The Trace Method in Algebraic K-theory

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Outline

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Algebraic K-theory

Question

Why do we care about the algebraic K-theory?

- **(s-cobordism Theorem)** If $(W; M, M')$ is an h-cobordism, then it is cylinder iff its Whitehead torsion $\tau(W, M) \in K_1(\mathbb{Z}[\pi_1 M])$ vanishes.

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- **(Vandiver Conjecture)** Prime p does not divide the class number of the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta_p)^+$.
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- **(Quillen–Lichtenbaum Conjecture)** There is a descent spectral sequence of some nice scheme with E^2-term given in terms of étale cohomology and converging to the algebraic K-theory of X.

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Many constructions. Let R be an associative unital ring. Quillen's "+":

$$K_n(R) = \pi_n(K(R)) = \pi_n\left(\coprod_{K_0(R)} BGL(R)^+\right),$$

where $BGL(R)^+$ is the "+" construction on $BGL(R)$.

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where $BGL(R)^+$ is the "+" construction on $BGL(R)$.

- $K_0(R)$ is the Grothendieck group of isomorphism class of finitely generated projective R-modules.
- $K_1(R) = GL(R)^{ab}$.
- $K_2(R) = Z(St(R))$.

Basic Motivation

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- (Quillen, 1972) $K_n(\mathbb{F}_p) = \mathbb{Z}/(p^i - 1)$ for $n = 2i - 1$, and 0 otherwise.
- $K_0(\mathbb{Z}) = \mathbb{Z}$, $K_1(\mathbb{Z}) = K_2(\mathbb{Z}) = \mathbb{Z}/2$, $K_3(\mathbb{Z}) = \mathbb{Z}/48$.

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- Extremely hard in general! Even $K_n(\mathbb{Z})$ is not fully known.
- Trace Method: let R be unital, associative ring. There are maps
  - **(Dennis trace)** $K(R) \to THH(R)$.
  - **(Cyclotomic trace)** $K(R) \to TC(R)$.

Trace Maps

Classically, the Dennis trace is given by

$$BGL_n(R) \to N^{cyc}(GL_n(R)) \to N^{cyc}(M_n(R)) \xrightarrow{\text{Morita}} N^{cyc}(R) \simeq HH(R),$$

leading a map from K to HH, the Hochschild homology of R.

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- The trace maps with codomains being THH and TC are topological refinements of these maps. These new trace maps are obtained similarly via Morita equivalence, but with an extra edgewise subdivision of cyclic bar construction of THH.

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Topological Hochschild homology

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Definition

$THH(R)$ is the geometric realization of (R is a ring over Sp, often an orthogonal spectrum)

$$R \leftleftarrows R \wedge R \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; R \wedge R \wedge R \cdots ,$$

where the smash is over $\mathbb{S}$, the sphere spectrum.

Topological Hochschild homology

The previous definition is equivalent to the traditional definition by Bökstedt, before the notion of symmetric monoidal structure even existed:

Definition

Let F be a functor with smash product (FSP). It aims to endow a monoidal structure on Top. Then, the topological Hochschild homology of a pointed space X is $THH(F;X)$, given by the geometric realization of

$$THH_\bullet(F;X) = ([p] \mapsto \operatorname{hocolim}_p \operatorname{Map}(S^{i_0} \wedge \cdots \wedge S^{i_0}, F(S^{i_0}) \wedge \cdots \wedge F(S^{i_p}) \wedge X))$$

where $i_0, \cdots, i_p \in \mathbb{N}$.

By choosing the appropriate F and $X = \text{pt}$, one has the THH of a ring R.

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where $i_0, \cdots, i_p \in \mathbb{N}$.

By choosing the appropriate F and $X = \mathrm{pt}$, one has the THH of a ring R. (In fact, $F(X) = |R\Delta_\bullet(X)/\Delta_\bullet(\mathrm{pt})|$.)

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Theorem (Nikolaus-Scholze, 2018, Theorem III.6.1)

Two definitions are essentially the same.

An Important Cofiber Sequence

- Assume G is a compact Lie group or a finite group from now on. Let $\mathcal{F}$ be the collection of non-trivial subgroups of G that are closed under conjugations and subgroups.

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- There is a G-space $E\mathcal{F}$ such that for each subgroup $H \leq G$, $\widetilde{E\mathcal{F}}^H$ is a one-point set for $H \in \mathcal{F}$ and empty otherwise.
- There is a cofiber sequence of particular interest:

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}},$$

where $\widetilde{E\mathcal{F}}$ is the cofiber, i.e. $\widetilde{E\mathcal{F}} = S^0 \cup CE\mathcal{F}_+$.

Genuine Cyclotomic Spectra

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Definition

A genuine cyclotomic spectrum is an S^1-spectrum with an S^1-equivariant equivalence

$$\Phi^{C_p} X \to X,$$

where $\Phi^G X = (\widetilde{E\mathcal{F}} \wedge X)^G$ is the geometric fixed points.

Theorem

$THH(F;X)$ is a cyclotomic spectrum.

Cyclotomic Spectra

- Nikolaus-Scholze used a different definition of cyclotomic spectra. Namely, a cyclotomic spectrum, in their settings, is an S^1-spectrum with an S^1-equivariant map $X \to X^{tC_p}$ for any prime p, where $X^{tC_p} = \mathrm{cofib}(X_{hC_p} \to X^{hC_p})$.

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- Let $\text{CycSp}^{\text{gen}}$ be the category of genuine cyclotomic spectra (the traditional one defined earlier), and CycSp be the cyclotomic spectra defined here.

Theorem (Nikolaus-Scholze, 2018)

There is an ∞-categorical equivalences:

$$\text{CycSp}^{\text{gen}} \cong \text{CycSp},$$

when restricting to bounded below spectra.

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Tate Diagram

Consider the cofiber sequence

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Let X be a cyclotomic spectrum. Apply functors $- \wedge X$, $- \wedge F(EG, X)$, and use the map $X \xrightarrow{\epsilon} F(EG_+, X)$, we get a diagram

$$\begin{array}{ccccc}
EG_+ \wedge X & \longrightarrow & X & \longrightarrow & \widetilde{EG} \wedge X \\
\downarrow{\scriptstyle \mathrm{id} \wedge \epsilon} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \mathrm{id} \wedge \epsilon} \\
EG_+ \wedge F(EG_+, X) & \longrightarrow & F(EG_+, X) & \longrightarrow & \widetilde{EG} \wedge F(EG_+, X)
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By Adams Isomorphism, after taking $(-)^G$, the Tate diagram becomes

$$\begin{array}{ccccc} X_{hC_{p^n}} & \longrightarrow & X^{C_{p^n}} & \longrightarrow & \Phi^{C_{p^n}} X \cong X^{C_{p^{n-1}}} \\ \downarrow \simeq & & \downarrow & & \downarrow \\ X_{hC_{p^n}} & \longrightarrow & X^{hC_{p^n}} & \longrightarrow & X^{tC_{p^n}} \end{array}$$

where X_{hG} is the homotopy orbits, X^{hG} is the homotopy fixed points, and X^{tG} is the Tate spectrum of X.

Bousfield-Kan Spectral Sequences

Theorem

1. $E^2_{s,t}(X_{hC_{p^n}}) = H_s(C_{p^n}; \pi_t X) \Rightarrow \pi_{s+t} X$.
2. $E^2_{s,t}(X^{hC_{p^n}}) = H^{-s}(C_{p^n}; \pi_t X) \Rightarrow \pi_{s+t} X$.
3. $E^2_{s,t}(X^{tC_{p^n}}) = \hat{H}^{-s}(C_{p^n}; \pi_t X) \Rightarrow \pi_{s+t} X$.

where $\hat{H}$ is the Tate cohomology.

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where $\hat{H}$ is the Tate cohomology.

The spectral sequences are obtained by "Greenlees filtration". They are related to each other:

1. $E^2_{s,t}(X^{tG}) = E^2_{s,t}(X^{hG})$, for $s < 0$.
2. $E^2_{s+1,t}(X^{tG}) = E^2_{s,t}(X_{hG})$, for $s \geq 1$.
3. For $s = 0, 1$, there is a short exact sequence of C_{p^n} in $(\pi_t X)$-coefficient:

$$0 \to \hat{H}^{-1} \to H_0 \xrightarrow{\text{Norm}} H^0 \to \hat{H}^0 \to 0.$$

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Topological Cyclic Homology

Definition

The topological cyclic homology of an FSP F at a prime p is

$$TC(F;p) = \left(\operatorname{hocolim}_{\Phi_p} THH(F)^{C_{p^n}}\right)^{hD},$$

where $\Phi, D : THH(F)^{C_{p^n}} \to THH(F)^{C_{p^{n-1}}}$ are different maps naturally arise from THH, and hD is the homotopy equalizer of D and id. Write $TC(F) = TC(F;p)\hat{}$ be the profinite completion of $TC(F;p)$.

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If we choose the new definition by N-S, then TC becomes an equalizer of some pair of maps.

Topological Cyclic Homology

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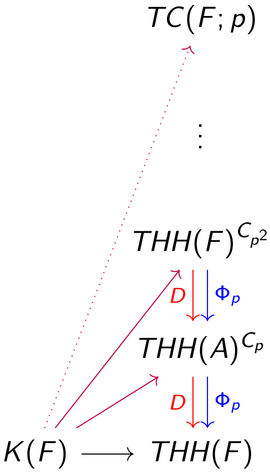

Witt Rings

Definition

Let A be a commutative ring. $W(A) = A^{\aleph_0}$, countably infinite product of A as sets. The Witt vectors are coordinates w_i of the image of the ghost map

$$w : W(A) \to A^{\aleph_0}, \quad a \mapsto (w_1(a), w_2(a), \cdots),$$

where $w_n = a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n$, for $a = (a_0, a_1, \cdots)$.

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- Verschiebung map V filters $W(A)$ by $W_n(A) = W(A)/V^n W(A)$, where each element in $W_n(A)$ has the form $(a_0, a_1, \cdots, a_n)$.

Some Results

Theorem (Dundas 97', McCarthy 97', Dundas-Goodwillie-McCarthy, 13')

Let p be a prime, and $R \to S$ be a surjection of rings with the nilpotent kernel. Then there is a homotopy Cartesian diagram after p-completion:

$$\begin{array}{ccc} K(R)\hat{}_p & \longrightarrow & TC(R)\hat{}_p \\ \downarrow & & \downarrow \\ K(S)\hat{}_p & \longrightarrow & TC(S)\hat{}_p \end{array}$$

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Corollary

Let k be a perfect field of characteristic $p > 0$. For finite $W(k)$-algebra A ($W(k)$ is the Witt ring of k), the cyclotomic trace

$$K_i(A; \mathbb{Z}_p) \to \pi_i(TC(A))\hat{}_p$$

Some Results

Theorem (Hesselholt-Madsen)

Let k be a perfect field of characteristic $p > 0$. Then for every $m > 0$,

$$K_{2m-1}(K[x]/(x^n); \mathbb{Z}_p) = W_{nm-1}(k)/V^n W_{m-1}(k),$$

and

$$K_{2m}(K[x]/(x^n); \mathbb{Z}_p) = 0,$$

where W is Witt ring, $W_n = W/V^n W$, and V is the Verschiebung map.

Most Recent Usage: Telescope Conjecture

Write $K(n)$ as the height n Morava K-theory. Let X be a type n spectrum, i.e. $K(k)_*(X) = 0$ for $k \leq n$, and non-zero at n. Let $f : X \to \Sigma^{-k} X$ be a v_n-self map, and $T(n) = f^{-1}X = (X \xrightarrow{f} \Sigma^{-k} X \xrightarrow{f} \Sigma^{-2k} X \xrightarrow{f} \cdots)$ be the invert of X w.r.t. f.

Telescope Conjecture (Ravenel 1984)

There is an equivalence of the Bousfield classes

$$\langle T(n)) \rangle = \bigvee \langle K(n) \rangle .$$

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Theorem (Burklund-Hahn-Levy-Schlank, Oct 26 2023, preprint)

FALSE for $n \geq 2$!

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- The key is to prove that the Bousfield localization of homotopy fixed points of the truncated Brown-Peterson spectrum $L_{T(n+1)}K(BP\langle n\rangle^{h\mathbb{Z}})$ is not $K(n+1)$-local.

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- After cyclotomic redshift, it suffices to show that $L_{T(2)}K(L^{hp^k\mathbb{Z}}) \to L_{T(2)}K(L)^{hp^k\mathbb{Z}}$ is not a cyclotomic completion, where L is the connective Adams summand of $ku_{(p)}$ for $k \gg 0$. The key technique here is the (cyclotomic) trace method:

Theorem (Mitchell 90', Dundas-Goodwillie-McCarthy 13', et al.)

Let R be an E_1-ring. For $n \geq 1$, there is a natural equivalence

$$L_{T(n+1)}K(R) \xrightarrow{\cong} L_{T(n+1)}TC(\tau_{\geq 0}R).$$

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With the theorem, use the "(cyclotomic) asymptotic constancy" to conclude the proof.

References

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Thank you!