Gamma_{(p)}$ satisfying $v_n^{-1} M = 0$ for $M \in \mathcal{C}\Gamma_{(p)}$ (clearly $\mathcal{C}_0 = \mathcal{C}\Gamma_{(p)}$). Let $\mathcal{C}$ be the thick subcategory of $\mathcal{C}\Gamma_{(p)}$. Then $\mathcal{C} = \mathcal{C}_n$ for some $n \geq 0$.*

Likewise,

Theorem 2.5 (Geometric TST). *Denote $\mathcal{D}_n$ as the full subcategory of $\mathsf{FH}_{(p)}$ satisfying $v_n^{-1}BP_*(X) = 0$ for $X \in \mathsf{FH}_{(p)}$ (clearly $\mathcal{D}_0 = \mathsf{FH}_{(p)}$). Let $\mathcal{D}$ be the thick subcategory of $\mathsf{FH}_{(p)}$. Then $\mathcal{D} = \mathcal{D}_n$ for some $n \geq 0$.*

There are two nested sequences of thick subcategories:

$$\begin{aligned} \mathcal{C}\Gamma_{(p)} &= \mathcal{C}_0 \supset \mathcal{C}_1 \supset \mathcal{C}_2 \supset \cdots, \\ \mathsf{FH}_{(p)} &= \mathcal{D}_0 \supset \mathcal{D}_1 \supset \mathcal{D}_2 \supset \cdots. \end{aligned}$$

Mitchell [7] proved that the inclusions above are strict. So BP_* (or MU_* if you prefer) sends each $\mathcal{D}_n$ to $\mathcal{C}_n$. The next section aims to prove the algebraic TST, which requires less tools to accomplish.

3. Algebraic thick subcategory theorem

The algebraic version of TST asks little knowledge in geometry and can be generalized pure algebraically in an easy manner. The main reference for this part is [4]. Fix a prime p.

Recall that there are several spectra related to BP. One is the Johnson-Wilson theory spectrum $BP\langle n\rangle$, which is obtained from BP by killing the ideal $(v_{n+1}, v_{n+2}, \cdots) \subset BP_*$. It is clear $BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n]$ and $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$, the Eilenberg-Mac Lane spectrum associated with $Z_{(p)}$. It has an associated fibration

$$\Sigma^{2(p^n-1)}BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \to BP\langle n+1\rangle.$$

If we invert the v_n-self map, i.e. taking the colimit of $v_n^\ell : BP\langle n\rangle \to \Sigma^{-2\ell(p^n-1)}BP\langle n\rangle$, we get a new spectrum $E(n)$ (not to confuse with the Morava E-theory!). We have $E(0) = H\mathbb{Q}$ and $E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n, v_n^{-1}]$ as expected. In fact, $BP\langle 1\rangle$ and $E(1)$ are summands of the connective and periodic complex K-theory localized at p.

Another relevant spectrum is $k(n)$, which is obtained from BP by killing the ideal $(p, v_1, v_2, \cdots, v_{n-1}, v_{n+1}, v_{n+2}, \cdots) \subset BP_*$. So $k(n)_* = \mathbb{Z}_{(p)}[v_n]$ without doubt. It also has an associated fibration

$$\Sigma^{2(p^n-1)}k(n) \xrightarrow{v_n} k(n) \to H\mathbb{Z}_{(p)}.$$

After inverting the v_n-self map, we obtain the Morava K-theory $K(n) = v_n^{-1}k(n)$ with $K(0) = H\mathbb{Q}$. Similarly, if we kill the ideal $(p, v_1, v_2, \cdots, v_{n-1}) \subset BP_*$, we can obtain a new spectrum $P(n)$ with $P(0) = BP$. The v_n-self map again induces an fibration

$$\Sigma^{2(p^n-1)}P(n) \xrightarrow{v_n} P(n) \to P(n+1).$$

Inverting v_n on $P(n)$, we get a new spectrum $B(n) = v_n^{-1}P(n)$.

One can regard $\mathcal{C}\Gamma_{(p)}$ as the abelian category of all BP_*BP-comodules finitely presented as BP_*-modules. See [1, 7]. If we replace $\mathcal{C}\Gamma_{(p)}$ by $\mathcal{P}(n)$, the abelian category of all $P(n)_*P(n)$-comodules finitely presented as $P(n)_*$-modules, and $\mathcal{C}_\|$ by $\mathcal{P}(n)_k$ in the statement of TST, we get a generalized TST as follows:

Theorem 3.1 (Generlized algebraic TST, version 1). *If $\mathcal{C}$ is a thick subcategory of $\mathcal{P}(n)$, then $\mathcal{C} = \mathcal{P}(n)_k$ for some $k \geq n$.*

In particular, if $n = 0$, it is identical to the classical algebraic TST.

One might further generalize this theorem. Let E_* be the commutative $P(n)_*$-algebra such that $E_* \otimes_{P(n)_*} (-)$ us exact on $\mathcal{P}(n)$. By Landweber exact functor theorem, $E_* \otimes_{P(n)_*} P(n)_*(-)$ is a homology theory [9]. Let $E_*E = E_* \otimes_{P(n)_*} P(n)_*P(n) \otimes_{P(n)_*} E_*$. We can make (E_*, E_*E) a Hopf algebroid by extending the structure maps of ones in $P(n)_*P(n)$. Since $P(n)_*P(n)$ is flat $P(n)_*$-module, and for every E_*-module N,

$$\begin{aligned} E_*E \otimes_{E_*} N &= E_* \otimes_{P(n)_*} P(n)_*P(n) \otimes_{P(n)_*} E_* \otimes_{E_*} N \\ &= E_* \otimes_{P(n)_*} P(n)_*P(n) \otimes_{P(n)_*} N, \end{aligned}$$

E_*E is then a flat E_*-module. Let M be an object in $\mathcal{P}(n)$. $E_* \otimes_{P(n)_*} M$ is an E_*E-comodule because

$$M \to P(n)_*P(n) \otimes_{P(n)_*} M \xrightarrow{\text{extension}} E_*E \otimes_{P(n)_*} M \to E_*E \otimes_{E_*} (E_* \otimes_{P(n)_*} M).$$

Let $\mathscr{S}$ be the category of $E_* \otimes_{P(n)_*} M$, where $M \in \mathcal{P}(n)$. The morphisms are given by $E_* \otimes f : E_* \otimes_{P(n)_*} M_1 \to E_* \otimes_{P(n)_*} M_2$, for $f : M_1 \to M_2 \in \mathcal{P}(n)$. It can be shown that $\mathscr{S}$ is abelian, and $E_* \otimes_{P(n)_*} (-) : \mathcal{P}(n) \to \mathscr{S}$ is exact. Write $\mathscr{S}_k = (E_* \otimes_{P(n)_*} (-))(\mathcal{P}(n)_k)$. As in the classical case, there is a nested sequence of categories:

$$\cdots \subset \mathscr{S}_{k+1} \subset \mathscr{S}_k \subset \cdots \subset \mathscr{S}_n = \mathscr{S}.$$

In contrast to the classical case, there is no evidence that the inclusions are strict. However, this is not important for our concern.

Theorem 3.2 (Generlized algebraic TST, version 2)**.** *If $\mathcal{C}$ is a thick subcategory of $\mathscr{S}$, then $\mathcal{C} = \mathscr{S}_k$ for some $k \geq n$.*

The proof of the generalized version of algebraic TST is identical to the one we are presenting now. The main tool to tackle the problem is the Landweber Filtration Theorem. We will take the validity of this theorem for granted. For interested readers, see [15].

Theorem 3.3 (Landweber Filtration Theorem)**.** *Any module M in $\mathcal{C}\Gamma_{(p)}$ admits a finite filtration*

$$0 = M_s \subset \cdots \subset M_1 \subset M_0 = M,$$

such that for any $0 \leq i \leq s-1$, we have $M_i/M_{i+1} \xrightarrow[\text{stable}]{\cong} BP_*/I_{n_i}$, where $I_{n_i} = (p, v_1, \cdots, v_{n_i-1})$ is the invariant prime ideal of BP, and "stable" means isomorphism eventually (i.e. after a dimension shift).

Here the term "invariant prime ideal" comes from a theorem by Landweber and Morava. Explicitly,

Theorem 3.4 (Morava-Landweber)**.** *The only invariant prime ideals in BP_* are $I_n = (p, v_1, v_2, \cdots, v_n)$ for $0 \leq n \leq \infty$. There is a short exact sequence of comodules*

$$0 \to \Sigma^{2(p^n-1)} BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0.$$

The proof of the theorem can be found in [14] and [2, 4.3].

Remark 3.5. The original statement of the theorem by Morava and Landweber was that the only Γ-invariant prime ideals in MU_*. However, there is no well-defined candidate for analogous Γ-action on BP_*. So the definition of invariance used in 3.3 goes as follows: a regular ideal $(x_1, x_2, \cdots, x_k)$ is *invariant* in BP_* if the sequence

$$0 \to BP_*/(x_1, x_2, \cdots, x_k) \xrightarrow{x_{k+1}} BP_*/(x_1, x_2, \cdots, x_k) \to BP_*/(x_1, x_2, \cdots, x_{k+1}) \to 0$$

is exact as comodules. This is slightly different from the one in the original statement. See [2, 4.3].

Let A be commutative unital ring. Recall that an A-module M is finitely presented if there is a surjective A-module map $\varphi : A^q \to M$ with a finite $\ker \varphi$ for some $q \geq 0$. A finitely presented module over a colimit of Noetherian rings admits filtrations. Notice that a finite generated M module over a Noetherian ring R admits a finite filtration

$$0 = F_0 M \subset F_1 M \subset \cdots \subset F_k M = M,$$

and $F_i M / F_{i-1} M \cong R/I_i$ for each prime ideal $I_i \subset R$, $1 \leq i \leq k$. Combining the fact that MU_* is a colimit of Noetherian rings implies Theorem 3.3.

Before we start to prove Theorem 2.4, we need one more corollary of Theorem 3.3.

Corollary 3.6. *Let $M \in \mathcal{C}\Gamma_{(p)}$. Then $v_n^{-1} M = 0$ implies $v_{n-1}^{-1} M = 0$ for all n.*

Proof. By Theorem 3.3, M admits a finite filtration

$$0 = M_s \subset \cdots \subset M_1 \subset M_0 = M$$

with $M_i/M_{i+1} \cong BP_*/I_m$ eventually, for $m > n$. $v_n^{-1} M = 0$ implies that any element $x \in M$ satisfies $v_n^{\ell_1}(x) = 0$ for some $\ell_1 \geq 1$. This x must fall into some M_i/M_{i+1}. Since $v_{n-1} \in I_m$ and I_m is invariant, it follows that $v_{n-1}^{\ell_2}(x) = 0$ for some $\ell_2 \geq 1$. Hence, $v_{n-1}^{-1} M = 0$. □

Proof of Theorem 2.4. Choose an arbitrary object $M \in \mathcal{C}\Gamma_{(p)}$. Define $\|M\| = \{m \geq 1 : v_{m-1}^{-1} M = 0\} \cup \{0\}$. We make the convention that $v_0 = p$. If $M \neq 0$, then $\|M\| = \{0, 1, 2, \cdots, N_M\}$ for some $N_M \geq 0$. Let $\mathcal{C}$ be a thick subcategory of $\mathcal{C}\Gamma_{(p)}$, and $k = \max \cap_{M \in \mathcal{C}} \|M\|$. By definition of k, it is clear that $\mathcal{C} \subset \mathcal{C}_k$ and $\mathcal{C} \not\subset \mathcal{C}_{k+1}$. It suffices to show that $\mathcal{C} \supset \mathcal{C}_k$.

Let $M \in \mathcal{C}_k$. M admits a filtration

$$0 = M_s \subset \cdots \subset M_1 \subset M_0 = M$$

by Theorem 3.3. For each i, there is an associated short exact sequence

$$0 \to M_{i+1} \to M_i \to M_i/M_{i+1} \to 0.$$

By induction on i, we know that all M_i and $M_i/M_{i+1} \cong BP_*/I_{n_i}$ are in $\mathcal{C}$. Since localization is an exact functor, $v_{k-1}^{-1} M_i = v_{k-1}^{-1} M_i/M_{i+1} = 0$. It is straightforward that $n_i \geq k$ holds for all $0 \leq i \leq s-1$. On the other hand, $v_k^{-1} M \neq 0$ implies that there is some $j \in [0, s-1]$ such that $v_k^{-1} BP_*/I_{n_j}$. So $n_j \leq k$. It follows that $n_j = k$, and $BP_*/I_k \in \mathcal{C}$. Consider the short exact sequence in $\mathcal{C}\Gamma_{(p)}$:

$$0 \to BP_*/I_{k+r} \xrightarrow{v_{k+r}} BP_*/I_{k+r} \to BP_*/I_{k+r+1} \to 0.$$

Inductively, we see that all $BP_*/I_{k+r} \in \mathcal{C}$ for $r \geq 0$.

We are ready to give the last shot to our problem. The previous discussion indicates that, inductively, for each associated short exact sequence

$$0 \to M_{i+1} \to M_i \to M_i/M_{i+1} \to 0,$$

we have $M_i/M_{i+1} \cong BP_*/I_{n_i} \in \mathcal{C}$ ($n_i \geq k$) and $M_{i+1} \in \mathcal{C}$ (inductive hypothesis). This yields that $M_i \in \mathcal{C}$ for all i. Thus, $M \in \mathcal{C}$, and hence $\mathcal{C}_k \subset \mathcal{C}$. □

4. Geometric background

The proof of geometric TST requires a background in Spanier-Whitehead duality and Bousfield classes. We will discuss them in details in this section. The readers are assumed to have a fair understanding of the category of spectra Sp.

4.1. Spanier-Whitehead duality.

Let X be a finite spectrum. We denote its Spanier-Whitehead (S-W) duality by DX (unique up to homotopy), which satisfies the following properties:

Theorem 4.1. *(1) $D^2X = D(DX) \simeq X$, and $[X,Y]_* \cong [DY, DX]_*$.*
(2) For any homology theory E_*, there is an natural isomorphism $E_kX \cong E^{-k}DX$.
(3) $D(-)$ is a contravariant functor.
(4) For every finite spectrum Y, $D(X \wedge Y) = DX \wedge DY$. In particular, $D\Sigma X = \Sigma^{-1}DX$.
(5) For any spectrum Y, there is an isomorphism $[X,Y]_* \cong \pi_*(DX \wedge Y)$ that is natural in X and Y.
(6) $K(n)_*(X)$ is free over $K(n)_*$, and

$$\hom_{K(n)_*}(K(n)_*(X), K(n)_*(Y)) \cong K(n)_*(DX \wedge Y).$$

The isomorphism still holds if we replace $K(n)_*$ by the ordinary homology H_*.

To keep in mind, the slogan of S-W duality is to rephrase the linear dual to a vector space over some algebraic closed field in the category Sp. For example, one see statement 1 in Theorem 4.1 as a reformulation of $(V^*)^* \cong V$. 4 in Theorem 4.1 is similar to $(V \otimes W)^* \cong V^* \otimes W^*$, and 5 in Theorem 4.1 is similar to $\hom(V,W) \cong V^* \otimes W$.

A proof of the theorem can be found in [8, III.5], omitted in our context for simplicity. Instead, we would like to demonstrate some of its geometric intuitions. Again, you may find rich content in [8, III.5].

The origin of S-W duality came from the Alexander duality. If $X \subset S^n$ is a compact, locally contractible subspace, and $X \neq S^n, \varnothing$, then Alexander duality says $\overline{H}^{n-i-1}(X;\mathbb{Z}) \cong \overline{H}_i(S^n - X;\mathbb{Z})$, where $0 \leq i \leq n-1$ and $\overline{H}$ is the reduced homology. From the isomorphism, one concludes that X determines the homology of its complement in S^n. However, there is nothing to say about the homotopy type of $S^n - X$. For instance, if we fix $n = 3$ and $X = S^1$, then $K = S^3 - X$ is a knot. In addition to the trivial embedding as a rigid circle S^1, the knot diagram can also be either the (right-handed) trefoil 3_1

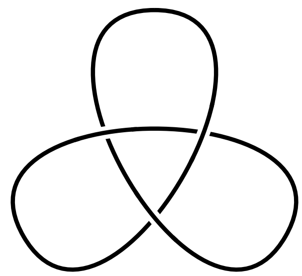


Figure 1. The (right-handed) trefoil 3_1.

or the left-handed trefoil 3_2, which is the mirror of 3_1,

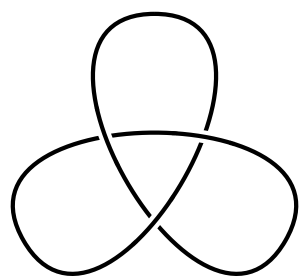


Figure 2. The left-handed trefoil 3_2.

In 1914, Dehn showed that 3_1 is chiral (i.e. $3_1 \not\simeq 3_2$) (see this post), so they cannot be transformed into each other by Reidemeister moves. Their geometric properties (e.g. Jones polynomial) are quite different. Passing to π_1, one shows $\pi_1(S^3 - K) = \langle x, y \mid x^3 = y^2 \rangle$ by Wirtinger presentation. This is far from the fundamental group of $S^3 - K$ if we choose K embedded as a rigid circle S^1.

It is for those and other similar and compelling reasons that the homotopy type of K depends on both X and the embedding. So how far does a bare condition on X determine anything about K beyond its homology?

Let $X \subset S^n$ be a compact and locally contractible subspace. Embed S^n as an equatorial sphere in S^{n+1}, and ΣX in S^{n+1} by joining to the two poles. It is clear that $S^{n+1} - \Sigma X \simeq S^n - X$. If $X \subset S^n$ and $Y \subset Y^m$ and $f : \Sigma^p X \xrightarrow{\simeq} \Sigma^q Y$ for some $p, q \geq 0$, one can embed $\Sigma^p X$ in S^{n+p} and $\Sigma^q Y$ in S^{m+q} simultaneously without changing the equivalence and the complements.

From now on, we assume the equivalence between spaces are piecewise linear (PL). Suppose $X' \subset S^{n'}$. Embed $S^{n'}$ as an equatorial sphere in $S^{n'+1}$ as before without changing X'. Then the complement of X' in $S^{n'+1}$ is the suspension of that in $S^{n'}$. Recall that the *join* of two space x and Y, denoted $X * Y$, is the line segments joining the points in X and Y. That is, $X * Y = X \times Y \times I/\{(x, y_1, 0) \simeq (x, y_2, 0), (x_1, y, 0) \simeq (x_2, y, 0)\}$. An easy exercise in undergraduate topology course shows that $S^{n'} * S^{m'} = S^{m'+n'+1}$. Let $Y' \subset S^{m'}$, $f' : X' \to Y'$ be a homotopy equivalence, and M be the mapping cylinder of f'. In the total sphere $S^{m'+n'+1}$ one has $S^{m'+n'+1} - X = \Sigma^{m'+1}(S^{n'} - X)$ and $S^{m'+n'+1} - Y = \Sigma^{n'+1}(S^{m'} - Y)$. The maps

$$S^{m'+n'+1} - X \xleftarrow{f} S^{m'+n'+1} - M \xrightarrow{g} S^{m'+n'+1} - Y$$

induced by the inclusions $X \xrightarrow{f} M \xleftarrow{g} Y$, induce isomorphisms in homology. By Alexander duality, they induce isomorphisms of cohomology. We can further embed $S^{m'}$ and $S^{n'}$ into higher dimensional spheres without changing the homotopy type of X and Y in the total sphere. Iterate the process until everything is simply-connected, the isomorphisms will hold and the homotopy type remains unchanged. This means that X, or even the stable homotopy type of X, determines the stable homotopy type of its complement.

The preceding recipe still depends on the embedding. The next step is to eliminate the effect of the embedding. Let K, L be two disjoint finite simplicial complexes embedded in S^n. Let $\mathfrak{L}$ be a PL path from K to L joining only one points in each of them, say $x \in K$ and $y \in L$. Take some point in the middle of the path as the point at ∞. We can now embed R and L into R^n by defining a map $\mu : K \times L \to S^{n-1}$ with $\mu(k, \ell) = (k - \ell)/\|k - \ell\|$. Since the restrictions of μ at $\{x\} \times L$ and $K \times \{L\}$ are null-homotopic, we get a map $\mu : K \wedge L \to S^{n-1}$.

Using the framework we have, we can set up everything in Sp. Let X be a finite spectrum, then we can form $[W \wedge X, \mathbb{S}]_0$, which is a Brown functor of W. By Brown representability, there is a unique spectrum DX up to homotopy such that

$$[W \wedge X, \mathbb{S}]_0 \xrightarrow[\cong]{\Phi} [W, DX]_0.$$

The isomorphism actually holds for all $[-,-]_*$. Take $W = DX$ and $\mathrm{id} : DX \to DX$, there is a map called *unit* $\eta = \Phi(\mathrm{id}) : DX \wedge X \to \mathbb{S}$. Dually, we have the *counit* $e : \mathbb{S} \to DX \wedge X$. For any $f : W \to DX$, we have

$$W \wedge X \xrightarrow{f \wedge \mathrm{id}} DX \wedge X \xrightarrow{\eta} \mathbb{S}.$$

Take $g : X \to Y$. It induces $g^* : DY \to DX$ with

$$\begin{array}{ccc} DY \wedge X & \xrightarrow{\mathrm{id} \wedge g} & DY \wedge Y \\ {\scriptstyle g^* \wedge \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \eta_Y} \\ DX \wedge X & \xrightarrow{\eta_X} & \mathbb{S} \end{array}$$

which further induces a natural transformation

$$\begin{array}{ccc} [W \wedge X, \mathbb{S}]_* & \xleftarrow{(\mathrm{id} \wedge g)^*} & [W \wedge Y, \mathbb{S}]_* \\ \cong \downarrow & & \downarrow \cong \\ [W, DX]_* & \xleftarrow{g^*} & [W, DY]_* \end{array}$$

It follows that we have an adjoint pair as expected: if W, X are finite spectra, then $[W, Z \wedge DX]_* \cong [W \wedge X, Z]_*$ for any spectrum Z.

We can directly check statement 1 in Theorem 4.1 as follows: since the smash product is commutative, the result follows from

$$[X, D^2X] \xrightarrow{\cong} [X \wedge DX, \mathbb{S}] = [DX \wedge X, \mathbb{S}] \xrightarrow{\cong} [DX, DX].$$

Adams [8, III.5] showed that if X is finite, then so is DX. The existence of DX is easy. Since every finite CW complexes can be built up by attaching spheres along the attaching maps, one can build the dual of finite CW complexes via the obvious relation $DS^n = S^{-n}$ (exercise!). By $D\Sigma X = \Sigma^{-1}DX$, the dual of finite spectra exist.

The statement 4 in Theorem 4.1 can also be checked easily. This is because $[W, DX \wedge DY]_* \cong [W \wedge Y, DX]_* \cong [W \wedge Y \wedge X, \mathbb{S}]_* \cong [W, D(Y \wedge X)]_*$, indicates $DX \wedge DY \cong D(X \wedge Y)$. Note that $\eta_{X\wedge Y} : D(X \wedge Y) \wedge (X \wedge Y) \to \mathbb{S}$ is the same as $\eta_X \wedge \eta_Y : DX \wedge X \wedge DY \wedge Y \to \mathbb{S} \wedge \mathbb{S} = \mathbb{S}$. Thus, we have a commutative diagram

$$\begin{array}{ccc} D(X\wedge Y)\wedge X\wedge Y & \xrightarrow{\eta_{X\wedge Y}} & \mathbb{S} \\ {\scriptstyle\cong}\uparrow & & \uparrow{\scriptstyle \eta_X\wedge\eta_Y} \\ DX\wedge DY\wedge X\wedge Y & \xrightarrow{=} & DX\wedge X\wedge DY\wedge Y \end{array}$$

We need one more lemma that is important in the proof of TST. Let W, X be finite spectra, and $f : W \to \mathbb{S}$ fit into a cofiber sequence $W \xrightarrow{f} \mathbb{S} \xrightarrow{e} DX \wedge X$. Such a map always exists in Sp. Let $C_f = C_{f^{(1)}} = DX \wedge X$ be the mapping cone.

Lemma 4.2. *For $k > 1$, then there is a cofiber sequence*

$$C_{f^{(k)}} \to C_{f^{(k-1)}} \to \Sigma W^{(k-1)} \wedge C_f,$$

where $f^{(k)}$ denotes the k-th smash of f, i.e. $f^{(k)} : W^{(k)} = \underbrace{W \wedge \cdots \wedge W}_{k} \to \underbrace{X \wedge \cdots \wedge X}_{k} = X^{(k)}$.

Proof. Recall that in classical homotopy theory, for any sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is a commutative diagram

$$\begin{array}{ccccc} C_f & \longrightarrow & * & \longrightarrow & \Sigma C_f \\ \uparrow & & \uparrow & & \uparrow \\ Y & \xrightarrow{g} & Z & \longrightarrow & C_g \\ {\scriptstyle f}\uparrow & & {\scriptstyle =}\uparrow & & \uparrow \\ X & \xrightarrow{gf} & Z & \longrightarrow & C_{gf} \end{array}$$

such that each column and row is a cofiber sequence. Let $X = W^{(k)}$, $Y = W^{(k-1)}$, $Z = \mathbb{S}$, and $g = f^{(k-1)}$, we have

$$\begin{array}{ccccc} W^{(k-1)}\wedge C_f & \longrightarrow & * & \longrightarrow & \Sigma W^{(k-1)}\wedge C_f \\ \uparrow & & \uparrow & & \uparrow \\ W^{(k-1)}\wedge \mathbb{S} = W^{(k-1)} & \xrightarrow{f^{(k-1)}} & \mathbb{S} & \longrightarrow & C_{f^{(k-1)}} \\ {\scriptstyle f}\uparrow & & {\scriptstyle =}\uparrow & & \uparrow \\ W^{(k-1)}\wedge W = W^{(k)} & \xrightarrow{f^{(k)}} & \mathbb{S} & \longrightarrow & C_{f^{(k)}} \end{array}$$

The last column is the desired cofiber sequence. □

4.2. **Bousfield localization.** Let E_* be a generalized homology theory. Recall that Y is E_*-*local* if for any spectrum X with $E_*X = 0$, $[X, Y]_* = 0$. An E_*-*localization* of X is a map $\eta : X \to L_E X$, where $L_E X$ is E_*-local, such that $E_*\eta$ is an isomorphism.

Proposition 4.3. *(1) For any cofiber sequence $W \to X \to Y$, if any two of the ingredients are E_*-local, then the rest is also E_*-local.*
(2) If $X \vee Y$ is E_*-local, then so are X and Y.

One might wish to compare this proposition with the definition of thick subcategories. Be warned that the colimit of a direct system of E_*-local spectra needs not to be E_*-local.

Theorem 4.4 (Bousfield localization)**.** *For any homology theory E_* and any spectrum X, there is a unique (up to homotopy) E_*-localization $L_E X$. In other words, $L_E(-)$ is functorial.*

Lemma 4.5. *Let E be a ring spectrum, then $E \wedge X$ is E_*-local for all $X \in \mathsf{Sp}$.*

Proof. We want to show that for all $W \in \mathsf{Sp}$, $E_*W = 0$, we have $[W, E \wedge X] = 0$. Choose an arbitrary map $f : W \to E \wedge X$. Denote the multiplication and unit of E by $m : E \wedge E \to E$ and $\eta : \mathbb{S} \to E$, respectively. Then from the commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{f} & E \wedge X & & \\ \downarrow{\scriptstyle \eta\wedge\mathrm{id}} & & \downarrow{\scriptstyle \eta\wedge\mathrm{id}} & \searrow^{\mathrm{id}} & \\ E \wedge W & \xrightarrow{\mathrm{id}\wedge f} & E \wedge E \wedge X & \xrightarrow{m\wedge\mathrm{id}} & E \wedge X \end{array}$$

we conclude $E \wedge W = 0$. So f is null-homotopic. □

We introduce some new concepts that will not be used in our context. They have good taste in the general picture described in §1.

Definition 4.6. Let E be a ring spectrum. Define an associated new class $\mathscr{A}$ that satisfies:

(1) $E \in \mathscr{A}$.
(2) If $N \in \mathscr{A}$. then $N \wedge X \in \mathscr{A}$ for all $X \in \mathsf{Sp}$.
(3) For any $f : X \to Y$ with $X, Y \in \mathscr{A}$, the cofiber of f is also in $\mathscr{A}$.
(4) Any retract of the element in $\mathscr{A}$ is also in $\mathscr{A}$.

Elements in $\mathscr{A}$ are called E_***-nilpotent**. If $X \in \mathsf{Sp}$ is E_*-equivalent to an E_*-nilpotent spectrum, then X is called E_***-prenilpotent**.

From the definition, we can immediately derive a corollary:

Corollary 4.7. *Any E_*-nilpotent spectrum is E_*-local.*

Definition 4.8. Let E, F be spectra. We say they are **Bousfield equivalent**, if for any spectrum X, $E \wedge X$ being contractible implies $F \wedge X$ being contractible, and vice versa. One can show that Bousfield equivalence is an equivalence relation. Denote the Bousfield equivalence class of E by $\langle E \rangle$, called the **Bousfield class** of E. We use the notation $\langle E \rangle \geq \langle F \rangle$ to indicate that the contractibility of $E \wedge X$ implies that of $F \wedge X$ for any $X \in \mathsf{Sp}$, and $\langle E \rangle > \langle F \rangle$ if $\langle E \rangle \geq \langle F \rangle$ but $\langle E \rangle \neq \langle F \rangle$.

Remark 4.9. Alternatively, one sees that

$$\langle E\rangle = \{X \in \mathsf{Sp} : E \wedge X = 0\}.$$

Therefore, $\langle E\rangle \geq \langle F\rangle$ can be interpreted as $E \wedge X = 0$ implying $F \wedge X = 0$. This is a more computation-friendly way to see the Bousfield classes. An easy corollary (exercise) is to show

- $\langle E\rangle \wedge \langle F\rangle = \langle E \wedge F\rangle$.
- $\langle E\rangle \vee \langle F\rangle = \langle E \vee F\rangle$.

It is not hard to show that the Bousfield classes satisfy the distribution laws. That is,

- $(\langle X\rangle \vee \langle Y\rangle) \wedge \langle Z\rangle = (\langle X\rangle \wedge \langle Z\rangle) \vee (\langle Y\rangle \wedge \langle Z\rangle)$,
- $(\langle X\rangle \wedge \langle Y\rangle) \vee \langle Z\rangle = (\langle X\rangle \vee \langle Z\rangle) \wedge (\langle Y\rangle \vee \langle Z\rangle)$.

From the definition, the following proposition is also immediate:

Proposition 4.10. *If $\langle E\rangle = \langle F\rangle$, then $L_E = L_F$. What's more, if $\langle E\rangle \leq \langle F\rangle$, then $L_E L_F = L_E$, and there is a natural transformation $L_F \Rightarrow L_E$.*

Corollary 4.11. *Let E be any spectrum, and 0 be the trivial (contractible) spectrum.*

(1) $\langle \mathbb{S}\rangle \geq \langle E\rangle \geq \langle 0\rangle$.
(2) $\langle E\rangle \wedge \langle \mathbb{S}\rangle = \langle E\rangle$.
(3) $\langle E\rangle \vee \langle \mathbb{S}\rangle = \langle \mathbb{S}\rangle$.
(4) $\langle E\rangle \wedge \langle 0\rangle = \langle 0\rangle$.
(5) $\langle E\rangle \vee \langle 0\rangle = \langle E\rangle$.

The corollary tells us, under the partial order $\geq$, $\langle \mathbb{S}\rangle$ is the biggest one, and $\langle 0\rangle$ is the smallest one. Any spectrum must be in the middle. This motivates us to think about whether every spectrum fits into some partially-ordered chain. Surprisingly, we have a famous theorem by Tetsusuke Ohkawa [17] stated as follows.

Theorem 4.12 (Ohkawa)**.** *$\{\langle E\rangle : E \in \mathsf{Sp}\}$ is a set.*

We refer the interested readers to [16].

Proposition 4.13. *Let $f : \Sigma^d X \to X$ be a self map, and $g : X \to Y$.*

(1) If $W \to X \xrightarrow{g} Y \to \Sigma W$ is a cofiber sequence, then $\langle W\rangle \leq \langle X\rangle \vee \langle Y\rangle$. If f is smash nilpotent, i.e. $f^{(k)}$ is null-homotopic for some $k > 1$, then $\langle W\rangle = \langle X\rangle \vee \langle Y\rangle$.

(2) Let $\widehat{X} = \operatorname{colim}\left(X \xrightarrow{f} \Sigma^{-d}X \xrightarrow{f} \Sigma^{-2d}X \xrightarrow{f} \cdots\right)$, and C_f be the cofiber of f. Then

$$\langle X\rangle = \left\langle \widehat{X}\right\rangle \vee \langle C_f\rangle,$$
$$\left\langle \widehat{X}\right\rangle \wedge \langle C_f\rangle = \langle 0\rangle.$$

Assume we have already proved TST. One famous theorem as a direct corollary is the class invariance proposition, as claimed in §1.

Theorem 4.14 (Class invariance theorem)**.** *Let X and Y be p-local finite spectra of types m and n, respectively. Then $\langle X\rangle = \langle Y\rangle$ iff $m = n$. In general, $\langle X\rangle \geq \langle Y\rangle$ if $m \leq n$.*

Proof. Let $\mathcal{C}_X$ (resp. $\mathcal{C}_Y$) be the smallest thick subcategories of $\mathsf{FH}_{(p)}$ containing X (resp. Y). In other words, $\mathcal{C}_X$ (resp. $\mathcal{C}_Y$) contains all finite complexes that can be built up from X (resp. Y) by taking cofibrations and retractions. So for any $X' \in \mathcal{C}_X$, $\langle X\rangle \geq \langle X'\rangle$.

Since X is of type m, $K(m-1)_*(X) = 0$, and so $K(m-1)_*(X') = 0$. By TST, $\mathcal{C}_X = \mathcal{D}_m$ for some $m \geq 0$. Similarly, $\mathcal{C}_Y = \mathcal{D}_n$ for some $n \geq 0$. Hence, $m = n$ is equivalent to $\mathcal{C}_X = \mathcal{D}_m = \mathcal{D}_n = \mathcal{C}_Y$, which is the same to say $\langle X\rangle = \langle Y\rangle$. Moreover, $\langle X\rangle < \langle Y\rangle$ follows from the strict inclusion $\mathcal{D}_m \supset \mathcal{D}_n$. □

Proposition 4.15. *Let $\mathbb{S}_{\mathbb{Q}}$, $\mathbb{S}_{(p)}$, and $\mathbb{S}/p$ be rational sphere spectrum, p-local sphere spectrum, and mod p Moore spectrum, respectively. Then*

- $\langle \mathbb{S}_{(p)}\rangle = \langle \mathbb{S}_{\mathbb{Q}}\rangle \vee \langle \mathbb{S}/p\rangle$.
- $\langle \mathbb{S}_{\mathbb{Q}}\rangle \wedge \langle \mathbb{S}/p\rangle = \langle 0\rangle$.
- $\langle \mathbb{S}/p\rangle \wedge \langle \mathbb{S}/q\rangle = \langle 0\rangle$, *if* $p \neq q$.
- $\langle \mathbb{S}\rangle = \langle \mathbb{S}_{\mathbb{Q}}\rangle \vee \bigvee_{p\in\mathcal{P}} \langle \mathbb{S}/p\rangle$.

This proposition leads to the critical theorem as follows. See [1, 2].

Theorem 4.16. *Let the spectra $B(n), K(n), E(n), BP\langle n\rangle, k(n)$ be as defined in §3.*

(1) *(Johnson-Wilson)* $\langle B(n)\rangle = \langle K(n)\rangle$.
(2) *(Johnson-Yosimura)* $\langle v_n^{-1}BP\rangle = \langle E(n)\rangle$.
(3) $\langle P(n)\rangle = \langle K(n)\rangle \vee \langle P(n+1)\rangle$.
(4) $\langle E(n)\rangle = \bigvee_{i=0}^{n} \langle K(i)\rangle$.
(5) $\langle k(n)\rangle = \langle K(n)\rangle \vee \langle H\mathbb{Z}/p\rangle$.
(6) $\langle BP\langle n\rangle\rangle = \langle E(n)\rangle \vee \langle H\mathbb{Z}/p\rangle$.
(7) $\langle K(m)\rangle \wedge \langle K(n)\rangle = \langle 0\rangle$ *if* $m \neq n$.
(8) $\langle K(n)\rangle \wedge \langle H\mathbb{Z}/p\rangle = \langle 0\rangle$.

From the theorem, we know that $\langle v_n^{-1}BP\rangle = \langle E(n)\rangle = \langle K(0)\rangle \vee \langle K(1)\rangle \vee \cdots \vee \langle K(n)\rangle$. So $v_n^{-1}BP_*(X) = 0$ in the statement of Theorem 2.5 is equivalent to $K(n)_*(X) = 0$.

5. Proof of thick subcategory theorem

We are now fully prepared to tackle Theorem 2.5. One last tool in need is the nilpotence theorem. Recall that the classical nilpotence theorem says

Theorem 5.1 (Nilpotence theorem)**.** *Let R be a connective ring spectrum of finite type, and $\pi_*R \xrightarrow{h} MU_*R$ be the Hurewicz map. Then any $\alpha \in \pi_*R$ is nilpotent if $h\alpha = 0$.*

What we need is another form of the classical nilpotence theorem, which states

Theorem 5.2 (Nilpotence theorem, smash product form)**.** *Let $f : F \to X$ be a map of spectra, and F be finite. If $MU \wedge f$ is null-homotopic, then f is smash nilpotent. If we localize at p, then the theorem still holds after replacing MU by BP.*

Proof of Theorem 5.1 implying Theorem 5.2. Under the assumption of Theorem 5.2, f is adjoint to $\hat{f} : \mathbb{S} \to X \wedge DF$. $E \wedge f$ being null-homotopic is equivalent to $E \wedge \hat{f}$ being null-homotopic, for each ring spectrum E for which Theorem 5.1 holds. It suffices to prove for the case $F = \mathbb{S}$. Our task is to prove the smash nilpotence of f, given

$$\mathbb{S} \xrightarrow{f} X = \mathbb{S} \wedge X \xrightarrow{\eta \wedge \mathrm{id}} E \wedge X$$

is null-homotopic. Recall that in Sp, every spectrum X is a homotopy colimit of its finite subspectra X_α. Both f and the desired null-homotopy factor through some finite X_α, i.e.

$$\mathbb{S} \xrightarrow{f_\alpha} X_\alpha = \mathbb{S} \wedge X_\alpha \xrightarrow{\eta \wedge \mathrm{id}} E \wedge X_\alpha$$

is null-homotopic. Let $Y = \Sigma^n X_\alpha$, where n is such that Y is 0-connected. Let $R = \bigvee_{j \geq 0} Y^{(j)}$. This is a connective ring spectrum of finite type with multiplication given by smashing with more copies of $Y^{(j)}$. Theorem 5.1 implies that $z \in \pi_* R$ is nilpotent if $hz = 0$. Let $E = MU$. Then $\mathbb{S} \xrightarrow{f_\alpha} X_\alpha \xrightarrow{\eta \wedge \mathrm{id}} MU \wedge X_\alpha$ is null-homotopic. Suspending n times yields

$$\Sigma^n \mathbb{S} \xrightarrow{f_\alpha} Y \xrightarrow{\eta \wedge \mathrm{id}} MU \wedge Y$$

being null-homotopic. If we further suspend ingredients of this map according the definition of R, we get $\pi_* R \to \pi_* MU \wedge R = MU_* R$. Now f (or f_α, if you prefer) corresponds to some $z \in \pi_* R$ such that $hz = 0$. Hence, f is itself smash nilpotent. □

The proof of TST needs the following key corollary of Theorem 5.2.

Corollary 5.3. *Let W, X, Y be p-local finite spectra, and $f : X \to Y$. If $K(n)_*(W \wedge f) = 0$ for all $n \geq 0$, then $W \wedge f^{(k)}$ is null-homotopic for large k.*

Proof. Write $R = DW \wedge W$. Let e, η be as defined in §4.1, i.e. the unit $\eta : R \to \mathbb{S}$ and the counit $e : \mathbb{S} \to R$. We claim that R is a ring spectrum. Indeed, the multiplication m is given by

$$\begin{aligned} m : R \wedge R &\xrightarrow{\equiv} DW \wedge W \wedge DW \wedge W \\ &\xrightarrow{\mathrm{id} \wedge De \wedge \mathrm{id}} DW \wedge \mathbb{S} \wedge W \\ &\xrightarrow{\equiv} DW \wedge W = R. \end{aligned}$$

Here we note $D^2 W = W$. f is adjoint to $\hat{f} : \mathbb{S} \to DX \wedge Y$, and $W \wedge f$ is adjoint to

$$\mathbb{S} \xrightarrow{\hat{f}} DX \wedge Y = \mathbb{S} \wedge DX \wedge Y \xrightarrow{e \wedge \mathrm{id} \wedge \mathrm{id}} R \wedge DX \wedge Y.$$

Denote $R \wedge DX \wedge Y$ by F, and the composite of the maps by g. Then $W \wedge f^{(k)} : W \wedge X^{(k)} \to W \wedge Y^{(k)}$ is adjoint to

$$\mathbb{S} \xrightarrow{g^{(k)}} F^{(k)} = R^{(k)} \wedge DX^{(k)} \wedge Y^{(k)} \xrightarrow{m^{k-1}} R \wedge DX^{(k)} \wedge Y^{(k)}.$$

The goal is to show $W \wedge f^{(k)}$ is null-homotopic for large k, which is the same to show $g^{(k)}$ is null-homotopic. By Theorem 5.2, it suffices to show $BP \wedge g^{(k)}$ is null-homotopic.

Let $T_k = R \wedge DX^{(k)} \wedge Y^{(k)}$ and $T = \operatorname{colim} \left(\mathbb{S} \xrightarrow{g} T_1 \xrightarrow{\mathrm{id} \wedge \hat{f}} T_2 \xrightarrow{\mathrm{id} \wedge \hat{f}} \cdots \right)$. $BP \wedge g^{(k)}$ being null-homotopic is equivalent to $BP \wedge T$ being contractible. From the fact ([1, 2, 4] and Theorem 4.16)

$$\langle BP \rangle = \langle K(0) \rangle \vee \langle K(1) \rangle \vee \cdots \vee \langle K(n) \rangle \vee \langle P(n+1) \rangle,$$

it suffices to show $P(m)\wedge T$ is contractible for large m, since $K(n)\wedge T$ is contractible for all n by assumption. Since we are working in the category of finite spectra, we have

$$K(m)_*(W \wedge f) = K(m)_* \otimes H_*(W \wedge f),$$
$$P(m)_*(W \wedge f) = P(m)_* \otimes H_*(W \wedge f).$$

By assumption, $H_*(W \wedge f) = 0$. So $P(m)_*(W \wedge f) = 0$, indicating that $P(m) \wedge T$ is contractible. We conclude our proof. □

Proof of Theorem 2.5. Let $\mathcal{C} \subset \mathsf{FH}_{(p)}$ be a thick subcategory, and n be the smallest integer such that $\mathcal{C}$ contains p-local finite spectra X with $K(n)_*(X) \neq 0$. Our goal is to show $\mathcal{C} = \mathcal{D}_n$. From the definition it follows that $\mathcal{C} \subset \mathcal{D}_n$. It suffices to prove the converse.

Let $Y \in \mathcal{D}_n$. For any p-local finite spectrum F, $X \wedge F \in \mathcal{C}$ from the thickness of $\mathcal{C}$, and so $X \wedge DX \wedge Y \in \mathcal{C}$. Let W be a finite spectra, and $f : W \to \mathbb{S}$ fit into a cofiber sequence $W \xrightarrow{f} \mathbb{S} \xrightarrow{e} DX \wedge X$. It is clear that $W \in \mathsf{FH}_{(p)}$. Let $C_f = C_{f^{(1)}} = DX \wedge X$ be the mapping cone. By Lemma 4.2, $C_{f^{(k)}} \wedge Y \in \mathbb{C}$ for $k \geq 1$. $K(i)_*(f) = 0$ when $K(i)_*(X) \neq 0$ because $DW \in \mathsf{FH}_{(p)} = \mathcal{D}_0$, and by statement 6 in Theorem 4.1 $K(i)_*(f) \cong K(i)_*(DW)$ for $i \geq n$. When $i < n$, $K(i)_*(Y) = 0$ by assumption. Thus, $K(i)_*(Y \wedge f) = 0$ for all i. By Corollary 5.3, $Y \wedge f^{(k)}$ is null-homotopic for some large k. Recall that in classical homotopy theory, the cofiber of a null-homotopy is equivalent to the wedge of its target and the suspension of its source, i.e.

$$Y \wedge C_{f^{(k)}} \simeq Y \vee \left(\Sigma Y \wedge W^{(k)}\right).$$

Since both $Y \wedge C_{f^{(k)}} \in \mathcal{C}$ and $\left(\Sigma Y \wedge W^{(k)}\right) \in \mathcal{C}$, we know that $Y \in \mathcal{C}$. Hence, $\mathcal{C} \supset \mathcal{D}_n$ as desired. □

References

[1] Ravenel, D. C. (1984). *Localization with respect to certain periodic homology theories.* American Journal of Mathematics, 106:351-414.

[2] Ravenel, D. C. (1986). *Complex cobordism and stable homotopy groups of spheres.* AMS Chelsea Publishing.

[3] Ravenel, D. C. (1992). *Nilpotence and Periodicity in Stable Homotopy Theory.* Princeton University Press.

[4] Jeanneret, A., Ravenel, D. C., Landweber, P. S. (1994). *A note on the thick subcategory theorem.* In: Broto, C., Casacuberta, C., Mislin, G. (eds) *Algebraic Topology: New Trends in Localization and Periodicity.* Progress in Mathematics, vol 136. Birkhäuser Basel.

[5] Lurie, J. (2010) *Lecture notes on chromatic homotopy theory.* Harvard University. `https://people.math.harvard.edu/lurie/252x.html`

[6] Nishida, G. (1973). *The nilpotency of elements of the stable homotopy groups of spheres.* J. Math. Soc. Japan. Vol 25, No 4.

[7] Mitchell, S. A. (1985). *Finite complexes with $A(n)$-free cohomology.* Topology, 24:227-248.

[8] Adams, J. F. (1974). *Stable Homotopy and Generalized Homology.* University of Chicago Press, Chicago.

[9] Rudyak, Y. B. (1986). *Exactness theorems in the cohomology theories MU, BP, and $P(n)$.* Mathematical notes of the Academy of Sciences of the USSR 40, 562-569.

[10] Behrens, M., Pemmaraju, S. (2004). *On the existence of the self map v_2^9 on the Smith-Toda complex $V(1)$ at the prime 3.* Contemp. Math, 346:9-49.

[11] Behrens, M., Hill, M., Hopkins, M. J., Mahowald, M. (2008). *On the existence of a v_2^{32} self map on $M(1,4)$ at the prime 2.* Homology, Homotopy, and Applications, 10:45-84.

[12] Devalapurkar, S. (2018) *Lecture notes on chromatic homotopy theory*. Harvard University. https://sanathdevalapurkar.github.io/files/iap-2018.pdf
[13] Devinatz, E., Hopkinsm M. J., Smith, J. (1988). *Nilpotence and Stable Homotopy Theory I.* Annals of Mathematics, 128(2):207–241.
[14] Landweber, P. S. (1973). *Annihilator ideals and primitive elements in complex bordism.* Illinois J. Math., 17:273-284.
[15] Landweber, P. S. (1976). *Homological properties of comodules over* $MU_*(MU)$ *and* $BP_*(BP)$. American Journal of Mathematics, 98:591–610.
[16] Casacuberta, C. (2015). *Depth and Simplicity of Ohkawa's Argument*. In: Ohsawa, T., Minami, N. (eds) *Bousfield Classes and Ohkawa's Theorem*. Springer Proceedings in Mathematics & Statistics, vol 309.
[17] Ohkawa, T. (1989). *The injective hull of homotopy types with respect to generalized homology functors*. Hiroshima Math. J. 19, no. 3, 631–639.