GEOMETRIC FIXED POINTS

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We will carry out the content of this talk based on the following three main aspects:

(1) What are the geometric fixed points?
(2) Why do we care about them?
(3) Are there any applications for the geometric fixed points?

The main references for this talk are [Blu17], [Sch18], and [May].

What are the geometric fixed points?

Let $\mathcal{F}$ be the set of non-empty subgroups H of G, such that $\mathcal{F}$ is closed under conjugation and taking subgroups. That is, if $H \in \mathcal{F}$, (H) is the conjugate class of H in G, then any $K \in (H)$ or $K \leq H$, $K \in \mathcal{F}$. There is a G-space $E\mathcal{F} \in \mathsf{GTop}$ such that for any $H \subset G$,

$$(E\mathcal{F})^H \simeq \begin{cases} * & , H \in \mathcal{F} \\ \varnothing & , H \notin \mathcal{F} \end{cases}$$

In particular, if H is trivial, then $(E\mathcal{F})^H = EG$.

Consider the following isotropy separation sequence:

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}.$$

This is a cofiber sequence, and $\widetilde{E\mathcal{F}}$ is the homotopy cofiber of $E\mathcal{F}_+ \to S^0$. To simplify the problem, in most cases of interest, e.g. $G = C_{p^n}$, G is good enough such that $E\mathcal{F} \simeq EG$. We will always assume this for the rest of the note.

Definition 0.1. Let $X \in \mathsf{GSU}$ be a genuine G-spectrum. The **geometric fixed point** of X is

$$\Phi^G(X) \coloneqq (\widetilde{E\mathcal{F}} \wedge X)^G = (\widetilde{EG} \wedge X)^G.$$

In fact, the geometric fixed point Φ^G is functorial, i.e. $\Phi^G : \mathsf{GSU} \to \mathsf{Sp}$.

Why do we care about them?

Let $V \in R[G]$, $X \in \mathsf{GSU}$. Recall that the categorical fixed points are the functor

$$(-)^G : \mathsf{GSU} \to \mathsf{Sp}$$

sending X to $(\imath^* X)^G$, where $\imath^* : \mathsf{GSU} \to \mathsf{GSU}^G$ is the change-of-universe functor, and $(\imath^* X)^G(V) = X(\imath^* V)$. Despite the easy definition, the categorical fixed point has a fatal problem: it does not commute with Σ^∞. So fixed points of spaces $X(V)^G$ cannot constitute a proper genuine G-spectrum. In fact, we have the following theorem by tom Dieck:

Theorem 0.2 (tom Dieck)**.** *Let G be a finite group, $X \in \mathsf{Top}_*$, one has*

$$(\Sigma^\infty X)^G \simeq \bigvee_{\{(H):H\leq G\}} \Sigma^\infty(EWH_+ \wedge_{WH} X^H),$$

where $WH = N_G H/H$ is the Weyl group.

Remark 0.3. Theorem 0.2 is not the original version by tom Dieck, but rather the one by Greenlees and May. The original version is the splitting of equivariant homotopy groups of spectra of both sides in the theorem 0.2. Namely,

$$\pi_*^G(\Sigma^\infty X) \cong \bigoplus_{\{(H):H\leq G\}} \pi_*^{WH}(\Sigma^\infty(EWH_+ \wedge_{WH} X^H)). \tag{0.4}$$

We will give a proof of (0.4), instead of the theorem 0.2 whose proof is quite different.

To prove (0.4), we first need the Wirthmüller isomorphism. Let $H \leq G$ be a subgroup. then the restriction of genuine equivariant spectra $\mathrm{Res}^G_H : \mathsf{GSU} \to \mathsf{HSU}$ has a left adjoint and a right adjoint:

$$\begin{array}{c} G_+\wedge_H - \\ \curvearrowleft \quad \perp \\ \mathsf{GSU} \xrightarrow{\mathrm{Res}^G_H} \mathsf{HSU} \\ \curvearrowleft \quad \perp \\ \mathrm{Map}^H(G_+,-) \end{array}$$

Explicitly, let $Y \in \mathsf{HSU}$, $V \in R[G]$, then

- $\mathrm{Map}^H(G_+,-)(V) = \mathrm{Map}^H(G_+, Y(\imath^* V))$, where $\imath$ is the change-of-universe functor. The structure map is given by
$$\begin{aligned}\sigma_{V,W} : \mathrm{Map}^H(G_+, Y(\imath^* V)) \wedge S^W &\to \mathrm{Map}^H(G_+, Y(\imath^* V) \wedge S^{\imath^* W}) \\ &\to \mathrm{Map}^H(G_+, Y(\imath^*(V \oplus W)))\end{aligned}$$
- $(G_+ \wedge_H Y)(V) = G_+ \wedge_H Y(\imath^* V)$. The structure maps are defined similarly.

In modern language, the left adjoint $G_+ \wedge_H -$ is called the induced G-spectrum, and the right adjoint $\mathrm{Map}^H(G_+, -)$ is called the coinduced G-spectrum. We are now ready to state the remarkable theorem.

Theorem 0.5 (Wirthmüller isomorphism)**.** *The induced G-spectrum is $\underline{\pi}_*$-isomorphic to the coinduced G-spectrum.*

We won't give a detailed proof of this theorem. Rather, the following corollary is of particular interest.

Corollary 0.6. *The composition*

$$\pi_*^G(G_+ \wedge_H Y) \xrightarrow{\mathrm{Res}^G_H} \pi_*^H(G_+ \wedge_H Y) \xrightarrow{\mathrm{proj}} \pi_*^H(Y)$$

is an isomorphism.

Proof. Let $\Phi_Y : G_+ \wedge_H Y \to \mathrm{Map}^H(G_+, Y)$ be the map inducing Wirthmüller isomorphism for $Y \in \mathsf{HSU}$. Let $\mathrm{ev} : \mathrm{Map}^H(G_+, Y) \to Y$ be the evaluation at 1. Then it corresponds to the counit of the adjunction between Res^G_H and $\mathrm{Map}^H(G_+, -)$. Composing ev and Φ_Y yields the desired result after passing to homotopy groups. $\square$

To prove (0.4), it will be helpful to break the whole map into each summand. Explicitly, the canonical quotient map $N_G H \to N_G H/H = WH$ induces

$$\begin{aligned}\theta_H : \pi_*^{WH}(\Sigma^\infty(EWH_+ \wedge_{WH} X^H)) &\to \pi_*^{N_G H}(\Sigma^\infty(EWH_+ \wedge_{WH} X^H)) \\ &\to \pi_*^{N_G H}(\Sigma^\infty(EWH_+ \wedge_{WH} X)),\end{aligned}$$

where the second map is induced by the inclusion $X^H \hookrightarrow X$. Using the corollary 0.6, we get

$$\theta_H : \pi_*^{WH}(\Sigma^\infty(EWH_+ \wedge_{WH} X^H)) \to \pi_*^{N_G H}(\Sigma^\infty(EWH_+ \wedge_{WH} X)) \tag{0.7}$$

$$\to \pi_*^G(G_+ \wedge_{N_G H} \Sigma^\infty(EWH_+ \wedge_{WH} X)) \tag{0.8}$$

$$\to \pi_*^G(\Sigma^\infty X). \tag{0.9}$$

The last map is induced by

$$\begin{aligned}&\mathrm{Map}_{\mathsf{GSU}}(G_+ \wedge_{N_G H} \Sigma^\infty(EWH_+ \wedge_{WH} X), \Sigma^\infty X) \xrightarrow{\cong} \\ &\qquad \mathrm{Map}_{\mathsf{N_GHSU'}}(G_+, \mathrm{Map}_{\mathsf{N_GHSU'}}(\Sigma^\infty(EWH_+ \wedge_{WH} X), \Sigma^\infty X)),\end{aligned}$$

and elements in $\mathrm{Map}_{\mathsf{N_GHSU'}}(\Sigma^\infty(EWH_+ \wedge_{WH} X), \Sigma^\infty X)$ is induced by $N_G H$-equivariant projections $EWH_+ \wedge_{WH} X \to X$.

Lemma 0.10. *For any $K \notin (H)$, $Y \in \mathsf{GTop}$, $Y^K = *$, and any $H \trianglelefteq G$,*

$$\mathrm{Map}^G(X, Y) \to \mathrm{Map}^{G/H}(X^H, Y^H)$$

is an acyclic fibration.

Again, the proof is omitted. Using the lemma 0.10, it is straightforward that θ_H in (0.7) is an isomorphism for any $K \notin (H)$ with $X^K = *$.

Now that we have shown for each conjugacy class (H), θ_H is an isomorphism. The natural question for the next is whether we can have some "induction" on (H) such that θ_H can be extended to all conjugacy classes, and the result follows. Luckily, we actually can.

Proposition 0.11. *Let $X^K = *$ for all $K \notin (H)$ and $H \leq G$. If one has $E_*(X) \cong E'_*(X)$ for two different $\mathbb{Z}$-graded homology theories, then $E_*(X) \cong E'_*(X)$ for all $X \in \mathsf{GTop}$.*

Note that both sides of θ_H are $\mathbb{Z}$-graded homology theories, and $\theta_K = 0$ for all $K \notin (H)$. Now we can induct on (H) by the proposition 0.11 and conclude the proof of (0.4).

Remark 0.12. The spectral version of the lemma 0.10 is the Adams isomorphism. Explicitly,

Theorem 0.13 (Adams isomorphism)**.** *Let $q : G \to G/N$ be the canonical projection, where $H \trianglelefteq G$ is a normal subgroup. Then the induced map $\jmath^* : \mathsf{Sp}_{G/N} \to \mathsf{Sp}_G$ of naive G-spectra induces an isomorphism*

$$[\jmath^* Y, X]_G \cong [Y, X/N]_{G/N},$$

where X/N is the N-orbit of X.

This is the key step to drawing the Tate diagram. We will see it soon. Moreover, the above result can be generalized:

Theorem 0.14 (Reich-Varisco, 2015)**.** *The same result holds for orthogonal G-spectra.*

There are some relations between the (additive) transfers and the tom Dieck theorem. Consider the nested subgroups $K \leq H \leq G$, with projection $\mathrm{proj} : G/K \to G/H$. Let $W \in R[H]$ be an H-representation. It corresponds to the map $i : H/K \to W$. By scaling i, one can assume WLOG the open balls around the image points $i(hK) = hw$ are pairwise disjoint. We can get an embedding from that: $H/K_+ \wedge D(W) \to W$. By one-point compactification, which sends the complement of $i(H/K_+ \wedge D(W))$ to the ∞ point, we get a new map

$$S^W \to H/K_+ \wedge S^W.$$

Apply the base-change functor $G_+ \wedge_H -$, we obtain the map

$$\mathrm{tr}_H^K : G/H_+ \wedge_H S^W \to G/K_+ \wedge S^W, \tag{0.15}$$

which is the **additive transfer**. To compute tr_H^K, observe that it induces homomorphisms in E_* and E^* for suitable (co)homology E. Then the composite

$$(\mathrm{tr}_H^K)^* \circ \mathrm{proj}^* : E(G/H_+ \wedge S^W) \to E(G/K_+ \wedge S^W) \to E(G/H_+ \wedge S^W)$$

is the multiplication by $|H/K|$, which is actually the degree map of $S^W \to H/K_+ \wedge S^W \to S^W$. Algebraically, after passing both sides of tr_H^K to some coefficient system, the resulting map is some "wrong way" map of different abelian groups. For example, let $\underline{A} : \mathsf{Orb}_G^{op} \to \mathsf{Ab}$ be a coefficient system with values and maps by

$$\begin{array}{c} A(G/G) \\ \big\downarrow \mathrm{Res}_e^G \\ A(G/e) \end{array}$$

then tr_e^G gives an arrow (colored in red) in the inverse direction:

$$\begin{array}{c} A(G/G) \\ \mathrm{tr}_e^G \big\uparrow \; \big\downarrow \mathrm{Res}_e^G \\ A(G/e) \end{array}$$

It then becomes a Mackey functor. An example of a Mackey functor is $\underline{\pi}_*$. Moreover, if we want to record the ring structure, then there is a multiplicative version of transfer called the "norm". Passing to some Mackey functor, the new resulting functor with information of multiplication is known as the Tambara functor. It is beyond our scope for this talk, so we will not discuss it in any further depth.

Consider the case when $X = S^0$ with the trivial G-action. Then by the tom Dieck theorem (0.4) we get

$$\pi_0^G(\mathbb{S}) \cong \bigoplus_{\{(H):H \leq G\}} \pi_0^{WH}(\Sigma_+^\infty EWH), \tag{0.16}$$

where $\mathbb{S} = \Sigma^\infty S^0$ is the sphere spectrum. For any finite group H, the corollary 0.6 yields

$$\pi_0(\mathbb{S}) \xrightarrow{\cong} \pi_0^H(H_+ \wedge \mathbb{S}) = \pi_0^H(\Sigma_+^\infty H).$$

In fact, the latter is isomorphic to $\pi_0^H(\Sigma_+^\infty EH)$, and the isomorphism is induced by the H-equivariant action map $h : H \to EH$ by sending any point x to $x \cdot p$ for the chosen basepoint $p \in EH$. To see why this is true, consider the skeletal filtration of EH with k-th skeleton denoted by $E^{(k)}H$. By standard construction of EH, $E^{(k)}H/E^{(k-1)}H$ is homeomorphic to $H \wedge (H^k \wedge S^k)$. The corollary 0.6 implies

$$\pi_*^H(H \wedge \Sigma^\infty(H^k \wedge S^k)) \cong \pi_*(\Sigma^\infty(H^k \wedge S^k)) \cong \pi_{*-k}(\Sigma^\infty H^k).$$

Thus the 0-th and 1-st H-equivariant stable homotopy groups of $E^{(k)}H/E^{(k-1)}H$ are 0 as $k \geq 2$, and $\pi_0^H(H \wedge \Sigma^\infty E^{(0)}H) \cong \pi_0(\Sigma^\infty\{\mathrm{pt}\}) = 0$. It follows that $E^{(1)}H \hookrightarrow EH$ induces an isomorphism

$$\pi_0^H(H \wedge \Sigma^\infty E^{(1)}H) = \pi_0^H(\Sigma_+^\infty E^{(1)}H) \cong \pi_0^H(\Sigma_+^\infty EH).$$

On the other hand, the sequence

$$\pi_1^H(\Sigma_+^\infty(E^{(1)}H/E^{(0)}H)) \xrightarrow{\delta} \pi_0^H(\Sigma_+^\infty E^{(0)}H) \to \pi_0^H(\Sigma_+^\infty E^{(1)}H) \to 0$$

is exact, and the connecting homomorphism δ is actually trivial. It follows that $E^{(0)}H \hookrightarrow EH$ induces an isomorphism

$$\pi_0^H(\Sigma_+^\infty E^{(0)}H) \cong \pi_0^H(\Sigma_+^\infty EH).$$

Since $E^{(0)}H$ is a discrete space with free H-action, $\Sigma_+^\infty E^{(0)}H \simeq H_+ \wedge \mathbb{S}$. Hence,

$$\pi_0(\mathbb{S}) \cong \pi_0^H(\Sigma_+^\infty EH).$$

Plugging into (0.16) with H replaced by WH, we get

$$\pi_0^G(\mathbb{S}) \cong \bigoplus_{\{(H):H\leq G\}} \pi_0(\mathbb{S}) \cong \bigoplus_{\{(H):H\leq G\}} \mathbb{Z}. \tag{0.17}$$

Note that $\pi_0^G(\mathbb{S})$ has a ring structure: for $f : S^V \to S^V$ and $g : S^W \to S^W$, $f \smile g : S^{V\oplus W} \to S^{V\oplus W}$. Define $A(G)$ to be the Grothendieck completion of the set of isomorphism classes of finite G-sets, equipped with direct sum as addition and product of G-sets as multiplication. $A(G)$ is then a well-defined ring, known as the Burnside ring.

Theorem 0.18. *There is a ring isomorphism*

$$A(G) \to \pi_0^G(\mathbb{S})$$
$$[G/H] \mapsto \mathrm{tr}_H^G(1)$$

where $1 \in \pi_0^G(\mathbb{S})$ *is the multiplicative unit.*

So far, we have shown that in genuine G-spectra, Φ^G and Σ_G^∞ interact well, while $(-)^G$ and Σ_G^∞ do not. Moreover, Φ^G satisfies the distributive law with respect to the smash product:

$$\Phi^G \circ (- \wedge -) = (\Phi^G \circ -) \wedge (\Phi^G \circ -).$$

However, we have not checked that $\Phi^G(X) = (EG \wedge X)^G$ really gives you a spectrum. To do that, we give a point-set model for Φ^G. Let $X \in \mathsf{GSU}$, and ρ be the regular representation of G with $\rho^G \cong \mathbb{R}$. For each n,

- $(\Phi^G X)_n = X(\rho \otimes \mathbb{R}^n)^G$, and

- the structure map is given by

$$\begin{aligned}(\Phi^G X)_n \wedge S^1 &\cong X(\rho \otimes \mathbb{R}^n)^G \wedge S^1 \\ &\cong (X(\rho \otimes \mathbb{R}^n) \wedge S^\rho)^G \\ &\to X((\rho \otimes \mathbb{R}^n) \oplus \rho)^G \cong X(\rho \otimes \mathbb{R}^{n+1})^G \\ &\cong (\Phi^G X)_{n+1}\end{aligned}$$

To see how this model is compatible with the one in the definition 0.1, we need appropriate models for EG and $\widetilde{EG}$. Note that for each $V \in R[G]$, let $\mathcal{F}_V$ be the collection of subgroups $H \leq G$ such that $V^H \neq 0$. Let $S(\infty V)$ be the unit sphere in the infinite-dimensional representation $\infty V = \oplus_{\mathbb{N}} V$; that is,

$$S(\infty V) = \bigcup_{n \geq 0} S(nV),$$

equipped with weak topology. Then $S(\infty V)^H = S(\infty(V^H))$. This is empty if $H \notin \mathcal{F}_V$ and contractible if $H \in \mathcal{F}_V$. Now taking $V = \rho$, the regular representation of G, and $\mathcal{F} \coloneqq \mathcal{F}_\rho$. Since $\rho^H \neq 0$ for all proper subgroup of G, while $\rho^G = 0$, we can take $E\mathcal{F} = S(\infty\rho)$ because $S(\infty\rho)$ is empty if $H \notin \mathcal{F}$ and contractible if $H \in \mathcal{F}$. The infinite representation sphere $S^{\infty\rho}$ is thus a model for $\widetilde{E\mathcal{F}}$, the homotopy cofiber of $E\mathcal{F}_+ \to S^0$. It is a fact that the inclusion

$$S^0 \to \widetilde{E\mathcal{F}}$$

induces an isomorphism of G-fixed points

$$(S^0)^G = S^0 \to (\widetilde{E\mathcal{F}})^G. \tag{0.19}$$

This is because $\widetilde{E\mathcal{F}}$ is the unreduced suspension of $E\mathcal{F}$, $(E\mathcal{F})^G = \varnothing$, and fixed points commute with mapping cones. So for every G-space M, the map

$$M \wedge S^0 = M \to \widetilde{E\mathcal{F}} \wedge M$$

induces an isomorphism of G-fixed points

$$M^G \to (\widetilde{E\mathcal{F}} \wedge M)^G.$$

Hence, for $X \in \mathsf{GSU}$, and $n \in \mathbb{N}$, there is an isomorphism

$$(\Phi^G(X))_n = X(\rho \otimes \mathbb{R}^n)^G \cong (\widetilde{E\mathcal{F}} \wedge X(\rho \otimes \mathbb{R}^n))^G = ((\widetilde{E\mathcal{F}} \wedge X)^G)_n,$$

coinciding with the definition 0.1.

The geometric fixed points have more nice properties. We list two of them:

(1) Φ^G preserves π_*-isomorphisms.
(2) Φ^G commutes with filtered colimits.

The proof is omitted.

Are there any applications for the geometric fixed points?

One extremely important usage of the geometric fixed points is the Tate diagram. Let X be a genuine G-spectrum. There is a map $\epsilon : X \to \mathrm{Map}(EG_+, X)$ inducing

the following diagram
(0.20)

$$\begin{array}{ccccc} EG_+ \wedge X & \longrightarrow & X & \longrightarrow & \widetilde{EG} \wedge X \\ \downarrow{\scriptstyle \mathrm{id}\wedge\epsilon} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \mathrm{id}\wedge\epsilon} \\ EG_+ \wedge \mathrm{Map}(EG_+, X) & \longrightarrow & \mathrm{Map}(EG_+, X) & \longrightarrow & \widetilde{EG} \wedge \mathrm{Map}(EG_+, X) \end{array}$$

To simplify, there is a theorem by Greenlees and May:

Theorem 0.21 (Greenlees-May)**.** $EG_+ \wedge X \simeq EG_+ \wedge \mathrm{Map}(EG_+, X)$.

Using the theorem, together with applying the G-fixed points functor $(-)^G$ to the diagram (0.20), and using the Adams isomorphism to conclude $(EG_+ \wedge X)^G \simeq X_{hG}$, we obtain the Tate diagram for X:

$$\begin{array}{ccccc} X_{hG} & \longrightarrow & X^G & \longrightarrow & (\widetilde{EG} \wedge X)^G = \Phi^G X \\ \downarrow{\scriptstyle =} & & \downarrow & & \downarrow \\ X_{hG} & \longrightarrow & X^{hG} & \longrightarrow & X^{tG} \end{array} \tag{0.22}$$

where $X_{hG} = EG_+ \wedge X$ is the homotopy orbits, $X^{hG} = (\mathrm{Map}(EG_+, X))^G$ is the homotopy fixed points, and $X^{tG} := (\widetilde{EG} \wedge \mathrm{Map}(EG_+, X))^G$ is the Tate spectrum of X.

The Tate diagram provides a computable way for people to understand X^G since there are spectral sequences for X_{hG}, X^{hG}, and X^{tG}. One modern usage of the Tate diagram (0.22) is the (cyclotomic) trace method, or in particular, the construction of the topological cyclic homology TC. Set $G = C_{p^n}$ for p a prime, and $X = THH(A)$ for $A \in \mathsf{CRing}$, the topological Hochschild homology spectrum of A. A theorem by Hesselholt and Madsen said that as a genuine S^1-spectrum, X is moreover a cyclotomic S^1-spectrum, i.e. $\rho_p^* \Phi^{C_p} X \simeq X$ for $\rho_p : S^1 \to S^1/C_p$. Write $X_n = THH(A)^{C_{p^{n-1}}}$ for $n \geq 1$. We can define three operations as follows:

(1) $F : X_{n+1} \to X_n$ induced by the inclusion of fixed points. This is known as the Frobenius.
(2) $F : X_n \to X_{n+1}$ is the transfer $\mathrm{tr}_{C_{p^{n-1}}}^{C_{p^n}}$. This is known as the Verschiebung.
(3) $R : X_{n+1} \to X_n$ is known as the restriction. It is given by

$$THH(A)^{C_{p^n}} \simeq (THH(A)^{C_p})^{C_{p^{n-1}}} \to (\Phi^{C_p} THH(A))^{C_{p^{n-1}}} \simeq THH(A)^{C_{p^{n-1}}}.$$

The three operations satisfy some good properties. For our interest, we only need to know $FR = RF$ and $VR = RV$ at this point. As suggested by the names, one would expect that X_n might have something to do with the Witt vectors. This turns out to be the case:

Theorem 0.23 (Hesselholt-Madsen)**.** *There is an isomorphism of rings $f : W_n(A) \cong \pi_0(X_n)$, where W_n is the ring of n-ary Witt vectors, such that R, V, F commute with f, respectively.*

Now we have a diagram

$$\cdots \underset{F}{\overset{R}{\rightrightarrows}} X_{n+1} \underset{F}{\overset{R}{\rightrightarrows}} X_n \underset{F}{\overset{R}{\rightrightarrows}} \cdots \underset{F}{\overset{R}{\rightrightarrows}} X_1 \underset{F}{\overset{R}{\rightrightarrows}} X_0$$

Let $TR(A) = \operatorname{holim}_R X_n$ and $TF(A) = \operatorname{holim}_F X_n$. Since $RF = FR$, F also induces $F : TR \to TR$ and similarly $R : TF \to TF$. Let $TC(A) = \operatorname{hofib}(F - \mathrm{id}) = \operatorname{hofib}(R - \mathrm{id})$. This is the topological cyclic homology of A. The cyclotomic trace is the map for $i \geq 0$,

$$K_i(A; \mathbb{Z}_p) \xrightarrow{\cong} \pi_i(TC(A)^{\wedge}_p)$$

of $W(k)$-algebras, where k is perfect and $\operatorname{char} k = p$, and $K_i(A; \mathbb{Z}_p)$ is the i-th algebraic K-theory group of A in $\mathbb{Z}_p$-coefficient. One thing to keep in mind is that $W(\mathbb{F}_p) \cong \mathbb{Z}_p$. This is a modern way to compute the algebraic K-theory. We end this section with the following remarkable theorem:

Theorem 0.24 (Goodwillie-Dundas-McCarthy). *Let $A \to B$ be a surjection of commutative rings with nilpotent kernel. Then there is a homotopy pullback of spectra*

$$\begin{array}{ccc} K(A) & \longrightarrow & TC(A) \\ \downarrow & & \downarrow \\ K(B) & \longrightarrow & TC(B) \end{array}$$

References

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