Homological Algebra of Quiver Representations

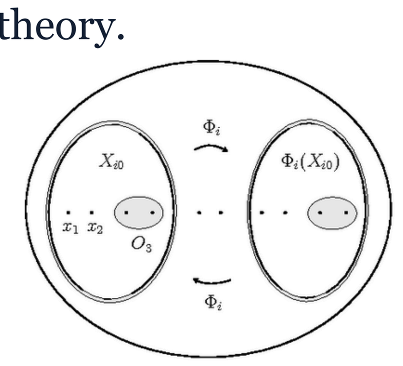
Jinghui Yang[1], Songling Zhou[2], Zac MacAdam[3], Alice Chudnovsky[4], Mosaad Al Thokair[5]

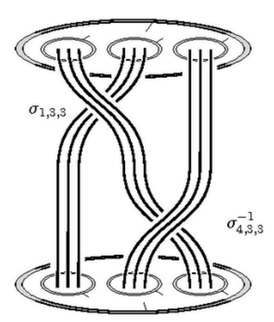
[1,3,4,5]Department of Mathematics, College of Liberal and Arts, University of Illinois at Urbana-Champaign, [2]Department of Economics, College of Liberal and Arts, University of Illinois at Urbana-Champaign

INTRODUCTION

In the study of "Geometric Group Theory", we usually want to find out finitely presented group acting on some surface. This leads to the consideration of **mapping class group**, which is defined to be the quotient of group of all homeomorphisms by isotopy relation. To achieve that, we should consider its action on the first homology group with coefficient $\mathbb{R}$, which appears to be a vector space, but this action is hardly faithful. However, in a special case called **Burau representation**, the **braid group** B_n is directly isomorphic to the mapping class group of a disk with n holes, which acts faithfully on the first homology group of the covering space corresponding to the projection from the first homology group of this surface to $\mathbb{Z}$, when n is less than 4. So we narrow down our interests to this situation to study how to generalize the faithfulness condition, which leads to the study of **Khovanov-Seidel quiver algebra**.

Our project is therefore to study the quiver algebra and quiver representation to obtain a comprehensive understanding of the Khovanov-Seidel algebra with the help of computer program. Eventually we hope to work out some specific achievable goals to observe better of the whole theory.





AIM

Theoretical Part

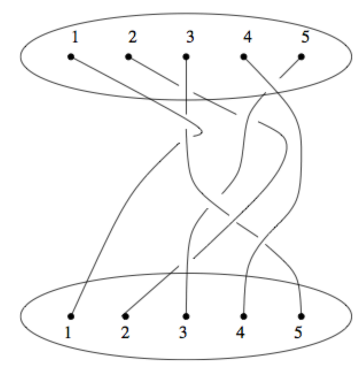
We will study the theory of quiver representations, with the tools from homological algebra. Later, we hope to use the knowledge to examine the Khovanov-Seidel algebra to see how the improvement of faithfulness will be developed by categorification. As a reachable goal, we will use the knowledge to work on the related recipe problems.

Computational Part

We have known Burau representation is not faithful in general, but Khovanov and Seidel found a related categorification to make it faithful (called Khovanov-Seidel Categorification). So our goal is hopefully to verify its faithfulness through computer programs. This involves the use of the programs, like Sage or GAP, to model the representations of quivers and to compute some chain complexes or mapping cones (as we currently work on). Also, we are working through the methods to crack the related recipe problems.

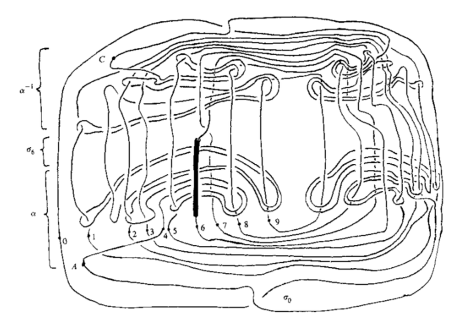
BACKGROUND

Definition. A **braid** is an intertwining collection of "strings" that are attached to top and bottom "bars". A **braid group** is a group whose elements are braids and group operation is concatenation.



Definition. A representation is known as **faithful**, if the trivial object of the group can only be represented by the trivial object of the other group.

Definition. The **(reduced) Burau representation** is corresponding to the action of braid groups on the 1st homology group of some complicated covering spaces. Unfortunately it is not faithful for braids of n≥5 strands (not clear when n=4).

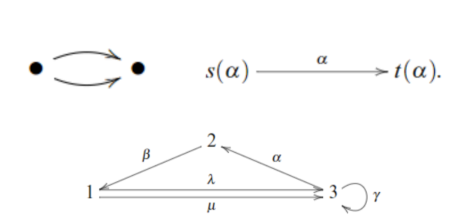


Definition. The **categorification** is the process of replacing set-theoretic theorems by category-theoretic analogues. Namely, we have the following correspondence:

elements	objects
equations between elements	isomorphisms between objects
sets	categories
functions	functors
equations between functions	natural isomorphisms between functors

Remark. By using this process, we can present each group element in an easier way for us to work with.

Definition. A **quiver** is just a collection of vertices, arrows and the assignments tell the starting points and ending points of arrows.



Definition. A **quiver representation** is the assignment of a vector space to each vertex and a linear map to each arrow of the quiver.

$$M \quad k \xrightarrow{1} k$$
$$M' \quad k \xrightarrow{0} k$$
$$M'' \quad k \xrightarrow{0} 0$$
$$M''' \quad k^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}} k^3$$

CURRENT PROGRESS

Theoretical Part

Currently we are studying the theory of quiver representation, including things like path algebras and projective resolutions. Meanwhile, we are working through the problems related to our topic. For example:

Consider the so-called **Beilinson quiver for the projective line**: $\bullet \rightrightarrows \bullet$

with the representations given by

$$P_0 = \left(0 \underset{0}{\overset{0}{\rightrightarrows}} \mathbb{C} \right),$$

$$P_1 = \left(\mathbb{C} \underset{e_2}{\overset{e_1}{\rightrightarrows}} \mathbb{C}^2 \right), \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We are trying to compute the hom complex hom*(P_0,P_1). Also, we are trying to construct the mapping cone Q with respect to some point in the previous complex and compute the hom complex hom*(P_0,Q). Generally, we want to show the how these process will continue in higher dimensions with what interesting properties appear later by studying the chain complexes generated by them and their cohomology spaces.

Computational Part

We want to develop a solution to the problems through not only the theoretical deductions but also the computer programs. To achieve this, we use computer software **Sage** and **GAP** to model the representations and help us with some easy calculations. Here is an example:

```
gap> LoadPackage("qpa");
gap> Q := Quiver(2, [ [1,2,"a"], [1,2, "b"]]);;
gap> PA := PathAlgebra( Rationals, Q );;
gap> rels := [ PA.a*PA.b ];;
gap> gb := GBNPGroebnerBasis( rels, PA );;
gap> I := Ideal( PA, gb );;
gap> alg := PA/I;;
gap> M := RightModuleOverPathAlgebra( alg, [0,1], [] );;
gap> N := RightModuleOverPathAlgebra( alg, [1,2], [ ["a", [[1, 0]]], ["b", [[0, 1]]]    ] );;
gap> f := RightModuleHomOverAlgebra(M, N, [ [[0]],[[1, 0]] ]);;
gap> lf := ComparisonLiftingToProjectiveResolution(f);
<chain map>
gap> H := MappingCone(lf);
[ --- -> -1:(1,2) -> ---, <chain map>, <chain map> ]
gap>
```

This is our initial work on cracking part of the problem stated above of construction of the cone Q, which we try to find the mapping cone using modules.

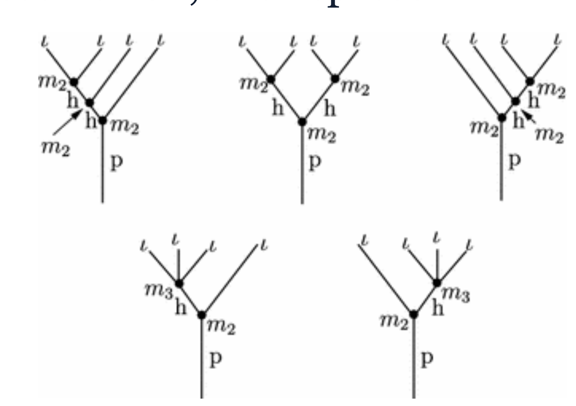
The use of Sage and GAP will visualize the process of every step, which is relatively easy to grasp than the theoretical knowledge. The high efficiency will save us a lot of time without loss of accuracy.

FUTURE PLANS

The majority of our project has been a crash course into Homological Algebra and Geometric Group Theory. Now that we've somewhat familiar with the background knowledge and the use of software, we're ready to take a deeper look of quiver representations and how they will play a role in research of Khovanov-Seidel algebras and Burau representations.

Some next steps:

- Solve the current problems in both theoretical and software (using Sage or GAP) ways, from which develop a solid foundation of observing the basic methods in studying quiver representations.
- Learn the concept of Burau representation in detail to see why it is not faithful in general.
- Learn the basic concept of Khovanov-Seidel algebras to get the inspiration on how it improves the results in Burau representation, and represent it in the computer if possible.



REFERENCES

[1] Seidel, P., *More about Vanishing Cycles and Mutation*, arXiv:math/0010032 [math.SG] (2000).
[2] Seidel, P. and Khovanov, M. *Quivers, Floer Cohomology, and Braid Group Action*, arXiv:math/0006056 [math.QA] (2001).
[3] Bigelow, S. *The Burau representation is not faithful for n=5*, arXiv:math/9904100 [math.GT] (1999).
[4] Schiffler R. *Quiver Representations*, Springer, Switzerland, 2014.

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